Parameterized Complexity and Local Search Approaches for the Stable Marriage Problem with Ties

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Abstract. We consider the variant of the classical Stable Marriage problem where preference lists can be incomplete and may contain ties. In such a setting, finding a stable matching of maximum size is NP-hard. We study the parameterized complexity of this problem, where the parameter can be the number of ties, the maximum or the overall length of ties. We also investigate the applicability of a local search algorithm for the problem. Finally, we examine the possibilities for giving an FPT algorithm or an FPT approximation algorithm for finding an egalitarian or a minimum regret matching.

1 Introduction

The Stable Marriage or Stable Matching problem was introduced by Gale and Shapley in [4]. In the classical problem setting, we are given a set of women, a set of men, and a preference list for each person, containing a strict ordering of the members of the opposite sex. The task is to find a matching between men and women that is stable in the following sense: there are no man m and woman w both preferring each other to their partners in the given matching. Gale and Shapley gave a linear time algorithm that always finds a stable matching [4, 6]. Moreover, it is easy to see that every stable matching must have the same size.

Practical applications motivated several reformulations of the classical problem in the recent decades [21, 22]. Among the most studied variants, the following two relaxations have significant practical importance, and thus have been investigated in many ways. First, preference lists may be "incomplete", meaning that a person may find some members of the opposite gender unacceptable. Second, the ordering in the preference lists may not be strict, resulting in "ties". On one hand, a stable matching can still be found in these modified situations by a simple extension of the Gale-Shapley algorithm [6], and if only one of the above relaxations is allowed then the stable matchings still have to be of the same size. On the other hand, in the case when we allow *both* incomplete lists and ties to be present in the model, then stable matchings of various size may exist. It has been proven in [11] that finding a stable matching of maximum size in this situation is NP-hard. Since then, several researchers have attacked the problem, most of them presenting approximation algorithms [10, 13].

We investigate this problem in the framework of parameterized complexity, introduced by Downey and Fellows [2]. In this approach, we assign a parameter to each problem instance, and we look for algorithms that have efficient running time if the parameter remains small. An algorithm is called *fixed-parameter tractable* or FPT, if it has running time $O(f(k)|I|^c)$, where k is the parameter assigned to the input I, f is a computable function, and c is a constant. If we allow ties in the Stable Marriage problem, then the number of ties or the maximum length of ties in an instance arise naturally as parameters.

We also consider a local search approach for this problem. Local search is a successful technique that has been applied in optimization problems for many decades [1]. However, there are only a few results investigating the connection of parameterized complexity and local search, although attention to this topic has been increasing recently [16]. The basic idea of local search is to improve a given initial solution step by step. The key procedure of this method, which is executed in each step, has the following tasks: given some initial solution S, find a better solution in the ℓ-neighborhood of S. Clearly, there is a tradeoff between the size of the neighborhood this procedure has to search through, and the expected value of the improvement. From this point of view, it is worth studying the running time of such a procedure as a function of ℓ. Consequentially, it is natural to ask whether we can give an FPT algorithm for this problem with parameter ℓ. This question has already been studied in a couple of optimization problems [12, 14, 18], and also in the context of Hospitals/Residents with Couples problem [19]. Considering the Maximum Stable Marriage with Ties and Incomplete Lists problem, where a maximum size stable matching has to be found, we present results stating that a local search algorithm for this problem cannot have FPT running time (under some standard complexity theoretic assumption).

Finally, we also study the possibility of giving an FPT approximation algorithm [17] for two problems in which we look for a stable matching that may not have maximum size, but is of minimum cost in some sense. Both of these problems (namely, finding an egalitarian or a minimum regret stable matching) are polynomial time solvable if no ties are allowed, but are inapproximable by polynomial time algorithms in a strong sense otherwise. We examine the possibilities of giving an approximation algorithm for these problems with running time that is not polynomial but is FPT, when considering some natural parameters.

The paper is organized as follows. Section 2 covers the preliminaries. We present our results in Sect. 3. The problem of finding a stable matching of maximum size is investigated in Subsect. 3.1, and contributions for the problem of finding an egalitarian or a minimum regret stable matching are discussed in Subsect. 3.2. A summary of our results can be found in Sect. 4.

2 Preliminaries

For an integer n, we use $[n]$ to denote the set $\{1, 2, \ldots, n\}$, and we write $\binom{[n]}{2}$ for the set $\{(i, j) \mid 1 \leq i < j \leq n\}$.

Parameterized complexity. A *parameterized problem* is a pair (Q, κ) where $Q \subseteq \Sigma^*$ is a decision problem over some alphabet Σ, and $\kappa : \Sigma^* \to \mathbb{N}$ is a *parameterization* of the problem, assigning a *parameter* to each instance of Q. An algorithm is *fixed-parameter tractable* or FPT, if it has running time at most $f(k)n^c$ for some computable function f and constant c, where n is the input length and k is the parameter assigned to the input. A parameterized problem is FPT, if it admits an FPT algorithm. Clearly, this notation can be extended to handle also *combined parameterizations* that assign not only one, but two (or more) parameters to each instance of a given problem. In this case, the running time of an FPT algorithm on an instance of length n and parameters k_1 and k_2 assigned to it must be at most $f(k_1, k_2)n^c$ for some function f and constant c.

Given two parameterized problems (Q_1, κ_1) and (Q_2, κ_2) over the alphabet Σ, an *FPT reduction* from (Q_1, κ_1) to (Q_2, κ_2) is a function $g : \Sigma^* \to \Sigma^*$, computable by an FPT algorithm, such that $I \in Q_1$ if and only if $g(I) \in Q_2$ and $\kappa_2(g(I)) \leq f(\kappa_1(I))$ for some computable function f, for every $I \in \Sigma^*$. FPT reductions are used in parameterized complexity theory to establish hardness results, analogously as polynomial reductions are used in classical complexity theory. The class of W[1]-hard problems are closed under FPT reductions, so an FPT reduction from a W[1]-hard problem to a given parameterized problem P shows that P is also W[1]-hard, and thus it is not FPT unless the widely believed W[1] $\neq$ FPT conjecture falls. Therefore, such a reduction is considered as a strong evidence for $P \notin$ FPT. The FPT reductions in this paper are from the W[1]-hard parameterized Clique problem, in which a graph G and a parameter k is given, and the task is to decide whether there is a clique of size k in G. For further details on parameterized complexity, see e.g. [2], [20] or [3].

Local search. Let Q be an optimization problem with an objective function T to be maximized. We suppose that some *distance* $d(S_1, S_2)$ is defined for each pair (S_1, S_2) of solutions for some instance I of Q. Using this distance function, we say that a solution S_1 is *ℓ-close* to a solution S_2 if $d(S_1, S_2) \leq \ell$. In this paper, we define the task of a local search algorithm in a permissive sense that has been introduced in [19]. Namely, a *permissive local search* algorithm for Q solves the following task:

Permissive local search for Q:	
Input:	(I, S_0, ℓ) where I is an instance of Q, S_0 is a solution for I, and $\ell \in \mathbb{N}$.
Task:	If there exists a solution S for I such that $d(S, S_0) \leq \ell$ and $T(S) > T(S_0)$, then output *any* solution S' for I with $T(S') > T(S_0)$.

According to this, given an instance I of Q, an initial solution S_0 for it and an integer ℓ, a *permissive local search* algorithm for Q is allowed to output any solution that is better than S_0, so its output does not have to be close to S_0. We remark that this differs from the usual definition of a local search algorithm, where the task is to output a solution that is ℓ-close to the given initial solution, and is better than that. Referring to such an algorithm as a *strict local search algorithm*, we clearly get that any strict local search algorithm is also a permissive one at the same time. Thus, all of our hardness results that consider permissive local search algorithms, immediately apply to strict local search algorithms as well. Note also that any algorithm that finds an optimal solution for any instance of Q can be considered as a permissive local search algorithm for Q.

3 Stable Marriage with Ties and Incomplete Lists

The input of the Stable Marriage with Ties and Incomplete Lists (or SMTI) problem is a triple (X, Y, r). Here X and Y are sets of *women* and *men*, respectively. A $p \in X \cup Y$ is a *person*, and for each person p we define $\mathcal{O}(p)$ to be the set containing the members of the opposite sex. The *ranking function* $r : (X \times Y) \cup (Y \times X) \to \mathbb{N} \cup \{\infty\}$ describes the ranking of the members of the opposite sex for each person. A person b is *acceptable* for a person $a \in \mathcal{O}(b)$ if $r(a,b) < \infty$. We assume that acceptance is mutual, i.e. either $r(a,b) = r(b,a) = \infty$ or a and b are acceptable for each other, forming an *acceptable pair*. We say that *a prefers b to c* if $r(a,b) < r(a,c)$. Ties may occur, meaning that $r(a,b) = r(a,c)$ is possible even if $b \neq c$. Formally, a *tie* with respect to a is a set $T \subseteq \mathcal{O}(a)$ of maximum cardinality such that $|T| \geq 2$ and $r(a,t_1) = r(a,t_2) \neq \infty$ for every $t_1, t_2 \in T$. A person p is *indifferent*, if there exists a tie w.r.t. p, and the *length* of a tie T is $|T|$.

For an instance $\mathcal{I}$ of SMTI, we will use the following parameterization functions:

- $\kappa_1(\mathcal{I})$ denotes the number of ties in $\mathcal{I}$.
- $\kappa_2(\mathcal{I})$ denotes the maximum length of a tie in $\mathcal{I}$.
- $\kappa_3(\mathcal{I})$ denotes the *total length of the ties* in $\mathcal{I}$, which is the sum of the length of each tie in the instance. Clearly, $\kappa_3(\mathcal{I}) \leq \kappa_1(\mathcal{I})\kappa_2(\mathcal{I})$.

In our proofs, we will also use a different notation to define an instance of the SMTI problem. Instead of determining the ranking function r by giving the value $r(a,b)$ for every possible pair (a,b), we will define the ranking function r implicitly by giving the *precedence list* $P(a)$ for each person a. The precedence list $P(a)$ is an ordered list containing the acceptable partners for a. Since ties may be involved, the ordering of these lists is not necessarily strict. For some $b \in \mathcal{O}(a)$, if b is not contained in $P(a)$ then we let $r(a,b) = \infty$, and if b is contained in $P(a)$ then we define $r(a,b)$ as the (possibly joint) ranking of b in $P(a)$ (i.e. one plus the number of persons strictly preceding b in $P(a)$).

A *matching* for (X, Y, r) is a subset M of the acceptable pairs w.r.t. r, where $|\{q \mid pq \in M\}| \leq 1$ for each person p. If $xy \in M$, then we say that x and y are *covered* by M, M *assigns* y to x and vice versa, which will be denoted by $M(x) = y$ and $M(y) = x$. We will use the notation $M(x) = \emptyset$ for the case when x is not covered by M, and we also extend r such that $r(p, \emptyset) = \infty$ for each person p. The *size* of a matching M, denoted by $|M|$, is the number of pairs contained in M. A pair xy is a *blocking pair for* M if $r(x,y) < r(x,M(x))$ and $r(y,x) < r(y,M(y))$, i.e. both x and y prefer each other to their partner in M (if exists). A matching is *stable* if no blocking pair exists for it. The task of the SMTI problem is to find a stable matching, if exists. We remark that other definitions of stability are also in use, such as strong and super-stability [9]. The definition of stability used by us, which received the most attention, is sometimes referred to as weak stability.

Although it is known that a stable matching exists for every instance of SMTI, there are several problems connected to stable matchings that are much harder. In Section 3.1, we study the Maximum Stable Marriage with Ties and Incomplete Lists problem, where the task is to find a stable matching of maximum size. In Section 3.2, we investigate two problems where we aim to find stable matchings that may not be of maximum size, but have some other useful properties.

3.1 Stable matchings of maximum size

If the preference lists are complete, meaning that each person finds every member of the opposite gender acceptable, or if no tie can be contained in the preference lists, then every stable matching must have the same size [6]. But if both ties and incomplete preference lists may occur, then stable matchings of different sizes may exist for a given instance [15]. The following problem, called Maximum Stable Marriage with Ties and Incomplete Lists (or shortly MaxSMTI), has been shown to be NP-hard [11]: given an instance $\mathcal{I}$ of SMTI and an integer s, find a stable matching for $\mathcal{I}$ of size at least s.

Moreover, it has been proven in [15] that MaxSMTI is NP-complete even in the special case when only women can be indifferent, each tie has length 2, and ties are only present at the end of the preference lists (i.e. if $t \in T$ for a tie T w.r.t. a, then $r(a,x) > r(a,t)$ implies $r(a,x) = \infty$). However, if no ties are involved in an instance of MaxSMTI, then a stable matching of maximum size can be found in linear time with an extension of the Gale-Shapley algorithm [4, 6]. This can be used when the total length of ties (i.e. $\kappa_3(\mathcal{I})$) is small for some instance $\mathcal{I}$, since we can apply a brute force algorithm that breaks ties in all possible ways and finds a stable matching of maximum size for all the instances obtained.

Theorem 1. MaxSMTI *is FPT with parameterization* κ_3.

Proof. Let $\mathcal{I} = (X,Y,r)$ be the instance given. We use a method for breaking ties as follows. Formally, let $\mathcal{I}$ contain the ties $\{T_i \mid i \in [\kappa_1(\mathcal{I})]\}$, and let t_i^j denote the j-th element of the tie T_i (according to some fixed order). If π_i is a bijection from T_i to $[|T_i|]$ (i.e. a permutation of T_i) for each $i \in [\kappa_1(\mathcal{I})]$, then the instance (X,Y,r') can be obtained from $\mathcal{I}$ by *breaking ties* according to $(\pi_1,\ldots,\pi_{\kappa_1(\mathcal{I})})$, if r' is defined such that $r'(a,b) < r'(a,c)$ if and only if either $r(a,b) < r(a,c)$ or b and c are both in the tie T_i w.r.t. a and $\pi_i(b) < \pi_i(c)$.

To produce a solution, we break ties in various ways. Let $P_i = \{\pi_i^j \mid j \in [|T_i|]\}$ where π_i^j is an arbitrary bijection from T_i to $[|T_i|]$ for which $\pi_i^j(t_i^j) = 1$ holds, i.e.

π_i^j puts the j-th element of T_i in the first place. Using this, we break ties according to each element of $P_1 \times \cdots \times P_{\kappa_1(\mathcal{I})}$, apply the Gale-Shapley algorithm for each instance obtained, and then output the stable matching having maximum size among the set $\mathcal{M}$ of stable matchings obtained.

We claim that all stable matchings for $\mathcal{I}$ can be obtained as a stable matching of an instance obtained from $\mathcal{I}$ by breaking ties this way. It is easy to verify that any matching in $\mathcal{M}$ is stable in M. Conversely, if M is a stable matching in $\mathcal{M}$, then it is also stable in the instance obtained by breaking ties according to $(\pi_1, \ldots, \pi_{\kappa_1(\mathcal{I})})$ where $\pi_i = \pi_i^j$ if T_i is a tie with respect to some a such that $M(a) = t_i^j$, otherwise π_i can be any permutation from P_i.

Clearly, as we have to break ties in at most

$$\prod_{i \in [\kappa_1(\mathcal{I})]} |T_i| \leq \left(\frac{\sum_{i \in [\kappa_1(\mathcal{I})]} |T_i|}{\kappa_1(\mathcal{I})} \right)^{\kappa_1(\mathcal{I})} \leq \frac{\kappa_3(\mathcal{I})}{\kappa_1(\mathcal{I})}^{\kappa_1(\mathcal{I})} \leq \kappa_3(\mathcal{I})^{\kappa_3(\mathcal{I})}$$

many ways, this method yields a solution in $O(\kappa_3(\mathcal{I})^{\kappa_3(\mathcal{I})} \cdot |\mathcal{I}|)$ time. □

Theorem 1 immediately raises the question of whether MaxSMTI is FPT if the parameter is the number of ties (κ_1). As claimed by Theorem 2, this problem turns out to be hard. Theorem 2 also states a negative result about the possibility of giving an efficient permissive local search algorithm for MaxSMTI. The proof of this result relies on a similar construction to the proof of the hardness result for MaxSMTI with parameterization κ_1, so we will prove them simultaneously.

Recall that the objective function to be maximized in the MaxSMTI problem is the size of the stable matching. We define the distance of two stable matchings M_1 and M_2 for $\mathcal{I}$ as the number of persons p in $\mathcal{I}$ such that $M_1(p) \neq M_2(p)$. We denote this value by $d(M_1, M_2)$. Accordingly, the task of a permissive local search algorithm for MaxSMTI, as defined in Sect. 2, is the following: given an instance $\mathcal{I}$ of MaxSMTI, a stable matching M_0 for $\mathcal{I}$, and an integer ℓ, if there is a stable matching M for $\mathcal{I}$ with $|M| > |M_0|$ and $d(M_0, M) \leq \ell$, then find any stable matching M' for $\mathcal{I}$ with $|M'| > |M|$.

Theorem 2 shows that no permissive local search algorithm can run in FPT time (assuming W[1] $\neq$ FPT), even if we regard not only the number of ties but also ℓ as a parameter for some input $(\mathcal{I}, S_0, \ell)$.

Theorem 2. *(a) The decision version of* MaxSMTI *is W[1]-hard with parameterization κ_1, even if only women can be indifferent.*
(b) If W[1] $\neq$ FPT, then there is no permissive local search algorithm for the MaxSMTI *problem that runs in FPT time with combined parameters $(\kappa_1(\mathcal{I}), \ell)$, even if only women can be indifferent.*

Proof. Let $G(V, E)$ be the input graph and k be the parameter for the Clique problem. We are going to construct an SMTI instance $\mathcal{I} = (X, Y, r)$ with $\kappa_1(\mathcal{I}) = \binom{k}{2} + k + 1$ ties, each being in the preference list of a woman, together with a stable matching M_0 for $\mathcal{I}$ of size $|X| - 1$ such that the following statements are equivalent:

(1) $\mathcal{I}$ has a stable matching of size at least $|M_0|+1$, and
(2) there is a clique of size k in G.

This immediately yields an FPT-reduction from Clique to MaxSMTI with parameterization κ_1, proving statement (a). Moreover, we will also show that every stable matching of size at least $|M_0|+1$ must be ℓ-close to M_0 for $\ell = 6\binom{k}{2}+4k+4$. Therefore, a permissive local search algorithm for MaxSMTI can be used to detect whether $\mathcal{I}$ has a stable matching of size at least $|M_0|+1$, i.e. whether G has a clique of size k. Therefore, this construction also proves (b).

By the nature of the SMTI problem, the presented reduction is more complex than a typical reduction that proves hardness for some graph theoretic problem, since we have to describe the preference list for each person of the constructed instance. To ease the understanding, we illustrate the construction in Fig. 1 by depicting the bipartite graph underlying the instance, where persons are represented by nodes and we connect two nodes if and only if the corresponding persons are acceptable for each other. Moreover, we use edge weights to represent ranks, and we use bold edges to mark the edges of a given matching.

We write $V(G)=\{v_1,v_2,\dots,v_n\}$ and $m=|E(G)|$. To define $\mathcal{I}=(X,Y,r)$, we construct a *node-gadget* $\mathcal{G}^i$ for each $i\in[k]$, an *edge-gadget* $\mathcal{G}^{i,j}$ for each $(i,j)\in\binom{[k]}{2}$, and a *path-gadget* $\mathcal{P}$. The node-gadget $\mathcal{G}^i$ consists of women $X^i\cup\{x_0^i\}$ with $X^i=\{x_u^i \mid u\in[n]\}$ and men $Y^i\cup\{y_0^i\}$ with $Y^i=\{y_u^i \mid u\in[n]\}$. Similarly, the edge-gadget $\mathcal{G}^{i,j}$ consists of women $X^{i,j}\cup\{x_0^{i,j}\}$ with $X^{i,j}=\{x_{u,z}^{i,j} \mid u<z, v_uv_z\in E(G)\}$ and men $Y^{i,j}\cup\{y_0^{i,j}\}$ with $Y^{i,j}=\{y_{u,z}^{i,j} \mid u<z, v_uv_z\in E(G)\}$. The path-gadget contains women $\{p_i \mid i\in[\binom{k}{2}+2]\}$ and men $\{q_i \mid i\in[\binom{k}{2}+2]\}$. The set of all these women and men define X and Y, respectively.

Let M_0 contain the pairs $x_u^iy_u^i$ and $x_{u,z}^{i,j}y_{u,z}^{i,j}$ for all possible i,j,u and z, and also the pairs p_hq_{h+1} for all $h\in[\binom{k}{2}+1]$. Note that $|M_0|=|X|-1$, since $p_{\binom{k}{2}+2}$ is the only unmatched woman. Let ν be a bijection from $[\binom{k}{2}]$ into the set $\binom{[k]}{2}$, and let $C(i,u)=\{x_{u,z}^{i,j} \mid i<j\le k, u<z, v_uv_z\in E(G)\}\cup\{x_{z,u}^{j,i} \mid 1\le j<i, z<u, v_zv_u\in E(G)\}$ for all $i\in[k], u\in[n]$. We define the ranking function r by giving the precedence list $P(a)$ for each person a below. A tie $T=\{t_1,\dots,t_i\}$ w.r.t. a is denoted by $(t_1,\dots,t_i)$ in $P(a)$, and we use $[s_1,\dots,s_i]$ to denote an arbitrary ordering of $s_1,\dots,s_i$. (If it is not confusing, we will simply write (S) or $[S]$ instead of listing the elements of S in the brackets.) Observe that there are indeed $\binom{k}{2}+k+1$ indifferent women, there is no indifferent man, and each indifferent woman has exactly one tie in her preference list. The indices i,j,u, and z take all possible values in the lists, unless otherwise stated. For brevity, we write k' for $\binom{k}{2}$.

$P(x_u^i): y_u^i, y_0^i$	$P(y_u^i): x_0^i, [C(i,u)], x_u^i$
$P(x_0^i): y_0^i, (Y^i)$	$P(y_0^i): [X^i], x_0^i$
$P(x_{u,z}^{i,j}): y_{u,z}^{i,j}, [y_u^i, y_z^j], y_0^{i,j}$	$P(y_{u,z}^{i,j}): x_0^{i,j}, x_{u,z}^{i,j}$
$P(x_0^{i,j}): y_0^{i,j}, (Y^{i,j})$	$P(y_0^{i,j}): [X^{i,j}], p_{\nu^{-1}(i,j)}, x_0^{i,j}$
$P(p_h): q_{h+1}, y_0^{\nu(h)}, q_h$ if $h\in[k']$	$P(q_h): p_h, p_{h-1}$ if $2\le h\le k'+1$
$P(p_{k'+1}): (q_{k'+1}, q_{k'+2})$	$P(q_1): p_1$
$P(p_{k'+2}): q_{k'+2}$	$P(q_{k'+2}): p_{k'+1}, p_{k'+2}$.

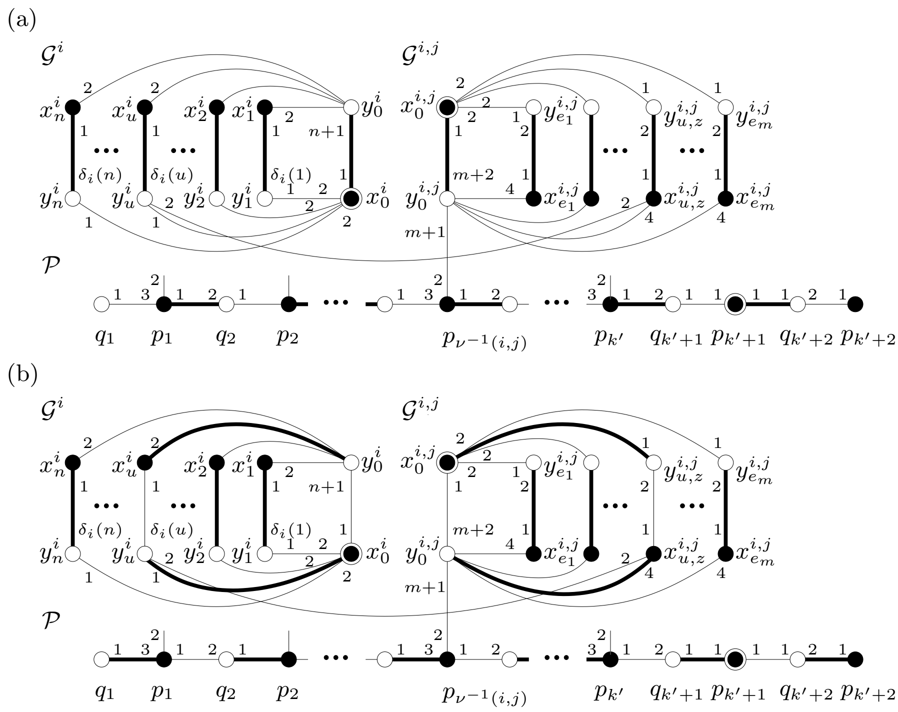

Fig. 1. Illustration for the SMTI instance $\mathcal{I}$ constructed in the proof of Theorem 2. White circles represent men, black circles represent women, and double black circles represent indifferent women. The bold edges in Fig. (a) show M_0, and the bold edges in (b) show a possible stable matching M that is larger than M_0. The small numbers on the edges represent ranks. We write $\delta_i(u)$ for $|C(i,u)|+2$, and also e_1 and e_m for two pairs in $\{(a,b) \mid v_a v_b \in E(G)\}$.

Observe that M_0 assigns each woman in $X \setminus \{p_{k'+2}\}$ to a man that she prefers the most, so they cannot be in a blocking pair for M_0. As $p_{k'+2}q_{k'+2}$ is also not a blocking pair, M_0 is indeed stable. By $|X| = |Y| = O(\binom{k}{2}) + \binom{k}{2}O(m) + kO(n)$, the construction takes polynomial time in n and m (using also $k \le n$). Since $\kappa_1(\mathcal{I}) \le \binom{k}{2} + k + 1$ also holds, this yields an FPT-reduction.

The basic idea of the above construction is the following. It is easy to see that we can only get a matching M larger than M_0 if we "swap" the matching M_0 along the path-gadget $\mathcal{P}$. However, the given ranks ensure that this can only result in a stable matching if we make a swap in each edge-gadget as well. (See Fig 1 (b). If the matching would include the edge $x_0^{i,j}y_0^{i,j}$, then $y_0^{i,j}p_{\nu^{-1}(i,j)}$ would be a blocking pair.) Such a swapping in the edge-gadget $\mathcal{G}^{i,j}$ can be done in m ways, as we can swap M_0 along the cycles formed by $x_0^{i,j}$, $y_0^{i,j}$, $x_{u,z}^{i,j}$, and $y_{u,z}^{i,j}$ for each $u < z$ where $v_u v_z$ is an edge. But the connections between the edge- and node-gadgets ensure that swapping M_0 along the cycle in $\mathcal{G}^{i,j}$ corresponding to some edge $v_u v_z$ can only result in a stable matching if we also swap it along the cycles in the node-gadgets $\mathcal{G}^i$ and $\mathcal{G}^j$ corresponding to the vertices v_u and v_z, respectively. As we can only make one swap in each gadget (because of the existence of x_0^i and y_0^i in the case of Fig. 1), this ensures that the $\binom{k}{2}$ edges of G that correspond to the swappings in the edge-gadgets have altogether at most k endpoints, as these endpoints must correspond to the swappings made in the k node-gadgets. Thus, we have a clique in G if and only if we can improve M_0.

Before going into the details, we remark that ties are unavoidable in the construction. First, swapping a stable matching along an alternating path of the underlying graph can only result in a stable matching if at least one node of the path corresponds to a person who is indifferent between its two possible partners on the path. Second, if there are two non-disjoint cycles C_1 and C_2 in the underlying graph such that swapping some stable matching along C_1 and along C_2 both result in a stable matching, then at least one person corresponding to a node in C_1 or C_2 must be indifferent. Thus, we need ties both for constructing an instance with a possibly improvable solution, and also for leaving enough space for the possible improvements to map the different cliques of the graph to different solutions.

To detail the proof of the reduction, we first show that the following are equivalent for any matching M for $\mathcal{I}$:

– property (p1): $p_1 q_1 \in M$ and M is stable,
– property (p2): $|M| = |M_0| + 1$ and M is stable, and
– property (p3): $|M| = |M_0| + 1$, M is stable, and M is ℓ-close to M_0.

Property (p3) $\Longrightarrow$ (p2) is trivial, and (p2) $\Longrightarrow$ (p1) should also be clear. To prove (p1) $\Longrightarrow$ (p3), suppose that M is a stable matching with $M(q_1) = p_1$. First, to prevent $p_1 q_2$ from being a blocking pair, M must assign p_2 to q_2. Applying this argument iteratively, we obtain that $M(q_h) = p_h$ for each $h \in [\binom{k}{2} + 1]$. Also, $q_{\binom{k}{2}+2} p_{\binom{k}{2}+2}$ must be contained in M, as otherwise this would be a blocking pair. Since $r(p_h, y_0^{\nu(h)}) > r(p_h, q_h)$ for each $h \in [\binom{k}{2}]$, we get that M can only be stable if $y_0^{\nu(h)}$ has a partner in M whom he prefers to p_h, implying $M(y_0^{i,j}) \in X^{i,j}$ for each $(i,j) \in \binom{k}{2}$. We denote by $\sigma(i,j)$ the pair (u,z) if $M(y_0^{i,j}) = x_{u,z}^{i,j}$, and similarly we let $\sigma(i) = u$ if $M(y_0^i) = x_u^i$.

As $r(x_{u,z}^{i,j}, y_{u,z}^{i,j}) = 1$ for every possible (u,z), we get $M(y_{\sigma(i,j)}^{i,j}) = x_0^{i,j}$, since otherwise $x_{\sigma(i,j)}^{i,j} y_{\sigma(i,j)}^{i,j}$ would be a blocking pair. Also, we obtain $M(x_{u,z}^{i,j}) = y_{u,z}^{i,j}$ for all $(u,z) \neq \sigma(i,j)$ for the same reason. Thus, each person in an edge-gadget $\mathcal{G}^{i,j}$ can only be assigned to a person in $\mathcal{G}^{i,j}$.

Suppose $\sigma(i,j) = (u^*, z^*)$. As $x_{\sigma(i,j)}^{i,j}$ prefers $y_{u^*}^i$ to $y_0^{i,j}$, and $y_{u^*}^i$ prefers $x_{\sigma(i,j)}^{i,j} \in C(i,u^*)$ to $x_{u^*}^i$, $M(x_{u^*}^i) = y_{u^*}^i$ is not possible, since then $y_{u^*}^i$ and $x_{\sigma(i,j)}^{i,j}$ would form a blocking pair. As $C(i,u^*)$ is a subset of persons in $\mathcal{G}^{i,j}$, we get $M(y_{u^*}^i) \notin C(i,u^*)$ by the argument above. This implies $M(y_{u^*}^i) = x_0^i$. Using again the stability of M, we also obtain $M(y_u^i) = x_u^i$ for every $u \neq u^*$, and $M(x_{u^*}^i) = y_0^i$. Note that this latter means $\sigma(i) = u^*$. Using the same arguments again, we also obtain $M(y_{z^*}^j) = x_0^j$, $M(y_0^j) = x_{z^*}^j$, and $M(y_z^j) = x_z^j$

for each $z \neq z^*$. This yields $\sigma(j) = z^*$, so we have $\sigma(i,j) = (\sigma(i), \sigma(j))$ for each $(i,j) \in \binom{[k]}{2}$.

Observe also that M covers each person of the instance, meaning $|M| = |M_0| + 1$. There are exactly 4 persons a in each node-gadget and in each edge-gadget for which $M(a) \neq M_0(a)$ holds. As $M(a) \neq M_0(a)$ holds for every person a in the path-gadget, we can conclude that M is $(6\binom{k}{2} + 4k + 4) = \ell$-close to M_0. Thus, (p1) indeed implies (p3), so the properties (p1), (p2) and (p3) are equivalent.

Now, one direction of the reduction follows from the observation that by $\sigma(i,j) = (\sigma(i), \sigma(j))$, the definition of $X^{i,j}$ implies $v_{\sigma(i)} v_{\sigma(j)} \in E(G)$ for each $(i,j) \in \binom{[k]}{2}$. Hence, we can conclude that $\{v_{\sigma(i)} \mid i \in [k]\}$ is a clique of size k in G, proving (1) $\Longrightarrow$ (2).

Finally, we prove (2) $\Longrightarrow$ (1). Suppose $\{v_{\sigma(i)} \mid i \in [k]\}$ is a clique in G. We construct a stable matching M of size $|M_0| + 1$ as follows (the indices take all the possible values):

$M(x_0^i) = y^i_{\sigma(i)}$ $\qquad$ $M(y_0^i) = x^i_{\sigma(i)}$

$M(x_0^{i,j}) = y^{i,j}_{\sigma(i),\sigma(j)}$ $\qquad$ $M(y_0^{i,j}) = x^{i,j}_{\sigma(i),\sigma(j)}$

$M(q_h) = p_h$.

By setting $M(a) = M_0(a)$ for every other person a, $|M| = |M_0| + 1$ is clear. It is straightforward to verify that M is stable, proving the theorem. $\square$

Observe that there is no bound on the length of the ties in the SMTI instance constructed in the proof of Theorem 2. Thus, we could hope that restricting κ_2 to be small yields an easier problem. But as already mentioned, NP-completeness has been shown in [15] for the special case of MaxSMTI when $\kappa_2(\mathcal{I}) = 2$ holds for every input $\mathcal{I}$. On the other hand, $\kappa_3(\mathcal{I}) \leq \kappa_1(\mathcal{I})\kappa_2(\mathcal{I})$, so Theorem 1 trivially implies that MaxSMTI is FPT with combined parameterization (κ_1, κ_2).

This latter fact trivially gives us a permissive local search algorithm for MaxSMTI with FPT running time, assuming the combined parameterization (κ_1, κ_2). Thus, it is natural to ask whether we can also give an FPT permissive local search algorithm by parameterizing the problem with only κ_2. The following theorem shows that no such algorithm can be given (supposing the standard assumption W[1] $\neq$ FPT holds). Moreover, the problem remains hard even if we restrict $\kappa_2 = 2$, and regard ℓ as a parameter.

Theorem 3. *If W[1] $\neq$ FPT, then there is no permissive local search algorithm for* MaxSMTI *that runs in FPT time with parameter ℓ, even if $\kappa_2 = 2$ and only women can be indifferent.*

Proof. The proof will be very similar to the proof of Theorem 2, so we will reuse some of the definitions and arguments used there. Clearly, we have to eliminate long ties in the constructed instance. Note that the instance constructed in the proof of Theorem 2 only contains ties longer than two in the preference lists of x_0^i and $x_0^{i,j}$ (where $i \in [k]$ and $(i,j) \in \binom{[k]}{2}$, respectively). Therefore, we break the ties in these lists. However, we must not narrow the number of possibilities for improving the initial matching. Thus, in order to avoid the presence of blocking

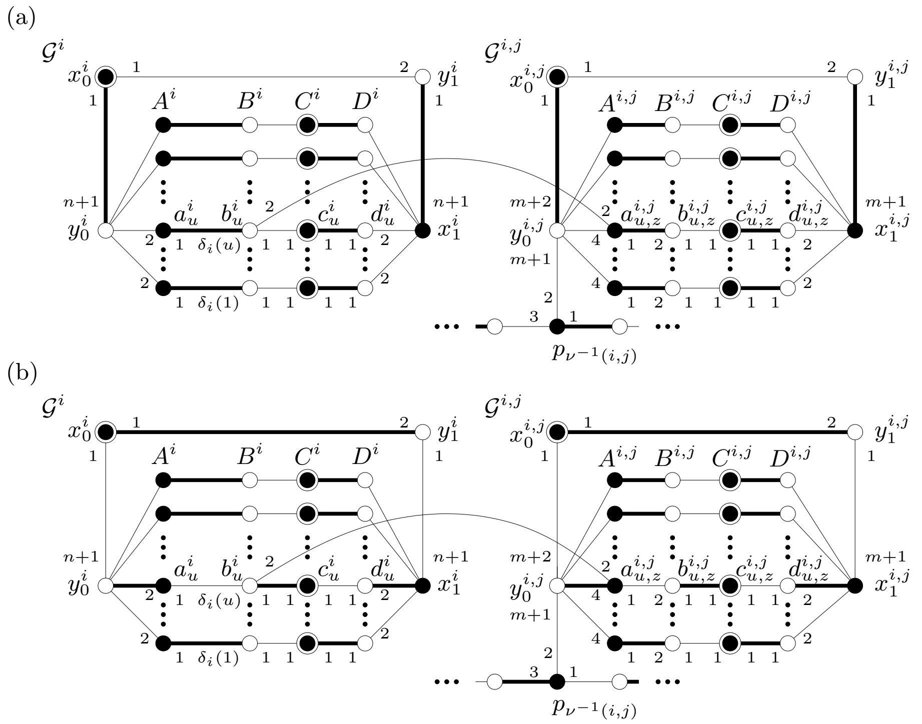

Fig. 2. The modified gadgets of the SMTI instance $\mathcal{I}$ constructed in the proof of Theorem 3. The bold edges in Fig. (a) represent M_0, and the bold edges in (b) show a possible stable matching M that is larger than M_0.

pairs for the swapped solutions, we have to place indifferent persons on each of the alternating cycles that might take part in a possible swapping.

Let $G(V, E)$ be the input graph and k be the parameter for the Clique problem. As before, we are going to construct an SMTI instance $\mathcal{I}$ with $\kappa_2 = 2$ together with a stable matching M_0 for $\mathcal{I}$ and the integer $\ell = 12\binom{k}{2} + 8k + 4$ such that the following three statements are equivalent:

(1) $\mathcal{I}$ has a stable matching of size at least $|M_0| + 1$,
(2) $\mathcal{I}$ has a stable matching M of size at least $|M_0| + 1$ that is ℓ-close to M_0, and
(3) there is a clique of size k in G.

Since the construction will take polynomial time, this clearly proves our theorem. Note that (2) $\Longrightarrow$ (1) is trivial.

Fig. 2 shows an illustration for the construction. Let $V(G) = \{v_i \mid i \in [n]\}$ and $m = |E(G)|$. The instance $\mathcal{I}$ consists of *node-gadgets* $\mathcal{G}^i$ for each $i \in [k]$, *edge-gadgets* $\mathcal{G}^{i,j}$ for each $(i, j) \in \binom{[k]}{2}$, and a *path-gadget* $\mathcal{P}$. The node-gadget $\mathcal{G}^i$ consists of women $A^i \cup C^i \cup \{x_0^i, x_1^i\}$ and men $B^i \cup D^i \cup \{y_0^i, y_1^i\}$, where $A^i =$

$\{a_u^i \mid u \in [n]\}$, and B^i, C^i, D^i are defined analogously to A^i. The edge-gadget $\mathcal{G}^{i,j}$ consists of women $A^{i,j} \cup C^{i,j} \cup \{x_0^{i,j}, x_1^{i,j}\}$ and men $B^{i,j} \cup D^{i,j} \cup \{y_0^{i,j}, y_1^{i,j}\}$, where $A^{i,j} = \{a_{u,z}^{i,j} \mid u < z, v_u v_z \in E(G)\}$, and $B^{i,j}, C^{i,j}, D^{i,j}$ are defined similarly. The path-gadget $\mathcal{P}$ is defined in the same way as in the proof of Theorem 2. Note that the number of men and women in $\mathcal{I}$ is $O(\binom{k}{2}) + \binom{k}{2}O(m) + kO(n)$, so the construction takes polynomial time in the size of G.

For each $i \in [k]$ we let $M_0(a_u^i) = b_u^i$ and $M_0(c_u^i) = d_u^i$ for each $u \in [n]$, and $M_0(x_h^i) = y_h^i$ for $h \in \{0,1\}$. Similarly, for each $(i,j) \in \binom{[k]}{2}$ we let $M_0(a_{u,z}^{i,j}) = b_{u,z}^{i,j}$ and $M_0(c_{u,z}^{i,j}) = d_{u,z}^{i,j}$ for each possible u and z, and $M_0(x_h^{i,j}) = y_h^{i,j}$ for $h \in \{0,1\}$. We define the ranking function for $\mathcal{I}$ by giving the preference lists, using the notation of the proof of Theorem 2. For each person p in $\mathcal{P}$ we let both $M_0(p)$ and its preference list $P(p)$ be defined as in the proof of Theorem 2. We also define $C(i,u) = \{a_{u,z}^{i,j} \mid i < j \leq k, u < z, v_u v_z \in E(G)\} \cup \{a_{z,u}^{j,i} \mid 1 \leq j < i, z < u, v_z v_u \in E(G)\}$ for all $i \in [k], u \in [n]$.

$P(a_u^i): b_u^i, y_0^i$ $\quad$ $P(b_u^i): c_u^i, [C(i,u)], a_u^i$
$P(c_u^i): (b_u^i, d_u^i)$ $\quad$ $P(d_u^i): c_u^i, x_1^i$
$P(x_0^i): (\{y_0^i, y_1^i\})$ $\quad$ $P(y_0^i): [A^i], x_0^i$
$P(x_1^i): [D^i], y_1^i$ $\quad$ $P(y_1^i): x_1^i, x_0^i$
$P(a_{u,z}^{i,j}): b_{u,z}^{i,j}, [b_u^i, b_z^j], y_0^{i,j}$ $\quad$ $P(b_{u,z}^{i,j}): c_{u,z}^{i,j}, a_{u,z}^{i,j}$
$P(c_{u,z}^{i,j}): (b_{u,z}^{i,j}, d_{u,z}^{i,j})$ $\quad$ $P(d_{u,z}^{i,j}): c_{u,z}^{i,j}, x_1^{i,j}$
$P(x_0^{i,j}): (y_0^{i,j}, y_1^{i,j})$ $\quad$ $P(y_0^{i,j}): [A^{i,j}], p_{\nu^{-1}(i,j)}, x_0^{i,j}$
$P(x_1^{i,j}): [D^{i,j}], y_1^{i,j}$ $\quad$ $P(y_1^{i,j}): x_1^{i,j}, x_0^{i,j}$.

It is easy to see that M_0 is indeed a stable matching for M, and covers every woman except for $p_{\binom{k}{2}+2}$. Observe that $\kappa_2(\mathcal{I}) = 2$ indeed holds, but $\kappa_1(\mathcal{I})$ is not bounded.

To prove (1) $\Longrightarrow$ (3), suppose that M is a stable matching that covers every man and woman. Using the same arguments as in the proof of Theorem 2, we obtain $M(q_h) = p_h$ for each $h \in [\binom{k}{2}+2]$ and $M(y_0^{i,j}) \in A^{i,j}$ for each $(i,j) \in \binom{[k]}{2}$. Following that argument and exploiting the stability of M, after defining $\sigma(i,j)$ to be (u,z) if $M(y_0^{i,j}) = a_{u,z}^{i,j}$ and $\sigma(i)$ to be u if $M(y_0^i) = a_u^i$, we can easily obtain $\sigma(i,j) = (\sigma(i), \sigma(j))$ proving that $\{v_{\sigma(i)} \mid i \in [k]\}$ is a clique in G. This proves (1) $\Longrightarrow$ (3).

To prove (3) $\Longrightarrow$ (2), let $\{v_{\sigma(i)} \mid i \in [k]\}$ be a clique in G. We define a stable matching M covering each person in $\mathcal{I}$ as follows.

$M(y_0^i) = a_{\sigma(i)}^i$ $\quad$ $M(y_0^{i,j}) = a_{\sigma(i),\sigma(j)}^{i,j}$
$M(b_{\sigma(i)}^i) = c_{\sigma(i)}^i$ $\quad$ $M(b_{\sigma(i),\sigma(j)}^{i,j}) = c_{\sigma(i),\sigma(j)}^{i,j}$
$M(d_{\sigma(i)}^i) = x_1^i$ $\quad$ $M(d_{\sigma(i),\sigma(j)}^{i,j}) = x_1^{i,j}$
$M(y_1^i) = x_0^i$ $\quad$ $M(y_1^{i,j}) = x_0^{i,j}$
$M(q_h) = p_h$.

For every other person p in $\mathcal{I}$ we let $M(p) = M_0(p)$. It is straightforward to check that M is a stable matching for $\mathcal{I}$ that is ℓ-close to M_0. $\square$

3.2 Egalitarian and minimum regret stable matchings

Theorems 2 and 3 show that a stable matching for an instance $\mathcal{I}$ of maximum size is hard to find even if $\kappa_1(\mathcal{I})$ or $\kappa_2(\mathcal{I})$ is small. Improving an initial stable matching also remains hard in these cases, even if we can restrict our attention to solutions that are close to the initial solution. However, we can still try to find stable matchings that may not be of maximum size, but have some other useful properties.

If M is a stable matching for an SMTI instance $\mathcal{I} = (X, Y, r)$, then the *cost for p w.r.t. M*, denoted by $c_M(p)$, is defined to be $r(p, M(p))$ if $M(p) \neq \emptyset$, and $1 + r(p, q^*)$ otherwise, where q^* has maximum rank according to p among all acceptable partners for p. Now, the *weight* of M is $w(M) = \sum_{p \in X \cup Y} c_M(p)$, and the *regret* of M is $r(M) = \max_{p \in X \cup Y} c_M(p)$. An *egalitarian* (*minimum regret*) stable matching for $\mathcal{I}$ is a stable matching for $\mathcal{I}$ that has the minimum weight (regret, respectively) among all stable matchings for $\mathcal{I}$. The task of the Egalitarian Stable Marriage with Ties and Incomplete Lists (or EgalSMTI) problem is to find an egalitarian stable matching for the given SMTI instance, and the Minimum Regret Stable Marriage with Ties and Incomplete Lists (or MinregSMTI) problem is defined analogously.

If P $\neq$ NP and $\varepsilon > 0$ then there is no polynomial time approximation algorithm with ratio $N(\mathcal{I})^{1-\varepsilon}$ for these problems, even if only women can be indifferent, each preference list has at most one tie, and $\kappa_2(\mathcal{I}) = 2$ [15]. Here, $N(\mathcal{I})$ is the number of men in $\mathcal{I}$. Moreover, it has also been shown in [7] that for some $\delta > 0$ it is NP-hard to approximate EgalSMTI and MinregSMTI within a ratio of $\delta N(\mathcal{I})$. However, if there are no ties, then a minimum regret or an egalitarian stable matching can be found in polynomial time [8, 5]. As Theorem 4 shows, this can be exploited to give an FPT algorithm for both EgalSMTI and MinregSMTI if we parameterize it by κ_3, or equivalently, by (κ_1, κ_2). On the other hand, Theorems 5 and 6 present some bounds on the approximability of these problems that hold even if we allow the approximation algorithm to run not in polynomial time but in FPT time with parameterization κ_1. Note that such an approximation algorithm would have a tractable running time if κ_1 is a small integer, even if the length of the ties is unbounded.

Theorem 4. EgalSMTI *and* MinregSMTI *are FPT with parameterization* κ_3.

Proof. We will use the method described in the proof of Theorem 1 for breaking ties, and we also adopt the notation T_i, P_i and π_i^j for some $i \in [\kappa_1(\mathcal{I})]$ and $j \in [|T_i|]$. Denoting the elements of $P_1 \times \cdots \times P_{\kappa_1(\mathcal{I})}$ by $p_1, \ldots, p_t$, we write $\mathcal{I}_i$ for the instance obtained by breaking ties according to p_i. Note that the instances $\mathcal{I}_1, \ldots, \mathcal{I}_t$ differ from $\mathcal{I}$ only by the ranks assigned to the persons that are contained in a tie.

In order to find a minimum regret or an egalitarian matching, we have to define the ranking functions of the instances $\mathcal{I}_1, \ldots, \mathcal{I}_t$ in a special way. More precisely, if M_i is an egalitarian (minimum regret) stable matching for $\mathcal{I}_i$ ($i \in [t]$), then we need the following to be true: the weight of an egalitarian (minimum regret) stable matching for $\mathcal{I}$ equals $\min_{i\in[t]}\{w(M_i)\}$ ($\min_{i\in[t]}\{r(M_i)\}$, respectively). To enforce this property, which we will call *weight conserving property*, we define the ranking function of $\mathcal{I}_i$ for some $p_i = (\pi_1, \dots, \pi_{\kappa_1(\mathcal{I})})$ as follows (allowing $b = \emptyset$ also):

$$r'(a,b) = \begin{cases} r(a,b) + \frac{\pi_k(b)-1}{|T_k|} & \text{if } T_k \text{ is a tie w.r.t. } a \text{ that contains } b, \\ r(a,b) & \text{if no such tie exists.} \end{cases}$$

First, suppose that M is an egalitarian matching for $\mathcal{I}$. Let π_i ($i \in [\kappa_1(\mathcal{I})]$) be such that $\pi_i(M(a)) = 1$ if T_i is a tie w.r.t. a containing $M(a)$, otherwise π_i is chosen arbitrarily. Observe that M is a stable matching of $\mathcal{I}_{(\pi_1,\dots,\pi_t)}$, and the cost for any person w.r.t. M is the same in $\mathcal{I}$ and in $\mathcal{I}_{(\pi_1,\dots,\pi_t)}$, by the definition of r'. On the other hand, if M_i is a stable matching in $\mathcal{I}_i$, then it is also stable in $\mathcal{I}$, and the cost for any person w.r.t. M_i in $\mathcal{I}$ is at most its cost in $\mathcal{I}_i$. Thus, we get that this tie breaking method indeed fulfils the weight conserving property.

Note that the instances obtained have a strict ranking for each person, but the ranks may be not only be integers but rational numbers as well. However, finding an egalitarian or a minimum regret stable matching for such instances can be done in $O(n^4 \log n)$ time by [8, 5] for an instance of size n, so this does not cause any problems. Hence, breaking ties as above, finding the egalitarian or minimum regret stable matching for the instances obtained, and choosing a stable matching among them having minimum weight or regret yields an algorithm for both problems with running time $\prod_{i\in[\kappa_1(\mathcal{I})]} |T_i| O(|\mathcal{I}|^4 \log |\mathcal{I}|) = O(\kappa_3(\mathcal{I})^{\kappa_3(\mathcal{I})} |\mathcal{I}|^4 \log |\mathcal{I}|)$. □

We remark that if $\kappa_1(\mathcal{I})$ is a fixed constant, then both the egalitarian and the minimum regret matching can be found in polynomial time, as the algorithm of Theorem 4 runs in $\prod_{i\in[\kappa_1(\mathcal{I})]} |T_i| O(|\mathcal{I}|^4 \log |\mathcal{I}|) \le |\mathcal{I}|^{\kappa_1(\mathcal{I})} O(|\mathcal{I}|^4 \log |\mathcal{I}|) = O(|\mathcal{I}|^{\kappa_1(\mathcal{I})+4} \log |\mathcal{I}|)$. Theorems 5 and 6 show that if κ_1 is not a constant but a parameter, then we have strong lower bounds on the ratio of any FPT approximation algorithm, for both of these problems.

Theorem 5. *There is a $\delta > 0$ such that if W[1] $\neq$ FPT, then there is no FPT-approximation with parameterization κ_1 for* EgalSMTI *that has ratio $\delta N(\mathcal{I})$, even if only women can be indifferent.*

Proof. We show that the theorem holds for $\delta = \frac{1}{14}$. Suppose that an FPT-approximation algorithm $\mathcal{A}$ with parameterization κ_1 and ratio $\delta N(\mathcal{I})$ exists for EgalSMTI. We will show that this yields an FPT algorithm for the Clique problem.

We are going to construct an SMTI instance $\mathcal{I}'$ by adding some new persons to the instance $\mathcal{I} = (X, Y, r)$ constructed in the proof of Theorem 2. See Fig. 3 for an illustration. The basic idea is that we complement the path-gadget $\mathcal{P}$ in a way such that the weight of the solution is mainly determined by the choices made for the persons in the path-gadget. Thus, we can only decrease the weight

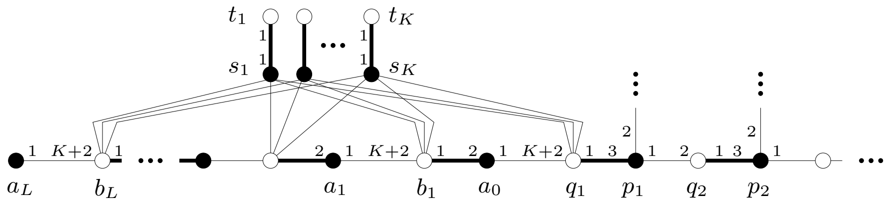

Fig. 3. The modified SMTI instance $\mathcal{I}'$ constructed in the proof of Theorem 5. Bold edges represent M'.

of the initial solution by swapping it along the path-gadget, which can be done exactly if there is a clique in the given graph (as we have already seen). Moreover, we ensure that swapping the initial solution along the path-gadget results in a decrease of the weight that is large enough, so that even an FPT-approximation algorithm can detect whether this is possible.

Clearly, these ideas are also useful for considering the MinregSMTI problem. Therefore, we describe the construction in a general way, by using some parameters in the classical sense (K and L) that can be used to "tune" the weights that appear in the weight or in the regret of the solutions.

Now, we describe the details of the proof. Let $G(V,E)$ and k be the input graph and the parameter for Clique, with $V(G) = \{v_i \mid i \in [n]\}$ and $|E(G)| = m$. To construct $\mathcal{I}'$ from $\mathcal{I}$, we add new women $A \cup S$ and men $B \cup T$, where $A = \{a_u \mid u \in \{0\} \cup [L]\}$, $B = \{b_u \mid u \in [L]\}$, $S = \{s_u \mid u \in [K]\}$, and $T = \{t_u \mid u \in [K]\}$. The value of the integers K and L will be specified later. The preference lists for the newly introduced persons and for q_1 are given below, all other preferences are as in $\mathcal{I}$.

$P(a_u): b_u, b_{u+1}$ if $u \in [L-1]$	$P(q_1): p_1, [S], a_0$
$P(a_0): q_1, b_1$	$P(s_u): t_u, [\{q_1\} \cup B]$
$P(a_L): b_L$	$P(t_u): s_u$
$P(b_u): a_{u-1}, [S], a_u$.	

Note that $N(\mathcal{I}') = K + L + |Y|$, where $|Y| = \binom{k}{2} + 2 + k(n+1) + \binom{k}{2}(m+1)$. Let $M_{\mathcal{A}}$ be the output of $\mathcal{A}$ on input $\mathcal{I}'$, and let M_E be an egalitarian stable matching for $\mathcal{I}'$. First, suppose there is a clique of size k in G. Let M be a stable matching containing p_1q_1 for the instance $\mathcal{I}$, such an M can be defined as in the last paragraph of the proof of Theorem 2. By adding the pairs $\{s_ut_u \mid u \in [K]\}$ and $\{b_ua_{u-1} \mid u \in [L]\}$ to M, we clearly obtain a stable matching M' for $\mathcal{I}'$.

Observe that the regret of M in $\mathcal{I}$ is at most the maximum rank R in $\mathcal{I}$, which is at most $\max\{m+2, n+1, (k-1)\Delta(G)+2\}$, with $\Delta(G)$ denoting the highest degree in G. Thus we get $w(M) \leq (|X|+|Y|)R = 2|Y|R$, implying $w(M') = 2+(2+1)L+(1+1)K+w(M) \leq 3L+2K+2R|Y|+2$. Since $w(M_E) \leq w(M')$ and the ratio of $\mathcal{A}$ is $\delta N(\mathcal{I}')$, we obtain that $w(M_{\mathcal{A}}) \leq \delta N(\mathcal{I}')w(M_E) \leq \delta(K+L+|Y|)(3L+2K+2R|Y|+2) =: w^1_{\mathcal{A}}$.

Now, if there is no clique of size k in G, then no stable matching for $\mathcal{I}'$ can contain p_1q_1. To see this, observe that the restriction of such a matching on $\mathcal{I}$ would also be stable. Now, recall that if p_1q_1 is contained in some stable matching for $\mathcal{I}$, then G contains a clique of size k, by the claims in the proof of Theorem 2. Thus, $q_1p_1 \notin M_{\mathcal{A}}$. Observe also that every stable matching must include the pairs $\{s_ut_u \mid u \in [K]\}$, as s_u and t_u prefer each other the most. Using this, we get that q_1a_0 must be in M', as otherwise it would be a blocking pair. Applying this argument repeatedly, we can easily see that a_ib_i is in M' for $i = 1, 2, \ldots, L$. These observations together imply $w(M_{\mathcal{A}}) \geq (K + 2 + 1)(L + 1) + 2K = KL + 3L + 3K + 3 =: w^2_{\mathcal{A}}$.

Now, if $w^1_{\mathcal{A}} < w^2_{\mathcal{A}}$ holds, then algorithm $\mathcal{A}$ can decide whether there is a clique of size k in G, by outputting 'Yes' if $w(M_{\mathcal{A}}) \leq w^1_{\mathcal{A}}$ and outputting 'No' if $w(M_{\mathcal{A}}) \geq w^2_{\mathcal{A}}$. Setting $K = 2L$ and $L = 2R|Y| + 2$ guarantees $w^1_{\mathcal{A}} < w^2_{\mathcal{A}}$, because $w^1_{\mathcal{A}} < \delta \cdot 3.5L \cdot 8L = 28\delta L^2 = 2L^2 < w^2_{\mathcal{A}}$. Finally, observe that $R = O(m + nk)$ and $|Y| = O(k^2m + nk)$ implies $N(\mathcal{I}') = K + L + |Y| = 3(2R|Y| + 2) + |Y| = O(k^2(m^2+n^2)+k^3nm)$, hence the instance $\mathcal{I}'$ can be created in polynomial time. So by $\kappa_1(\mathcal{I}') = \binom{k}{2} + k + 1$, the presented algorithm for the Clique problem indeed runs in FPT time.

We remark that the theorem also holds for any $\delta < \frac{1}{5+2\sqrt{6}}$. This can be proven by an easy calculation similar to the one above, by setting $K = \lceil \sqrt{\alpha^2 + 5\alpha + 3} \rceil L$ and $L = \lceil (R|Y| + 1)/\alpha \rceil$ for some α that is close enough to 0. □

Theorem 6. *If W[1] $\neq$ FPT and $\varepsilon > 0$, then there is no FPT-approximation with parameterization κ_1 for* MinregSMTI *that has ratio $N(\mathcal{I})^{1-\varepsilon}$.*

Proof. Again, as in the proof of Theorem 5, we suppose that there exists such an FPT-approximation algorithm $\mathcal{A}$, and we show that this can be used to give an FPT algorithm for the Clique problem. Let G and k be the input graph and the parameter given for the Clique problem. We will use the construction $\mathcal{I}'$, introduced in the proof of the Theorem 5, with the restriction that we set $L = 0$ and $K = \max\{|Y|, \lceil (2R)^{1/\varepsilon} \rceil\}$. Recall that $N(\mathcal{I}') = K + L + |Y|$, $R = O(m + nk)$ and $Y = O(k^2m + nk)$. Since ε is a constant, $\mathcal{I}'$ can again be constructed from G and k in polynomial time.

Let $M_{\mathcal{A}}$ denote the output of $\mathcal{A}$ on input $\mathcal{I}'$. Suppose that G has a clique of size k. If we define M' as in the proof of Theorem 5, then $p_1q_1 \in M'$, and $r(M') \leq R$ is easy to see. By the ratio of $\mathcal{A}$, and using also $K \geq |Y|$ we get $r(M_{\mathcal{A}}) \leq N(\mathcal{I}')^{1-\varepsilon}R = (K + |Y|)^{1-\varepsilon}R \leq 2K^{1-\varepsilon}R$.

On the other hand, if there is no clique of size k in G, then p_1q_1 cannot be contained $M_{\mathcal{A}}$, as we have already shown in the proof of Theorem 2 that this would imply a clique of size k in G. Since s_u must be assigned to t_u in any stable matching of $\mathcal{I}'$, we get $q_1a_0 \in M_{\mathcal{A}}$, implying $c_{M_{\mathcal{A}}}(q_1) = K + 2$. Thus, $r(M_{\mathcal{A}}) \geq K + 2$ in this case.

It is easy to check that $K + 2 > K \geq K^{1-\varepsilon}2R$ holds by the choice of K. Therefore, $\mathcal{A}$ indeed can decide whether G has a clique of size k, and by $\kappa_1(\mathcal{I}') = \binom{k}{2} + k + 1$, this yields an FPT algorithm for Clique. □

	Parameters		
	$\kappa_2 = 2$ (and ℓ)	κ_1 (and ℓ)	κ_1, κ_2
MaxSMTI	NP-hard ([15])	W[1]-hard (Theorem 2)	FPT (Theorem 1)
Local Search for MaxSMTI	No FPT alg. (Theorem 3)	No FPT alg. (Theorem 2)	FPT (Theorem 1)
Approximation for EgalSMTI	No poly. alg. has ratio $N^{1-\varepsilon}$ if $\varepsilon > 0$ ([15])	No FPT alg. has ratio δN for some $\delta > 0$ (Theorem 5)	FPT, exact (Theorem 4)
Approximation for MinregSMTI	No poly. alg. has ratio $N^{1-\varepsilon}$ if $\varepsilon > 0$ ([15])	No FPT alg. has ratio δN for some $\delta > 0$ (Theorem 6)	FPT, exact (Theorem 4)

Table 1. Summary of our results (assuming W[1] $\neq$ FPT and P $\neq$ NP). The parameter ℓ is only defined in the local search problem for MaxSMTI, and N denotes the number of men.

4 Conclusions

We have shown that MaxSMTI remains W[1]-hard if we parameterize the problem with the number of ties, but becomes FPT if parameterized with the total length of ties. We have also shown that if W[1] $\neq$ FPT, then no local search algorithm with FPT running time can be given for this problem when the size of the neighborhood to be examined is considered as parameter, even if each tie has length 2, or even if the number of ties is also considered as a parameter.

We also proved that no FPT algorithm can approximate the EgalSMTI or the MinregSMTI problem, when the parameter is the number of ties, unless W[1] = FPT. On the other hand, both problems can be solved in FPT time if we parameterize them by the total length of ties.

A summary of our results is shown in Table 1.

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References

1. E.H.L. Aarts, J.K. Lenstra (Editors): *Local Search in Combinatorial Optimization*, Wiley, New York, 1997.
2. R.G. Downey, M.R. Fellows: *Parameterized complexity*. Springer, 1999.
3. J. Flum, M. Grohe: *Parameterized Complexity Theory*, Springer-Verlag, Berlin, 2006. Springer, 2006.

4. D. Gale, L. S. Shapley: College admissions and the stability of marriage. *American Mathematical Monthly*, 69:9–15, 1962.
5. D. Gusfield: Three fast algorithms for four problems in stable marriage. *SIAM J. Comput.*, 16(1):111–128, 1987.
6. D. Gusfield, R. W. Irving: *The Stable Marriage Problem: Structure and Algorithms.* MIT Press, 1989.
7. M.M. Halldórsson, R.W. Irving, K. Iwama, D.F. Manlove, S. Miyazaki, Y. Morita, S. Scott: Approximability results for stable marriage problems with ties. *Theor. Comput. Sci.*, 306(1-3):431–447, 2003.
8. R. W. Irving, P. Leather, D. Gusfield: An efficient algorithm for the "optimal" stable marriage. *J. ACM* 34(3):532–543, 1987.
9. R. W. Irving: Stable marriage and indifference. *Discrete Appl. Math.*, 48(3):261–272, 1994.
10. R. W. Irving, D. F. Manlove: Approximation algorithms for hard variants of the stable marriage and hospitals/residents problems. *J. Comb. Optim.* 16(3):279–292, 2008. (DOI: 10.1007/s10878-007-9133-x)
11. K. Iwama, D. Manlove, S. Miyazaki, Y. Morita: Stable marriage with incomplete lists and ties. *ICALP '99, LNCS 1644* 443–452, 1999.
12. S. Khuller, R. Bhatia, R. Pless: On local search and placement of meters in networks. *SIAM J. Comput.*, 32(2):470–487, 2003.
13. Z. Király: Better and simpler approximation algorithms for the stable marriage problem. *ESA 2008, LNCS 5193*, 623–634, 2008.
14. A. Krokhin, D. Marx: On the hardness of losing weight. *ICALP 2008, LNCS 5125*, 662–673, 2008.
15. D. F. Manlove, R. W. Irving, K. Iwama, S. Miyazaki, Y. Morita: Hard variants of stable marriage. *Theor. Comput. Sci.*, 276(1-2):261–279, 2002.
16. D. Marx: Local search. *Parameterized Complexity News*, 3:7–8, 2008.
17. D. Marx: Parameterized complexity and approximation algorithms. *The Computer Journal*, 51(1):60–78, 2008.
18. D. Marx: Searching the k-change neighborhood for TSP is W[1]-hard. *Oper. Res. Lett.*, 36(1):31–36, 2008.
19. D. Marx, I. Schlotter: Stable assignment with couples: parameterized complexity and local search. *Manuscript.*
20. R. Niedermeier: *Invitation to Fixed-Parameter Algorithms.* Oxford University Press, 2006.
21. A.E. Roth: The evolution of the labor market for medical interns and residents: a case study in game theory. *J. Polit. Econ.*, 92:991–1016, 1984.
22. A.E. Roth, M. Sotomayor: *Two Sided Matching: A Study in Game-Theoretic Modelling and Analysis.* Cambridge University Press, Cambridge, 1990.