Computing the deficiency of housing markets with duplicate houses*

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Abstract. The model of a housing market, introduced by Shapley and Scarf in 1974 [14], captures a fundamental situation in an economy where each agent owns exactly one unit of some indivisible good: a house. We focus on an extension of this model where duplicate houses may exist. As opposed to the classical setting, the existence of an economical equilibrium is no longer ensured in this case. Here, we study the *deficiency* of housing markets with duplicate houses, a notion measuring how close a market can get to an economic equilibrium. We investigate the complexity of computing the deficiency of a market, both in the classical sense and also in the context of parameterized complexity.

We show that computing the deficiency is NP-hard even under several severe restrictions placed on the housing market, and thus we consider different parameterizations of the problem. We prove W[1]-hardness for the case where the parameter is the value of the deficiency we aim for. By contrast, we provide an FPT algoritm for computing the deficiency of the market, if the parameter is the number of different house types.

Keywords. Housing market, Economic equilibrium, Parameterized complexity.

1 Introduction

The standard mathematical model of a housing market was introduced in the seminal paper of Shapley and Scarf [14], and has successfully been used in the analysis of real markets such as campus housing [15], assigning students to schools [1], and kidney transplantation [13]. In a housing market there is a set of agents, each one owns one unit of a unique indivisible good (house) and wants to exchange it for another, more preferred one; the preference relation of an agent is a linearly ordered list (possibly with ties) of a subset of goods. Shapley and Scarf proved that in such a market an economic equilibrium always exists. A

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constructive proof in the form of the Top Trading Cycles algorithm is attributed to Gale (see [14]).

However, if we drop the assumption that each agent's house is unique, it may happen that the economic equilibrium no longer exists, and it is even NP-complete to decide its existence, see Fekete, Skutella, and Woeginger [8]. Further studies revealed that the border between easy and hard cases is very narrow: if agents have strict preferences over house types then a polynomial algorithm to decide the existence of an equilibrium is possible, see Cechlárová and Fleiner [4]. Alas, the problem remains NP-complete even if each agent only distinguishes between three classes of house types (trichotomous model): better house types, the type of his own house, and unacceptable house types [4]. So it becomes interesting to study the so-called *deficiency* of the housing market, i.e. the minimum possible number of agents who cannot get a most preferred house in their budget set under some prices of the house types.

In the present paper we give several results concerning the computation of the deficiency of housing markets, also from the parameterized complexity viewpoint. First, we show that the deficiency problem is NP-hard even in the case when each agent prefers only one house type to his endowment, and the maximum number of houses of the same type is two. This result is the strongest possible one in the sense that each housing market without duplicate houses admits an equilibrium [14]. Then we show that the deficiency problem is W[1]-hard with the parameter α describing the desired value of the deficiency, even if each agent prefers at most two house types to his own house, and the preferences are strict. Notice that the parameterized complexity of the case when each agent prefers only one house type to his endowment remains open. On the other hand, assuming that the preferences are strict, we provide a brute force algorithm that decides whether the deficiency is at most α in polynomial time for each fixed constant α. This shows that the problem is contained in XP when parameterized by α. This is in a strict contrast with the trichotomous model where even the case $\alpha = 0$ is NP-hard [4]. Finally, we provide an FPT algorithm for computing the deficiency (that works irrespectively of the type of preferences) if the parameter is the number of different house types.

To put our results into a broader context, let us mention that for general markets with divisible goods the celebrated Arrow–Debreu Theorem [2] guarantees the existence of an equilibirum under some mild conditions on agents' preferences. By contrast, it is well-known that in case of indivisible goods an equilibrium may not exist. From many existing approaches trying to cope with this nonexistence, let us mention Deng, Papadimitriou, and Safra [6] who introduced a notion of an ε-approximate equilibrium as one where the market "clears approximately", and the utility of each agent is within ε from the maximum possible utility in his budget set. They concentrated on approximation possibilites, and as far as we know such questions have not been studied yet from the parameterized complexity viewpoint.

2 Preliminaries

The paper is organized as follows. First, we introduce the model under examination, and give a brief overview of the basic concepts of parameterized complexity. In Section 3 we present some hardness results, whilst Section 4 is devoted to the proposal of two algorithms concerned with the computation of deficiency.

2.1 Description of the model

Let A be a set of N *agents*, H a set of M *house types*. The *endowment function* $\omega : A \to H$ assigns to each agent the type of house he originally owns. In the classical model of Shapley and Scarf [14], $M = N$ and ω is a bijection. If $N > M$ we say that the housing market has duplicate houses. Preferences of agent a are given in the form of a linear preference list $P(a)$. The house types appearing in the preference list of agent a are said to be *acceptable*, and we assume that $\omega(a)$ belongs to the least preferred acceptable house types for each $a \in A$. The notation $i \succeq_a j$ means that agent a *prefers* house type i to house type j. If $i \succeq_a j$ and simultaneously $j \succeq_a i$, we say that house types i and j are in a *tie* in a's preference list; if $i \succeq_a j$ and not $j \succeq_a i$, we write $i \succ_a j$ and say that agent a *strictly* prefers house type i to house type j. (If the agent is clear from the context, the subscript will be omitted.) The N-tuple of preferences $(P(a), a \in A)$ will be denoted by $\mathcal{P}$ and called the *preference profile*.

The *housing market* is the quadruple $\mathcal{M} = (A, H, \omega, \mathcal{P})$. We also define the *submarket* of $\mathcal{M}$ restricted to some agents of $S \subseteq A$ in the straightforward way.

We say that $\mathcal{M}$ is a housing market with *strict preferences* if there are no ties in $\mathcal{P}$. The *maximum house-multiplicity* of a market $\mathcal{M}$, denoted by $\beta(\mathcal{M})$, is the maximum number of houses of the same type, i.e. $\beta(\mathcal{M}) = \max_{h \in H} |\{a \in A : \omega(a) = h\}|$. The *maximum number of preferred house types* in the market, denoted by $\gamma(\mathcal{M})$, is the maximum number of house types that any agent might strictly prefer to its own house, i.e. $\gamma(\mathcal{M}) = \max_{a \in A} |\{h \in H : h \succ_a \omega(a)\}|$. We say that the market $\mathcal{M}$ is *simple*, if $\gamma(\mathcal{M}) = 1$.

The set of types of houses owned by agents in $S \subseteq A$ is denoted by $\omega(S)$. For each agent $a \in A$ we denote by $f_T(a)$ the set of the most preferred house types from $T \subseteq H$. For a set of agents $S \subseteq A$ we let $f_T(S) = \bigcup_{b \in S} f_T(b)$. For one-element sets of the form $\{h\}$ we often write simply h in expressions like $\omega(S) = h$, $f_T(S) = h$, etc.

We say that a function $x : A \to H$ is an *allocation* if there exists a permutation π on A such that $x(a) = \omega(\pi(a))$ for each $a \in A$. Notation $x(S)$ for $S \subseteq A$ denotes the set $\bigcup_{a \in S}\{x(a)\}$. In the whole paper, we assume that allocations are *individually rational*, meaning that $x(a)$ is acceptable for each $a \in A$. Notice that for each allocation x, the set of agents can be partitioned into directed cycles (*trading cycles*) of the form $K = (a_0, a_1, \ldots, a_{\ell-1})$ in such a way that $x(a_i) = \omega(a_{i+1})$ for each $i = 0, 1, \ldots, \ell - 1$ (here and elsewhere, indices for agents on cycles are taken modulo ℓ). We say that agent a *is trading* in allocation x if $x(a) \neq \omega(a)$.

Given a price function $p : H \to \mathbb{R}$, the *budget set* of agent a according to p is the set of house types that a can afford, i.e. $\{h \in H : p(h) \leq p(\omega(a))\}$. A pair (p, x), where $p : H \to \mathbb{R}$ is a price function, and x is an allocation is an *economic equilibrium* for market $\mathcal{M}$ if $x(a)$ is among the most preferred house types in the budget set of a.

It is known that if (p, x) is an economic equilibrium, then x is *balanced* with respect to p, i.e. $p(x(a)) = p(\omega(a))$ for each $a \in A$ (see [8, 4]).

As a housing market with duplicate houses may admit no equilibrium, we are interested in price-allocation pairs that are "not far" from the equilibrium. One possible measure of this distance was introduced in [4] by the notion of *deficiency* of the housing market.

An agent is said to be *unsatisfied* with respect to (p, x) if $x(a)$ is not among the most preferred house types in his budget set according to p. We denote by $\mathcal{D}_{\mathcal{M}}(p, x)$ the set of unsatisfied agents in $\mathcal{M}$ w.r.t. (p, x); more formally

$$\mathcal{D}_{\mathcal{M}}(p, x) = \{a \in A : \exists h \in H \text{ such that } h \succ_a x(a) \text{ and } p(h) \leq p(\omega(a))\}.$$

Given a price function p and an allocation x balanced w.r.t. p, we say that (p, x) is an α-*deficient equilibrium*, if $|\mathcal{D}_{\mathcal{M}}(p, x)| = \alpha$. Clearly, an economic equilibrium is a 0-deficient equilibrium. The deficiency of a housing market $\mathcal{M}$, denoted by $\mathcal{D}(\mathcal{M})$ is the minimum α such that $\mathcal{M}$ admits an α-deficient equilibrium. Given a housing market $\mathcal{M}$ and some $\alpha \in \mathbb{N}$, the task of the DEFICIENCY problem is to decide whether $\mathcal{D}(\mathcal{M}) \leq \alpha$.

We shall deal with the computational complexity of DEFICIENCY. For computational purposes, we shall say that the *size of the market* is equal to the total length of all preference lists of the agents, denoted by L.

2.2 Parameterized complexity

The aim of parameterized complexity theory is to study the computational complexity of NP-hard problems in a more detailed manner than in the classical setting. In this approach, we regard the running time of an algorithm as a function that depends not only on the size but also on some other crucial properties of the input. To this end, for each input of a given problem we define a so-called *parameter*, usually an integer, describing some important feature of the input.

Given a parameterized problem, we say that an algorithm is *fixed-parameter tractable* or *FPT*, if its running time on an input I with parameter k is at most $f(k)|I|^{O(1)}$ for some computable function f that only depends on k, and not on the size $|I|$ of the input. The intuitive motivation for this definition is that such an algorithm might be tractable even for large instances, if the parameter k is small. Hence, looking at some parameterized version of an NP-hard problem, an FPT algorithm may offer us a way to deal with a large class of typical instances.

The parameterized analysis of a problem might also reveal its W[1]-hardness, which is a strong argument showing that an FPT algorithm is unlikely to exist. Such a result can be proved by means of an FPT-reduction from an already known W[1]-hard problem such as Clique. Instead of giving the formal definitions, we refer to the books by Flum and Grohe [9] or by Niedermeier [12]. For a comprehensive overview, see the monograph of Downey and Fellows [7].

Considering the Deficiency problem, the most natural parameters, each describing some key property of a market $\mathcal{M}$, are as follows: the number of different houses types $|H| = M$, the maximum house-multiplicity $\beta(\mathcal{M})$, and the maximum number of preferred house types $\gamma(\mathcal{M})$ in the market. The value α describing the deficiency of the desired equilibrium can also be a meaningful parameter, if we aim for a price-allocation pair that is "almost" an economic equilibrium. The next sections investigate the influence of these parameters on the computational complexity of the Deficiency problem.

3 Hardness results

We begin with a simple observation which will be used repeatedly later on.

Lemma 1. *Let $\mathcal{M} = (A, H, \omega, \mathcal{P})$ be a housing market, p a price function and x a balanced allocation for p. Suppose $\omega(U) = u$ and $\omega(Z) = z$ for some sets $U, Z \subseteq A$ of agents. Suppose also that $f_H(Z) = u$ and $f_T(U) = z$ where $T \subseteq H$ contains the budget sets of all agents in U. Then $p(u) \neq p(z)$ implies that at least $\min\{|U|, |Z|\}$ agents in $U \cup Z$ are unsatisfied with respect to (p, x).*

Proof. If $p(u) \neq p(z)$ and the allocation is balanced, agents from the two sets cannot trade with each other. Therefore, due to the assumptions, if $p(u) > p(z)$ then all the agents in U are unsatisfied; if $p(z) > p(u)$ then all the agents in Z must be unsatisfied, and the assertion follows. □

Theorem 1. *The* Deficiency *problem is NP-complete even for simple markets $\mathcal{M}$ with $\beta(\mathcal{M}) = 2$.*

Proof. We provide a reduction from the Directed Feedback Vertex Set. We shall take its special version where the out-degree of each vertex is at most 2, which is also NP-complete, see Garey and Johnson [10], Problem GT7.

Given a directed graph $G = (V, E)$ with vertex set V and arc set E such that the outdegree of each vertex is at most 2, and an integer k, we construct a simple housing market $\mathcal{M}$ with $\beta(\mathcal{M}) = 2$ such that $\mathcal{D}(\mathcal{M}) \leq k$ if and only if G admits a feedback vertex set of cardinality at most k.

First, there are two house types $\hat{v}, \hat{v}'$ for each vertex $v \in V$ and $k + 1$ house types $\hat{e}_1, \ldots, \hat{e}_{k+1}$ for each arc $e \in E$. The agents and their preferences are given in Table 1. Here and later on, we write $[n]$ for $\{1, 2, \ldots, n\}$. The last entry in the list of each agent represents its endowment.

It is easy to see that $\mathcal{M}$ is simple, $\beta(\mathcal{M}) = 2$, the number of house types in $\mathcal{M}$ is $2|V| + (k + 1)|E|$ and the number of agents $|V| + (2k + 3)|E|$. To make the following arguments more straightforward, let us imagine $\mathcal{M}$ as a directed multigraph $\bar{G}$, where vertices are house types, and an arc from vertex $h \in H$ to vertex $h' \in H$ corresponds to an agent a with $\omega(a) = h$ and $h' \succ_a h$. Now,

agent	preference list
one agent $\bar{v}$ for each $v \in V$	$\hat{v}' \succ \hat{v}$
one agent $\bar{e}$ for each $e = vu \in E$	$\hat{e}_1 \succ \hat{v}'$
two agents $\bar{e}_i$ for each $e = vu \in E; i \in [k]$	$\hat{e}_{i+1} \succ \hat{e}_i$
two agents $\bar{e}_{k+1}$ for each $e = vu \in E$	$\hat{u} \succ \hat{e}_{k+1}$

Table 1. Endowments and preferences of agents in the market.

each directed cycle C in G has its counterpart $\bar{C}$ in $\bar{G}$, but each arc $e = vu$ on C corresponds to a "thick path" $\bar{v} \to \bar{u}$ containing $k+1$ consecutive pairs of parallel arcs in $\bar{G}$ (agents $\bar{e}_i, i \in [k+1]$). We shall also say that agents $\bar{e}, \bar{e}_i, i = 1, 2, \ldots, k+1$ are associated with the arc $e = vu$.

Now suppose that G contains a feedback vertex set W with cardinality at most k. For each $v \in W$ we remove agent $\bar{v}$ (together with its endowed house of type $\hat{v}$) from $\mathcal{M}$. The obtained submarket is acyclic, so assigning prices to house types in this submarket according to a topological ordering, we get a price function and an allocation with no trading in $\mathcal{M}$, where the only possible unsatisfied agents are the agents $\{\bar{v} \mid v \in W\}$.

Conversely, suppose that $\mathcal{M}$ admits a k-deficient equilibrium (p, x). If x produced any trading, then each trading cycle would necessarily involve some thick path $\bar{v} \to \bar{u}$ and thus exactly one agent from each pair $\bar{e}_i, i \in [k+1]$ on this thick path, making at least $k+1$ agents unsatisfied. Hence, there is no trading in x. Now, take any cycle $C = (v_1, v_2, \ldots, v_r, v_1)$ in G. Since it is impossible that all the inequalities $p(\hat{v}_1) < p(\hat{v}_2)$, $p(\hat{v}_2) < p(\hat{v}_3)$,..., $p(\hat{v}_r) < p(\hat{v}_1)$ along the vertices of C are fulfilled, at least one agent in $\bar{C}$ is unsatisfied. If this agent is $\bar{v}$ or belongs to the set of agents associated to an arc $e = vu$, we choose vertex v into a set W. It is easy to see that W is a feedback vertex set and $|W| \leq k$. □

Theorem 1 yields that Deficiency remains NP-hard even if $\gamma(\mathcal{M}) = 1$ and $\beta(\mathcal{M}) = 2$ holds for the input market $\mathcal{M}$. This immediately implies that Deficiency is not in the class XP w.r.t. the parameters $\beta(\mathcal{M})$, describing the maximum house-multiplicity, and $\gamma(\mathcal{M})$, denoting the maximum number of preferred house types. Next, we show that regarding α (the desired value of deficiency) as a parameter is not likely to yield an FPT algorithm, not even if $\gamma(\mathcal{M}) = 2$.

Theorem 2. *The* Deficiency *problem for a market $\mathcal{M}$ with strict preferences and with $\gamma(\mathcal{M}) = 2$ is W[1]-hard with the parameter α.*

Proof. We are going to show a reduction from the W[1]-hard Clique problem, parameterized by the size of the solution. Given a graph G and an integer k as the input of Clique, we will construct a housing market $\mathcal{M} = (\mathcal{A}, H, \omega, \mathcal{P})$ with strict preferences and with $\gamma(\mathcal{M}) = 2$ in polynomial time such that $\mathcal{M}$ has deficiency at most $\alpha = k^2$ if and only if G has a clique of size k. Since α depends only on k, this construction yields an FPT-reduction, and we obtain that Deficiency is W[1]-hard with the parameter α.

agent	preferences	"multiplicity"
$a \in A$	$\hat{c} \succ \hat{a}$	$\lvert A\rvert = n - k$
$b \in B$	$\hat{a} \succ \hat{d} \succ \hat{b}$	$\lvert B\rvert = 2m - k(k-1)$
$b \in B'$	$\hat{a} \succ \hat{b}$	$\lvert B'\rvert = t - (2m - k(k-1))$
$f \in F_1^c$	$\hat{c} \succ \hat{f}_2^c \succ \hat{f}_1^c$	$\lvert F_1^c\rvert = k$
$f \in F_2^c$	$\hat{f}_1^c \succ \hat{f}_2^c$	$\lvert F_2^c\rvert = k+1$
$f \in F_1^d$	$\hat{d} \succ \hat{f}_2^d \succ \hat{f}_1^d$	$\lvert F_1^d\rvert = k(k-1)$
$f \in F_2^d$	$\hat{f}_1^d \succ \hat{f}_2^d$	$\lvert F_2^d\rvert = k(k-1)+1$
$c_i \in C$	$\hat{a} \succ \hat{q}_i \succ \hat{c}$	$\lvert C\rvert = n$
$d \in D$	$\hat{b} \succ \hat{s}_i \succ \hat{d}$ if $d \in \{d_i^1, d_i^2\}$	$\lvert D\rvert = 2m$
$q_i \in Q$	$\hat{f}_1^c \succ \hat{d} \succ \hat{q}_i$	$\lvert Q\rvert = n$
$s_i^1 \in S$	$\hat{q}_x \succ \hat{f}_1^d \succ \hat{s}_i$ where $e_i = v_x v_y \in E$, $x < y$	$\lvert\{s_i^1 \mid i \in [m]\}\rvert = m.$
$s_i^2 \in S$	$\hat{q}_y \succ \hat{f}_1^d \succ \hat{s}_i$ where $e_i = v_x v_y \in E$, $x < y$	$\lvert\{s_i^2 \mid i \in [m]\}\rvert = m.$

Table 2. The preference profile of the market $\mathcal{M}$.

Let $G = (V, E)$ with $V = \{v_1, v_2 \ldots, v_n\}$ and $E = \{e_1, e_2 \ldots, e_m\}$. We can clearly assume $n > k^2 + k$, as otherwise we could simply add the necessary number of isolated vertices to G, without changing the answer to the Clique problem. Similarly, we can also assume $m > k^2$, as otherwise we can add the necessary number of independent edges (with newly introduced endvertices) to G.

The set of house types in $\mathcal{M}$ is $H = \{\hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{f}_1^c, \hat{f}_2^c, \hat{f}_1^d, \hat{f}_2^d\} \cup \hat{Q} \cup \hat{S}$, where $\hat{Q} = \{\hat{q}_i \mid i \in [n]\}$ and $\hat{S} = \{\hat{s}_i \mid i \in [m]\}$. Let $t = \max\{2m-k(k-1), n-k+\alpha+1\}$. First, we define seven sets of agents, $A, B, B', F_1^c, F_2^c, F_1^d$ and F_2^d. The cardinality of these agent sets are shown in Table 2; note that there might be zero agents in the set B'. Any two agents will have the same preferences and endowments if they are contained in the same set among these seven sets. Additionally, we also define agents in $C \cup D \cup Q \cup S$, where $C = \{c_i \mid i \in [n]\}$, $Q = \{q_i \mid i \in [n]\}$, $D = \{d_i^1, d_i^2 \mid i \in [m]\}$, and $S = \{s_i^1, s_i^2 \mid i \in [m]\}$. The preference profile of the market is shown on Table 2. Again, the endowment of an agent is the last house type in its preference list.

First, suppose that $\mathcal{M}$ admits a balanced allocation x for some price function p such that (p, x) is α-deficient. Observe that $f_H(c) = \hat{a}$ for each $c \in C$, $f_H(a) = \hat{c}$ for each $a \in A$. By $|C| > |A| > \alpha$ and Lemma 1, we obtain that $p(\hat{a}) = p(\hat{c})$ must hold. Moreover, by $|C| = |A| + k$ we also know that there are at least k agents in C who cannot obtain a house of type $\hat{a}$, let $C^* \subseteq C$ be a set containing k such agents. Clearly, agents in C^* are unsatisfied. Moreover, if all agents in $C \setminus C^*$ are satisfied, then they must trade with the agents of A.

Second, note that $f_H(b) = \hat{a}$ for each $b \in B \cup B'$, so $|B \cup B'| > |A| + \alpha$ (which follows from the definition of t) implies that $p(\hat{a}) > p(\hat{b})$ must hold, as otherwise more than α agents in $B \cup B'$ could afford a house of type $\hat{a}$ but would not be able to buy one. Thus, the budget set of the agents $B \cup B'$ does not contain the house type $\hat{a}$. In particular, we get that no agent in B' is trading in x. Note also that $f_{H\setminus\{\hat{a}\}}(b) = \hat{d}$ and $f_H(d) = \hat{b}$ for each $b \in B$ and $d \in D$, so

Lemma 1 and $|D| > |B| > \alpha$ yield that only $p(\hat{b}) = p(\hat{d})$ is possible. Taking into account that $|B| = |D| - k(k-1)$, we know that there must be at least $k(k-1)$ unsatisfied agents in D who are not assigned a house of type $\hat{b}$; let D^* denote this set of unsatisfied agents. Notice that if all the agents in $D \setminus D^*$ are satisfied, then they must be trading with the agents of B.

As $C^* \cup D^*$ contains α unsatisfied agents w.r.t. (p, x), and the deficiency of (p, x) is at most α, we get that no other agent can be unsatisfied. By the above arguments, this implies $x(A) = \hat{c}$, $x(C \setminus C^*) = \hat{a}$, $x(B) = \hat{d}$, and $x(D \setminus D^*) = \hat{b}$.

Next, we will show that $x(f) = \hat{c}$ for each $f \in F_1^c$ and $x(f) = \hat{d}$ for each $f \in F_1^d$. We will only prove the first claim in detail, as the other statement is symmetric. First, observe that $p(\hat{f}_2^c) \geq p(\hat{f}_1^c)$ is not possible, because by $f_H(F_2^c) = \hat{f}_1^c$ and $|F_2^c| > |F_1^c|$ such a case would imply at least one unsatisfied agent in F_2^c. Thus, we know $p(\hat{f}_2^c) < p(\hat{f}_1^c)$, which means that $\hat{f}_2^c$ is in the budget set of each agent in F_1^c. But since they do not buy such a house (as x is balanced), and they cannot be unsatisfied, we obtain that they must prefer their assigned house to $\hat{f}_2^c$. Thus, for each agent f in F_1^c we obtain $x(f) = \hat{c}$, proving the claim. The most important consequence of these facts is that every agent in $C^* \cup D^*$ must be trading according to x, as otherwise the agents in F_1^c and in F_1^d would not be able to get a house of type $\hat{c}$ or $\hat{d}$, respectively.

Recall that agents in C^* are unsatisfied, as they do not buy houses of type $\hat{a}$. But since they are trading, they must buy k houses from $\hat{Q}$; let $\hat{q}_{i_1}, \hat{q}_{i_2}, \ldots, \hat{q}_{i_k}$ be these houses. The agents $F_1^c, C^*, Q^* = \{q_{i_j} \mid j \in [k]\}$ trade with each other at price $p(\hat{c})$, yielding $x(F_1^c) = \hat{c}$, $x(C^*) = \omega(Q^*)$ and $x(Q^*) = \hat{f}_1^c$.

Similarly, the $k(k-1)$ agents in D^* must be trading, buying $k(k-1)$ houses of the set $\hat{S}$; let S^* denote the owners of these houses. Now, it should be clear that exactly $2m - k(k-1)$ houses of type $\hat{d}$ are assigned to the agents B, and the remaining $k(k-1)$ such houses are assigned to the agents F_1^d.

It should also be clear that the agents S^* are trading with agents F_1^d, so we obtain $x(F_1^d) = \omega(D^*) = \hat{d}$ and $x(D^*) = \omega(S^*)$. Thus, agents of $Q \setminus Q^*$ can neither be assigned a house of type $\hat{d}$ (as those are assigned to the agents $B \cup F_1^d*$), nor a house of type $\hat{f}_1^c$ (as those are assigned to agents in Q^*). As agents of $Q \setminus Q^*$ cannot be unsatisfied, we have that $p(\hat{q}_i) < p(\hat{d}) < p(\hat{f}_1^c)$ holds for each $q_i \in Q \setminus Q^*$, meaning that these agents do not trade according to x. (Recall that $p(\hat{d}) = p(\hat{b}) < p(\hat{a}) = p(\hat{c}) = p(\hat{f}_1^c)$.)

Now, if $x(d) = s_i$ for some agent $d \in D^*$ and $i \in [m]$, then we know that $p(\hat{s}_i) = p(\hat{d}) = p(\hat{f}_1^d)$. As neither of s_i^1 and s_i^2 can be unsatisfied, but neither of them can get a house from $\hat{Q}$, it follows that both of them must obtain a house of type $\hat{f}_1^d$. Therefore, the set S^* must contain pairs of agents owning the same type of house, i.e. $S^* = \{s_{j_i}^1, s_{j_i}^2 \mid i \in [k(k-1)]\}$.

Let us consider the agents s_j^1 and s_j^2 in S^*, and let v_x and v_y denote the two endpoints of the edge e_j, with $x < y$. Since s_j^1 prefers $\hat{q}_x$ to $x(s_j^1) = f_1^d$, we must have $p(\hat{s}_j) < p(\hat{q}_x)$, since s_j^1 must not be unsatisfied. Similarly, s_j^2 prefers $\hat{q}_y$ to $x(s_j^2) = f_1^d$, implying $p(\hat{s}_j) < p(\hat{q}_y)$. Taking into account that $p(\hat{s}_j) = p(\hat{d}) >$

$p(\hat{q}_i)$ for each $q_i \in Q \setminus Q^*$, we get that both q_x and q_y must be contained in Q^*. Hence, each edge in the set $E^* = \{e_j \mid s_j^1, s_j^2 \in S^*\}$ in G must have endpoints in the vertex set $V^* = \{v_i \mid q_i \in Q^*\}$. This means that the $\binom{k}{2}$ edges in E^* have altogether k endpoints, which can only happen if V^* induces a clique of size k in G. This finishes the soundness of the first direction of the reduction.

For the other direction, suppose that V^* is a clique in G of size k. We construct an α-deficient equilibrium (p, x) for $\mathcal{M}$ as follows. Let $I^* = \{i \mid v_i \in V^*\}$ and $J^* = \{j \mid e_j = v_x v_y, v_x \in V^*, v_y \in V^*\}$ denote the indices of the vertices and edges of this clique, respectively. We define $Q^* = \{q_i \mid i \in I^*\}$, $C^* = \{c_i \mid i \in I^*\}$, $S^* = \{s_j^1, s_j^2 \mid j \in J^*\}$, and $D^* = \{d_j^1, d_j^2 \mid j \in J^*\}$. Now, we are ready to define the price function p as follows.

$$
\begin{aligned}
&p(\hat{a}) = p(\hat{c}) = p(\hat{f}_1^c) = p(\hat{q}_i) = 4 \text{ for each } q_i \in Q^*,\\
&p(\hat{b}) = p(\hat{d}) = p(\hat{f}_1^d) = p(\hat{s}_i) = 3 \text{ for each } i \text{ where } s_i^1, s_i^2 \in S^*,\\
&p(\hat{q}_i) = 2 \text{ for each } q_i \in Q \setminus Q^*,\\
&p(h) = 1 \text{ for each remaining house type } h.
\end{aligned}
$$

It is straightforward to verify that the above prices form an α-deficient equilibrium with the allocation x, defined below.

$$
\begin{array}{lll}
x(A) = \omega(C \setminus C^*), & x(C \setminus C^*) = \hat{a}, & \\
x(B) = \omega(D \setminus D^*), & x(D \setminus D^*) = \hat{b}, & \\
x(F_1^c) = \omega(C^*), & x(C^*) = \omega(Q^*), & x(Q^*) = \hat{f}_1^c,\\
x(F_1^d) = \omega(D^*), & x(D^*) = \omega(S^*), & x(S^*) = \hat{f}_1^d,\\
\multicolumn{3}{l}{x(a) = \omega(a) \text{ for each remaining agent } a.}
\end{array}
$$

It is easy to see that $\mathcal{D}(p, x) = C^* \cup D^*$, implying that (p, x) is indeed α-deficient by $|C^* \cup D^*| = k + k(k-1) = \alpha$. The only non-trivial observation we need during this verification is that $p(\hat{q}_x) > p(\hat{s}_i)$ and $p(\hat{q}_y) > p(\hat{s}_i)$ for any s_i, where v_x and v_y are the endpoints of e_i. These inequalities trivially hold if $s_i \notin S^*$. In the case $s_i \in S^*$ we know $v_x, v_y \in V^*$ (since e_i is an edge in the clique V^*), which yields $p(\hat{q}_x) = p(\hat{q}_y) = p(\hat{s}_i) + 1$.

Hence, the reduction is correct, proving the theorem. □

4 Algorithms for computing the deficiency

Theorem 2 implies that we cannot expect an algorithm with running time $f(\alpha)L^{O(1)}$ for some computable function f for deciding whether a given market has deficiency at most α. However, we present a simple brute force algorithm that solves the Deficiency problem for strict preferences in $O(L^{\alpha+1})$ time, which is polynomial if α is a fixed constant. This means that Deficiency is in XP with respect to the parameter α. Recall that due to the results of [4], no such algorithm is possible if ties are present in the preference lists, as even the case $\alpha = 0$ is NP-hard in the trichotomous model.

Theorem 3. *If the preferences are strict, then the* Deficiency *problem can be solved in $O(L^{\alpha+1})$ time.*

Proof. Let $\mathcal{M} = (A, H, \omega, \mathcal{P})$ be the market given, and let α denote the deficiency what we aim for. Suppose (p, x) is an α-deficient equilibrium for $\mathcal{M}$, and let $\mathcal{D}_{\mathcal{M}}(p, x) = \{a_1, a_2, \ldots, a_\alpha\}$ be the set of unsatisfied agents. Let also $h_i = x(a_i)$ denote the house type obtained by the unsatisfied agent a_i for each $i \in [\alpha]$.

Now, we define a set of modified preference lists $\mathcal{P}[p, x]$ as follows: for each agent $a \in \mathcal{D}_{\mathcal{M}}(p, x)$ we delete every house type from its preference list, except for $x(a)$ and $\omega(a)$. We claim that (p, x) is an equilibrium allocation for the modified market $\mathcal{M}[p, x] = (A, H, \omega, \mathcal{P}[p, x])$. First, it is easy to see that x is balanced with respect to the price function p and for $\mathcal{M}[p, x]$, as neither the prices nor the allocation was changed. Thus, we only have to see that there are no unsatisfied agents in $\mathcal{M}[p, x]$ according to (p, x). By definition, in the market $\mathcal{M}[p, x]$ we know $x(a_i) = f_H(a_i)$ for each agent $a_i \in \mathcal{D}_{\mathcal{M}}(p, x)$. It should also be clear that for each other agent $b \notin \mathcal{D}_{\mathcal{M}}(p, x)$, we get that $x(b)$ is the first choice of b in its budget set according to p, since b was satisfied according to (p, x) in $\mathcal{M}$. Thus, b is also satisfied according to (p, x) in $\mathcal{M}[p, x]$. This means that (p, x) is indeed an equilibrium allocation for $\mathcal{M}[p, x]$.

For the other direction, it is also easy to verify that any equilibrium allocation (p', x') for $\mathcal{M}[p, x]$ results in an equilibrium for $\mathcal{M}$ with deficiency at most α, as only agents in $\mathcal{D}_{\mathcal{M}}(p, x)$ can be unsatisfied in $\mathcal{M}$ with respect to (p', x').

These observations directly indicate a simple brute force algorithm solving the Deficiency problem. For any set $\{a_1, a_2, \ldots, a_\alpha\}$ of α agents, and for any α-tuple $h_1, h_2, \ldots, h_\alpha$ of house types such that h_i is in the preference list of a_i (for each $i \in [\alpha]$), find out whether there is an economic equilibrium for the modified market, constructed by deleting every house type except for h_i and $\omega(a_i)$ from the preference list of a_i, for each $i \in [\alpha]$. Finding an economic equilibrium for such a submarket can be carried out in $O(L)$ time using the algorithm provided by Cechlárová and Jelínková [5].

Note that we have L possibilities for choosing an arbitrary agent together with a house type from its preference list (as L is exactly the number of "feasible" agent-house pairs), so we have to apply the algorithm of [5] at most $\binom{L}{\alpha}$ times. Therefore, the running time of the whole algorithm is $O(L^{\alpha+1})$. The correctness of the algorithm follows directly from the above discussion. □

Finally, we provide an FPT algorithm for the case where the parameter is the number of house types in the market.

Theorem 4. *There is a fixed-parameter tractable algorithm for computing the deficiency of a housing market with arbitrary preferences, where the parameter is the number M of house types in the market. The running time of the algorithm is $O(M^M \sqrt{N} L)$.*

Proof. Let $\mathcal{M} = (A, H, \omega, \mathcal{P})$ be a given housing market. If there is an α-deficient equilibrium (p, x) for $\mathcal{M}$ for some α, then we can modify the price function p to p' such that all prices are integers in $[M]$, and (p', x) forms an α-deficient equilibrium. Thus, we can restrict our attention to price functions from H to $[M]$.

The basic idea of the algorithm is the following: for each possible price function, we look for an allocation maximizing the number of satisfied agents. As a result, we get the minimum number of unsatisfied agents over all possible price functions. Note that we have to deal with exactly M^M price functions.

Given a price function $p : H \to [M]$ and an agent a, we denote by $T(a)$ the house types having the same price as $\omega(a)$, and by $B(a)$ the budget set of a.

Clearly, for any balanced allocation x w.r.t. p, we know $x(a) \in T(a)$. Thus, we can reduce the market by restricting the preference list of each agent a to the house types in $T(a)$; let $P'(a)$ denote the resulting list. The reduced market now defines a digraph G with vertex set A and arcs ab for agents $a, b \in A$ where b owns a house of type contained in $P'(a)$; note that each vertex has a loop attached to it. It is easy to see that any balanced allocation x indicates a cycle cover of G, and vice versa. (A cycle cover is a collection of vertex disjoint cycles covering each vertex.)

By definition, a is satisfied in some allocation x with respect to p, if $x(a) \in f_{B(a)}(a)$. We call an arc ab in G *important*, if $\omega(b)$ is contained in $f_{B(a)}(a)$. Hence, an agent a is satisfied in a balanced allocation if and only if the arc leaving a in the corresponding cycle cover is an important arc. By assigning weight 1 to each important arc in G and weight 0 to all other arcs, we get that any maximum weight cycle cover in G corresponds to an allocation with the maximum possible number of satisfied agents with respect to p.

To produce the reduced preference lists and construct the graph G, we need $O(L)$ operations. For finding the maximum weight cycle cover, a folklore method reducing this problem to finding a maximum weight perfect matching in a bipartite graph can be used (see e.g. [3]). Finding a maximum weight perfect matching in a bipartite graph with $|V|$ vertices, $|E|$ edges, and maximum edge weight 1 can be accomplished in $O(\sqrt{|V|}|E|)$ time [11]. With this method, our algorithm computes the minimum possible deficiency of a balanced allocation in time $O(\sqrt{N}L)$, given the fixed price function p. As the algorithm checks all possible price functions from H to $[M]$, the total running time is $O(M^M\sqrt{N}L)$. □

5 Conclusion

We have dealt with the computation of the deficiency of housing markets. We showed that in general, if the housing market contains duplicate houses, this problem is hard even in the very restricted case where the maximum house-multiplicity in the market $\mathcal{M}$ is two ($\beta(\mathcal{M}) = 2$) and each agent prefers only one house type to his own ($\gamma(\mathcal{M}) = 1$).

To better understand the nature of the arising difficulties, we also looked at this problem within the context parameterized complexity. We proposed an FPT algorithm for computing the deficiency in the case where the parameter is the number of different house types. We also presented a simple algorithm that decides in $O(L^{\alpha+1})$ time if a housing market with strict preferences has deficiency at most α, where L is the length of the input. By contrast, we showed W[1]-hardness for the problem where the parameter is the value α describing the deficiency of the equilibrium we are looking for.

This W[1]-hardness result holds if $\gamma(\mathcal{M}) = 2$, leaving an interesting problem open: if each agent prefers only one house type to his endowment (i.e. $\gamma(\mathcal{M}) = 1$), is it possible to find an FPT algorithm with parameter α that decides whether the deficiency of the given market $\mathcal{M}$ is at most α? Looking at the digraph underlying such a market where vertices correspond to house types and arcs correspond to agents, and using the characterization of housing markets that admit an economic equilibrium given by Cechlárová and Fleiner [4], it is not hard to observe that this problem is in fact equivalent to the following natural graph modification problem: given a directed graph G, can we delete at most α edges from it such that each strongly connected component of the remaining graph is Eulerian?

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