MATRIX ANALYSIS OF V- OR Y-SUPPORTED CONTINUOUS BRIDGE GIRDERS

Géza Tassi * *– Pál Rózsa* ** *– Ildikó Schlotter* ***

ABSTRACT

Advantageous moment distribution can be achieved by V- or Y-supported bridges. The structural model is a continuous girder with alternating spans. The application of the force method leads to a linear system of equations. The coefficient matrix of the system is a periodic tridiagonal matrix with alternating elements in the codiagonals. After rearranging the equations and the unknowns, the blocks of the inverse of the coefficient matrix can be expressed by applying the Sherman-Morrison formula. The elements of the Green-matrix are obtained in an explicit form. This enables the description of the influence line ordinates.

1. INTRODUCTION

"Selecting the openings is one of the most important factors of bridge economy. We have to aspire to the minimum common (sub- and superstructure) cost." [2] This rule – among others – has been guiding bridge engineers to find new shapes of the bridge. There was a tendency to reduce the spans of girder bridges without significant increase of the expenses for the support.

The endeavour for a good solution was first all over the world realized in Hungary [1]. The arrangement under appropriate clearance conditions under the bridge enables a more advantageous moment distribution in the girder and a relatively slender shape of the supporting system (Fig. 1). The essence of the structure doesn't change if the rear leg of the V-support is embedded in the slope (Fig. 2) [3]. The basic idea was spread in the world and has found a wide range of applications in many countries.

The principle was adopted at different bridges. One direction of the improvement of the structure concluded in the application of V-shaped supports for multi-span continuous bridges [6] (Fig. 3). Another version of concrete continuous bridge superstructures was implemented in case of tall piers, where Y-shaped supports seemed to be fit for the same target [5] (Fig. 4).

In this study we shall only deal with the forces of the bridge girder under given conditions. Otherwise the presented method is suitable to produce the internal forces for several other superstructure shapes and arbitrary load cases.

In the following a matrix analysis will be presented for calculating the internal forces in an explicit form.

*D. Sc., Professor, BME Department of Structural Engineering
**D. Sc., Professor, BME Department of Computer Science and Information Theory
***E. Eng., PhD Student, BME Department of Computer Science and Information Theory

Figure 1: The world's first bridge with V-shaped supports

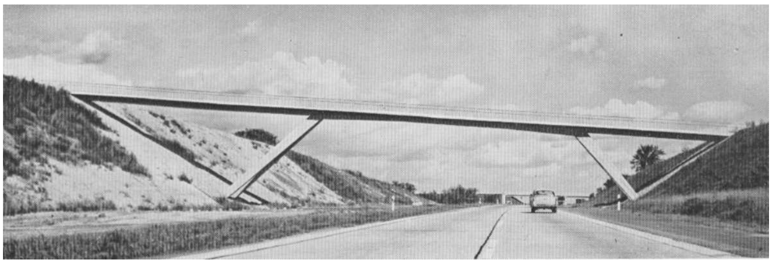

Figure 2: Pedestrian bridge with embedded rear stalk of the V-support

Figure 3: Continuous high speed railway viaduct with V-shape supports

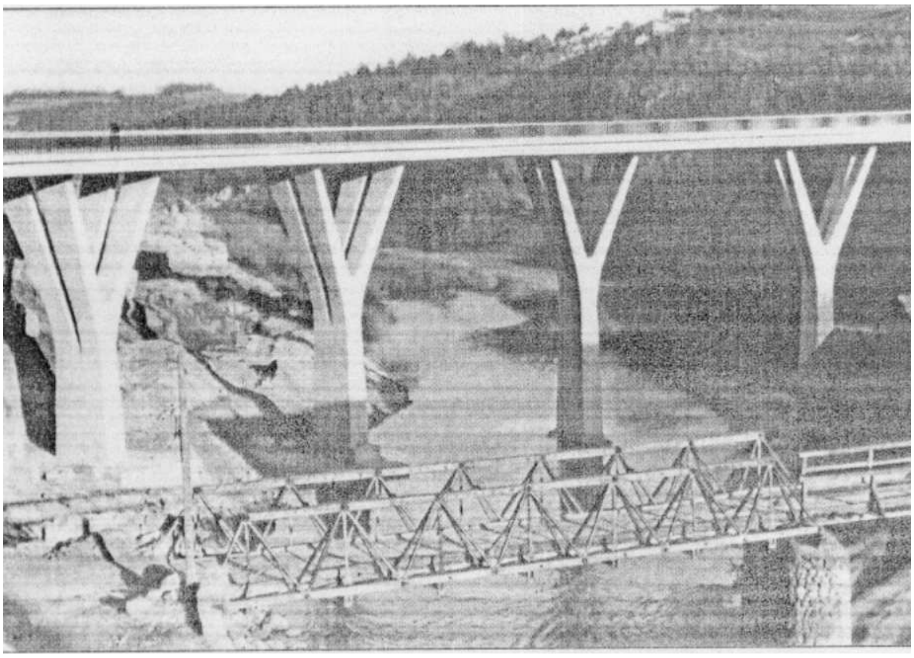

Figure 4: Highway viaduct with continuous girder and Y-shaped supports

2. BASIC ASSUMPTIONS

In the present analysis we shall suppose the following: the material of the superstructure is linearly elastic, the stiffness of the girder is constant (therefore the EJ value doesn't play any role). The girder is continuous, the spans are alternating, each second span has the same length, the first and the last spans are the same. The supports allow no vertical displacement but arbitrary horizontal slip except at a single (arbitrary) support. Fig. 5 gives the structural system. (N. B. it is indifferent in this calculation whether a bearing allows a horizontal shift or this is possible because of the slenderness of the columns.) No frame effect will be supposed.

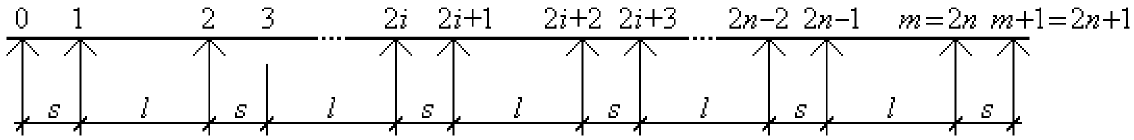

Figure 5: The structural system of the bridge girder

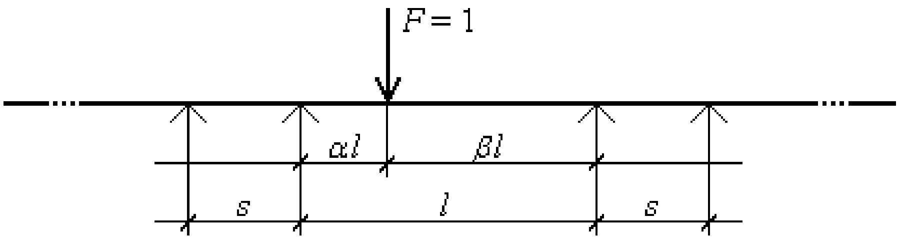

Figure 6: The unit force acting at a long span

3. SOLUTION OF THE STATICALLY INDETERMINATE STRUCTURE USING THE FORCE METHOD

The primary system for the force method is produced by setting a hinge above all intermediate supports. This way the number of redundant forces (moments above the supports) is m, the number of bays is $m + 1$, i. e. the number of the shorter spans (s) is $(m + 2)/2$ and that of the longer ones (l) is $m/2$. In order to distinguish the spans s and l (Fig. 6) the used numbering will be as follows: $m = 2n$. As a consequence, the index of the left side support of span s is an even number, $2i$, while that of the right side support of s is $2i + 1$, where $i = 0, 1, \ldots, n$.

The unit coefficients, i. e. the elements of the $2n \times 2n$ coefficient matrix in the main diagonal are as seen in matrix $\mathbf{A}$ in section 4.1., and the elements of the codiagonals are alternating as seen there as well.

There are different forms of the load vectors depending on the application point of the unit force: if the force is in a span l, the load vector contains two non-zero elements.

If the load acts at a span s the load vector has two further forms. In the intermediate bays there are two non-zero elements, in the case of an extreme bay only one. If the force is in a span l then $\mathbf{x}$ gives the unknown moments; if the force is in the bay s then the unknown moments are the elements of $\mathbf{x}'$.

We deal with the influence lines. Therefore it is considered that the unit concentrated load is acting at an arbitrary point of a bay as shown in Fig. 6. The values are given in vector $\mathbf{b}$ and $\mathbf{b}'$ respectively (see section 4.1.).

4. THE SYSTEM OF EQUATIONS AND ITS SOLUTION

4.1. The coefficient matrix and the load vector

The force method leads to a system of linear equations

$$\begin{aligned} \mathbf{A}\mathbf{x} &= \mathbf{b} && (1) \\ \mathbf{A}\mathbf{x}' &= \mathbf{b}'. && (2) \end{aligned}$$

The coefficient matrix of this is a tridiagonal marix:

$$\mathbf{A} = \begin{bmatrix} \frac{1}{3}(s+l) & \frac{1}{6}l & & & & \\ \frac{1}{6}l & \frac{1}{3}(s+l) & \frac{1}{6}s & & & \\ & \frac{1}{6}s & \frac{1}{3}(s+l) & \frac{1}{6}l & & \\ & & \frac{1}{6}l & \frac{1}{3}(s+l) & \ddots & \\ & & & \ddots & \ddots & \frac{1}{6}l \\ & & & & \frac{1}{6}l & \frac{1}{3}(s+l) \end{bmatrix}$$

and the elements of the load vector $\mathbf{b} = [b_k]$ and $\mathbf{b}' = [b'_k]$,respectively are

$$b_k = \begin{cases} \frac{1}{6}l^2(1+\beta)\alpha\beta & \text{if } k = 2j-1 \\ \frac{1}{6}l^2(1+\alpha)\alpha\beta & \text{if } k = 2j \\ 0 & \text{otherwise} \end{cases}$$

$$b'_k = \begin{cases} \frac{1}{6}s^2(1+\beta)\alpha\beta & \text{if } k = 2j \\ \frac{1}{6}s^2(1+\alpha)\alpha\beta & \text{if } k = 2j+1 \\ 0 & \text{otherwise} \end{cases}$$

Here $\mathbf{A}$ is a $2n$ by $2n$ matrix, $\mathbf{b}$ and $\mathbf{b}'$ are vectors of size $2n$ and j refers to the loaded span. We will investigate the two cases in a parallel way. In order to get a nicer form we can mulitply the equations by $6/l$. Then the coefficient martix will be

$$\frac{6}{l}\mathbf{A} = \begin{bmatrix} 2\left(1+\frac{s}{l}\right) & 1 & & & & \\ 1 & 2\left(1+\frac{s}{l}\right) & \frac{s}{l} & & & \\ & \frac{s}{l} & 2\left(1+\frac{s}{l}\right) & 1 & & \\ & & 1 & 2\left(1+\frac{s}{l}\right) & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & 1 & 2\left(1+\frac{s}{l}\right) \end{bmatrix}$$

while on the right side in the first case we get

$$\frac{6}{l}\mathbf{b} = \alpha\beta l \begin{array}{cl} \begin{bmatrix} 0 \\ \vdots \\ 1+\beta \\ 1+\alpha \\ \vdots \\ 0 \end{bmatrix} & \begin{array}{l} \\ \\ \leftarrow 2j-1 \\ \leftarrow 2j \\ \\ \\ \end{array} \end{array}$$

and in the second case

$$\frac{6}{l}\mathbf{b}' = \alpha\beta l \begin{array}{cl} \begin{bmatrix} 0 \\ \vdots \\ \left(\frac{s}{l}\right)^2(1+\beta) \\ \left(\frac{s}{l}\right)^2(1+\alpha) \\ \vdots \\ 0 \end{bmatrix} & \begin{array}{l} \\ \\ \leftarrow 2j \\ \leftarrow 2j+1 \\ \\ \\ \end{array} \end{array}$$

4.2. The Green-matrix of the problem

Based on the analogy to the differential equations, the inverse of the coefficient matrix can be considered as the Green-matrix of the problem.

As it can be seen, the coefficient matrix is a periodic tridiagonal matrix. In order to get the inverse of it, the rows and columns should be rearranged in such a way that first the odd numbered rows and columns, and afterwards the even numbered ones should follow each other. As a consequence, the periodic tridiagonal matrix will be transformed into a 2×2 block matrix of the following form, where the permutation matrix $\mathbf{P}$ corresponds to the rearrangement of the rows and columns:

$$\frac{6}{l}\left(\mathbf{P}^T\mathbf{A}\mathbf{P}\right)\left(\mathbf{P}^T\mathbf{x}\right) = \frac{6}{l}\left(\mathbf{P}^T\mathbf{b}\right)$$

where

$$\widetilde{\mathbf{A}} = \tfrac{6}{l}\mathbf{P}^T\mathbf{A}\mathbf{P} = \left[\begin{array}{cccc|cccc} 2\left(1+\frac{s}{l}\right) & 0 & 0 & & 1 & 0 & 0 & \\ 0 & 2\left(1+\frac{s}{l}\right) & 0 & & \frac{s}{l} & 1 & 0 & \\ 0 & 0 & 2\left(1+\frac{s}{l}\right) & \cdots & 0 & \frac{s}{l} & 1 & \\ & \vdots & & \ddots & & & \ddots & \ddots \\ \hline 1 & \frac{s}{l} & 0 & & 2\left(1+\frac{s}{l}\right) & 0 & 0 & \\ 0 & 1 & \frac{s}{l} & & 0 & 2\left(1+\frac{s}{l}\right) & 0 & \\ 0 & 0 & 1 & \ddots & 0 & 0 & 2\left(1+\frac{s}{l}\right) & \cdots \\ & & & \ddots & & \vdots & & \ddots \end{array}\right].$$

Since the rearrangement of the rows and columns corresponds to an orthogonal transformation by $\mathbf{P}$, the unknowns and the elements of the load vector will be rearranged the same way. That means the vector of the unknowns, which has size $2n$, becomes

$$\mathbf{P}^T\mathbf{x} = \begin{bmatrix} [x_{2i-1}] \\ \\ [x_{2i}] \end{bmatrix}, i = 1, 2, \ldots, n,$$

and the load vectors become

$$\widetilde{\mathbf{b}} = \frac{6}{l}\mathbf{P}^T\mathbf{b} = \alpha\beta l \begin{bmatrix} (1+\beta)\mathbf{e}_j \\ (1+\alpha)\mathbf{e}_j \end{bmatrix} \text{ and}$$

$$\widetilde{\mathbf{b}}' = \frac{6}{l}\mathbf{P}^T\mathbf{b}' = \alpha\beta l \left(\frac{s}{l}\right)^2 \begin{bmatrix} (1+\alpha)\mathbf{e}_{j+1} \\ (1+\beta)\mathbf{e}_j \end{bmatrix}$$

where by $\mathbf{e}_j$ we denote the jth unit vector. If the load vector is applied at the left and right side span respectively, the load vector is

$$\widetilde{\mathbf{b}}'' = \frac{6}{l}\mathbf{P}^T\mathbf{b}'' = \alpha\beta l \left(\frac{s}{l}\right)^2 \begin{bmatrix} (1+\alpha)\mathbf{e}_1 \\ 0 \end{bmatrix} \text{ and}$$

$$\widetilde{\mathbf{b}}''' = \frac{6}{l}\mathbf{P}^T\mathbf{b}''' = \alpha\beta l \left(\frac{s}{l}\right)^2 \begin{bmatrix} 0 \\ (1+\beta)\mathbf{e}_n \end{bmatrix}.$$

If we introduce the notation

$$\mathbf{H} = \begin{bmatrix} 0 & 1 & 0 & 0 & & 0 \\ 0 & 0 & 1 & 0 & & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ & & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

then we have

$$\widetilde{\mathbf{A}} = \begin{bmatrix} 2\left(1+\frac{s}{l}\right)\mathbf{E} & \mathbf{E}+\frac{s}{l}\mathbf{H}^T \\ \mathbf{E}+\frac{s}{l}\mathbf{H} & 2\left(1+\frac{s}{l}\right)\mathbf{E} \end{bmatrix} \tag{3}$$

where $\mathbf{E}$ denotes the n by n identity matrix.

Making use of the commutativity of the blocks of the matrix $\widetilde{\mathbf{A}}$, except for those in the secondary diagonal, it is easy to verify that

$$\begin{bmatrix} \mathbf{U} & \mathbf{V} \\ \mathbf{W} & \mathbf{U} \end{bmatrix}^{-1} = \begin{bmatrix} (\mathbf{U}^2-\mathbf{VW})^{-1}\mathbf{U} & -(\mathbf{U}^2-\mathbf{VW})^{-1}\mathbf{V} \\ -(\mathbf{U}^2-\mathbf{WV})^{-1}\mathbf{W} & (\mathbf{U}^2-\mathbf{WV})^{-1}\mathbf{U} \end{bmatrix} \tag{4}$$

if $\mathbf{VW} \neq \mathbf{WV}$ (see [4]).

Now in order to solve our linear equations (1) and (2), we will determine the inverse of $\widetilde{\mathbf{A}}$ by using the formula (4).We only have to check whether the following two conditions hold:

1. there exists the inverse of $\mathbf{N} = \left[\left(2\left(1+\frac{s}{l}\right)\mathbf{E}\right)^2 - \left(\mathbf{E}+\frac{s}{l}\mathbf{H}^T\right)\left(\mathbf{E}+\frac{s}{l}\mathbf{H}\right)\right]$

2. there exists the inverse of $\mathbf{M} = \left[\left(2\left(1+\frac{s}{l}\right)\mathbf{E}\right)^2 - \left(\mathbf{E}+\frac{s}{l}\mathbf{H}\right)\left(\mathbf{E}+\frac{s}{l}\mathbf{H}^T\right)\right]$.

In order to calculate the above mentioned inverses, we need the simple facts

$$\mathbf{H}^T\mathbf{H} = \begin{bmatrix} 0 & 0 & 0 & & 0 & \\ 0 & 1 & 0 & & 0 & \\ 0 & 0 & 1 & & 0 & \\ & \vdots & & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & & 1 \end{bmatrix} = \mathbf{E} - \mathbf{e_1}\mathbf{e_1}^T,$$

and $\mathbf{H}\mathbf{H}^T = \mathbf{E} - \mathbf{e_n}\mathbf{e_n}^T$. We can write

$$\mathbf{N} = \left(3 + 8\frac{s}{l} + 3\left(\frac{s}{l}\right)^2\right)\mathbf{E} + \left(\frac{s}{l}\right)^2 \mathbf{e_1}\mathbf{e_1}^T - \frac{s}{l}\left(\mathbf{H}+\mathbf{H}^T\right),$$

and similarly

$$\mathbf{M} = \left(3 + 8\frac{s}{l} + 3\left(\frac{s}{l}\right)^2\right)\mathbf{E} + \left(\frac{s}{l}\right)^2 \mathbf{e_n}\mathbf{e_n}^T - \frac{s}{l}\left(\mathbf{H}+\mathbf{H}^T\right).$$

It can be seen that $\mathbf{N}$ and $\mathbf{M}$ are tridiagonal matrices modified by one dyad. In order to get their inverse, the Sherman-Morrison formula can be applied.

The well known fact that the elements of the inverse can be expressed by Chebyshev polynomials of second kind enables us to find an explicit formula for the elements of the Green-matrix (see e. g. [4]).

Let us introduce $\mathbf{Q} = [q_{ij}] = \left(\frac{l}{s}\mathbf{N}\right)^{-1}$ and $2\cosh\psi = 8 + 3\left(\frac{s}{l}+\frac{l}{s}\right)$. Then we get

$$\mathbf{Q} = \left[\left(8 + 3\left(\frac{s}{l}+\frac{l}{s}\right)\right)\mathbf{E} - \left(\mathbf{H}+\mathbf{H}^T\right) + \frac{s}{l}\mathbf{e_1}\mathbf{e_1}^T\right]^{-1} =$$

$$= \left[\begin{bmatrix} 2\cosh\psi & -1 & 0 & & 0 \\ -1 & 2\cosh\psi & -1 & & 0 \\ 0 & -1 & 2\cosh\psi & \ddots & \\ & & \ddots & \ddots & -1 \\ 0 & 0 & 0 & -1 & 2\cosh\psi \end{bmatrix} + \frac{s}{l}\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\begin{bmatrix} 1 \; 0 \; 0 \cdots 0 \end{bmatrix}\right]^{-1}.$$

We can apply the formula

$$\begin{bmatrix} 2\cosh\psi & -1 & 0 & & 0 \\ -1 & 2\cosh\psi & -1 & & 0 \\ 0 & -1 & 2\cosh\psi & \ddots & \\ & & \ddots & \ddots & -1 \\ 0 & 0 & 0 & -1 & 2\cosh\psi \end{bmatrix}^{-1}_{ij} = \begin{cases} \dfrac{\sinh i\psi}{\sinh\psi}\dfrac{\sinh(n+1-j)\psi}{\sinh(n+1)\psi} & \text{if } i \le j \\ \dfrac{\sinh j\psi}{\sinh\psi}\dfrac{\sinh(n+1-i)\psi}{\sinh(n+1)\psi} & \text{if } i \ge j \end{cases}$$

Introducing the notation $\mathbf{S} = [s_{ij}] = \left(2\cosh\psi\mathbf{E} - \left(\mathbf{H} + \mathbf{H}^T\right)\right)^{-1}$ and using the Sherman-Morrison formula (see [4]) we get

$$\mathbf{Q} = \mathbf{S} - \frac{1}{\frac{l}{s} + s_{11}} \begin{bmatrix} s_{11} \\ s_{21} \\ \vdots \\ s_{n1} \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1n} \end{bmatrix} = \begin{bmatrix} s_{ij} - \dfrac{s_{i1}s_{1j}}{\frac{l}{s} + s_{11}} \end{bmatrix}.$$

Substituting the corresponding hyperbolic functions for s_{ij} after algebraic manipulations we get

$$q_{ij} = \begin{cases} \dfrac{\sinh(n+1-j)\psi}{\sinh\psi} \dfrac{\frac{l}{s}\sinh i\psi + \sinh(i-1)\psi}{\frac{l}{s}\sinh(n+1)\psi + \sinh n\psi} & \text{if } i \le j \\ \dfrac{\sinh(n+1-i)\psi}{\sinh\psi} \dfrac{\frac{l}{s}\sinh j\psi + \sinh(j-1)\psi}{\frac{l}{s}\sinh(n+1)\psi + \sinh n\psi} & \text{if } i \ge j \end{cases}$$

In a similar way when $\mathbf{R} = [r_{ij}] = \left(\dfrac{l}{s}\mathbf{M}\right)^{-1}$ the elements of the inverse

$$\mathbf{R} = [r_{ij}] = \left[\left(8 + 3\left(\frac{s}{l} + \frac{l}{s}\right)\right)\mathbf{E} - \left(\mathbf{H} + \mathbf{H}^T\right) + \frac{s}{l}\mathbf{e_n}\mathbf{e_n}^T\right]^{-1}$$

are obtained in the form

$$r_{ij} = \begin{cases} \dfrac{\sinh i\psi}{\sinh\psi} \dfrac{\frac{l}{s}\sinh(n+1-j)\psi + \sinh(n-j)\psi}{\frac{l}{s}\sinh(n+1)\psi + \sinh n\psi} & \text{if } i \le j \\ \dfrac{\sinh j\psi}{\sinh\psi} \dfrac{\frac{l}{s}\sinh(n+1-i)\psi + \sinh(n-i)\psi}{\frac{l}{s}\sinh(n+1)\psi + \sinh n\psi} & \text{if } i \ge j \end{cases}$$

Now that we know the inverse of $\mathbf{N}$ and $\mathbf{M}$, the solutions of the rearranged equations can easily be obtained. According to (3) the coefficient matrix can be considered as a block matrix, therefore we can calculate its inverse using the formula (4):

$$\widetilde{\mathbf{A}}^{-1} = \begin{bmatrix} 2(1 + \frac{s}{l})\mathbf{N}^{-1} & -\mathbf{N}^{-1}(\mathbf{E} + \frac{s}{l}\mathbf{H}^T) \\ -\mathbf{M}^{-1}(\mathbf{E} + \frac{s}{l}\mathbf{H}) & 2(1 + \frac{s}{l})\mathbf{M}^{-1} \end{bmatrix} = \begin{bmatrix} \mathbf{B}^{11} & \mathbf{B}^{12} \\ \mathbf{B}^{21} & \mathbf{B}^{22} \end{bmatrix}$$

where the blocks of $\widetilde{\mathbf{A}}^{-1}$ can be written as follows.

$$\begin{aligned}
\left(\mathbf{B}^{11}\right)_{ij} &= 2\left(1 + \frac{s}{l}\right)\left(\mathbf{N}^{-1}\right)_{ij} = 2\frac{s}{l}\left(1 + \frac{s}{l}\right) q_{ij} \\
\left(\mathbf{B}^{12}\right)_{ij} &= -\left(\mathbf{N}^{-1}\right)_{ij} - \frac{s}{l}\left(\mathbf{N}^{-1}\right)_{i,j+1} = -\frac{s}{l}\left(q_{ij} + \frac{s}{l}q_{i,j+1}\right) \\
\left(\mathbf{B}^{21}\right)_{ij} &= -\left(\mathbf{M}^{-1}\right)_{ij} - \frac{s}{l}\left(\mathbf{M}^{-1}\right)_{i,j-1} = -\frac{s}{l}\left(r_{ij} + \frac{s}{l}r_{i,j-1}\right) \\
\left(\mathbf{B}^{22}\right)_{ij} &= 2\left(1 + \frac{s}{l}\right)\left(\mathbf{M}^{-1}\right)_{ij} = 2\frac{s}{l}\left(1 + \frac{s}{l}\right) r_{ij}
\end{aligned}$$

4.3. The solution of the system of equations

Now we can calculate the solution of the equations (1) and (2):

$$\widetilde{\mathbf{x}} = \widetilde{\mathbf{A}}^{-1}\widetilde{\mathbf{b}} = \begin{bmatrix} \alpha\beta l\left((1+\beta)\left(\mathbf{B}^{11}\right)_{ij} + (1+\alpha)\left(\mathbf{B}^{12}\right)_{ij}\right) \\ \alpha\beta l\left((1+\beta)\left(\mathbf{B}^{21}\right)_{ij} + (1+\alpha)\left(\mathbf{B}^{22}\right)_{ij}\right) \end{bmatrix}$$

$$\widetilde{\mathbf{x}}' = \widetilde{\mathbf{A}}^{-1}\widetilde{\mathbf{b}}' = \begin{bmatrix} \alpha\beta\frac{s^2}{l}\left((1+\alpha)\left(\mathbf{B}^{11}\right)_{i,j+1} + (1+\beta)\left(\mathbf{B}^{12}\right)_{ij}\right) \\ \alpha\beta\frac{s^2}{l}\left((1+\alpha)\left(\mathbf{B}^{21}\right)_{i,j+1} + (1+\beta)\left(\mathbf{B}^{22}\right)_{i,j}\right) \end{bmatrix}$$

where $i = 1, 2, \ldots, n$. (This notation also applies henceforth, but we will omit it.)

This means that if the loaded span is a longer one, i. e. an l span, then the vector of the moments is the following:

$$\widetilde{\mathbf{x}} = \alpha\beta s \begin{bmatrix} \left(2\left(1+\frac{s}{l}\right)(1+\beta) - (1+\alpha)\right) q_{ij} - (1+\alpha)\frac{s}{l}q_{i,j+1} \\ \left(2\left(1+\frac{s}{l}\right)(1+\alpha) - (1+\beta)\right) r_{i,j} - (1+\beta)\frac{s}{l}r_{i,j-1} \end{bmatrix}$$

where j refers to the loaded span, and can have values $1, 2, \ldots, n$. If the loaded span is a shorter one and it is not extreme, i. e. j has value between 1 and $n-1$, then the moments are given in the vector

$$\widetilde{\mathbf{x}}' = \alpha\beta s\left(\frac{s}{l}\right)^2 \begin{bmatrix} \left(2\left(1+\frac{s}{l}\right)(1+\alpha) - \frac{s}{l}(1+\beta)\right) q_{i,j+1} - (1+\beta)q_{ij} \\ \left(2\left(1+\frac{s}{l}\right)(1+\beta) - \frac{s}{l}(1+\alpha)\right) r_{ij} - (1+\alpha)r_{i,j+1} \end{bmatrix}.$$

Finally, if the loaded span is the first or the last short one, respectively, we have

$$\widetilde{\mathbf{x}}'' = \widetilde{\mathbf{A}}^{-1}\widetilde{\mathbf{b}}'' = (1+\alpha)\alpha\beta s\left(\frac{s}{l}\right)^2 \begin{bmatrix} 2\left(1+\frac{s}{l}\right) q_{i1} \\ -r_{i1} \end{bmatrix} \quad \text{and}$$

$$\widetilde{\mathbf{x}}''' = \widetilde{\mathbf{A}}^{-1}\widetilde{\mathbf{b}}''' = (1+\beta)\alpha\beta s\left(\frac{s}{l}\right)^2 \begin{bmatrix} -q_{in} \\ 2\left(1+\frac{s}{l}\right) q_{in} \end{bmatrix}.$$

Let us remark here that the special case of equidistant piers (where $l = s$) can be handled much more simply. In this case our equations (1) and (2) have the following coefficient matrix:

$$\mathbf{A} = \begin{bmatrix} \frac{2}{3}l & \frac{l}{6} & & & \\ \frac{l}{6} & \frac{2}{3}l & \frac{l}{6} & & \\ & \frac{l}{6} & \frac{2}{3}l & \ddots & \\ & & \ddots & \ddots & \frac{l}{6} \\ & & & \frac{l}{6} & \frac{2}{3}l \end{bmatrix} = -\frac{l}{6} \begin{bmatrix} -4 & -1 & & & \\ -1 & -4 & -1 & & \\ & -1 & -4 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & -4 \end{bmatrix}$$

This case corresponds to the well known Clapeyron equations. Since this $2n$ by $2n$ matrix is symmetric uniform tridiagonal, we can directly get its inverse as follows:

$$\left(\frac{6}{l}\mathbf{A}\right)^{-1}_{ij} = \begin{cases} (-1)^{i+j}\dfrac{\sinh i\phi}{\sinh\phi}\dfrac{\sinh(2n+1-j)\phi}{\sinh(2n+1)\phi} & \text{if } i \le j \\ (-1)^{i+j}\dfrac{\sinh j\phi}{\sinh\phi}\dfrac{\sinh(2n+1-i)\phi}{\sinh(2n+1)\phi} & \text{if } i \ge j \end{cases}$$

where $2\cosh\phi = 4$. Our results naturally implicate this fact.

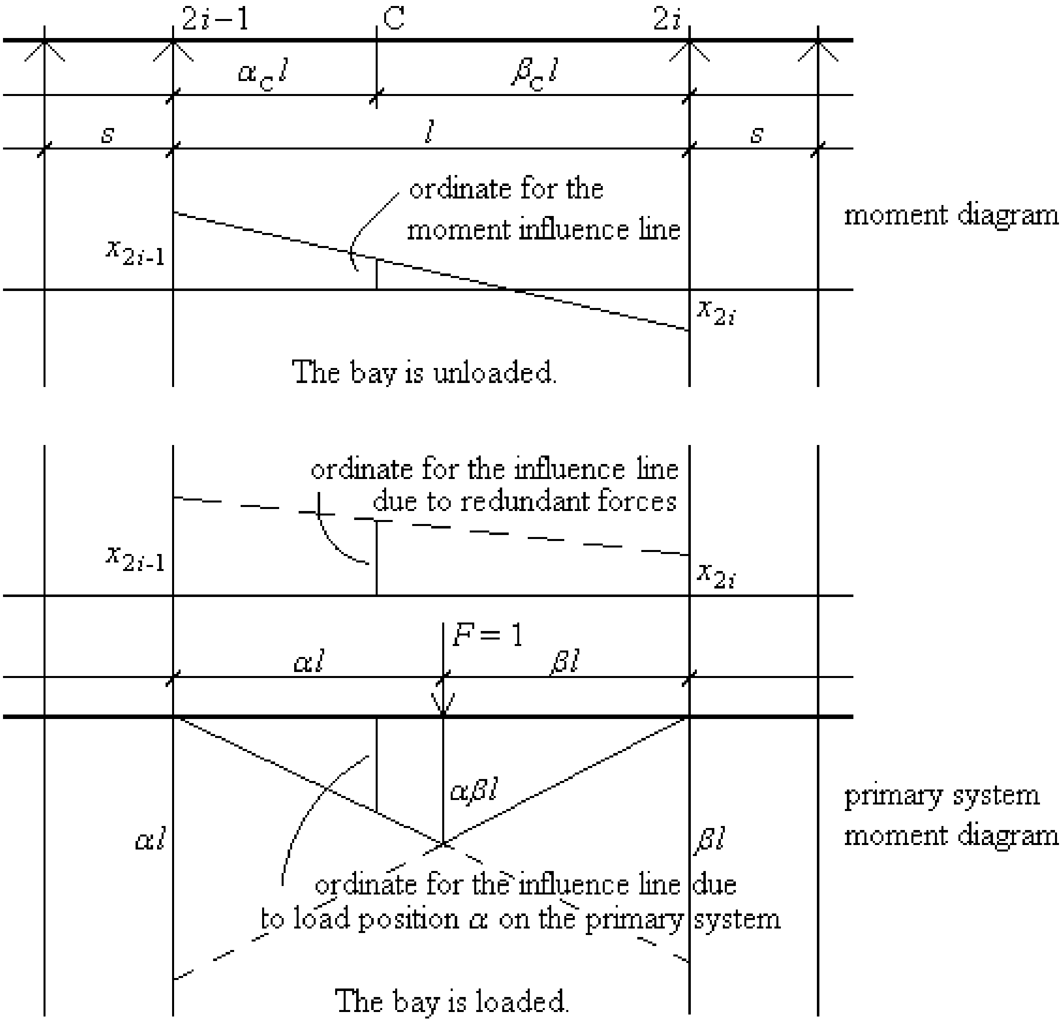

Figure 7: Built up of an influence line ordinate from redundant and primary system moments

4.4. The influence lines

The ordinates of the moment influence lines for cross sections above supports i are given directly by the unknowns x_i, x'_i. For cross sections C situated as seen in Fig. 7, the ordinates can be written as follows.

If the unit force is acting off the bay containing the cross section C, the ordinate is

$$\eta_{M_C} = x_{2i-1}\frac{\beta_C}{l} + x_{2i}\frac{\alpha_C}{l}$$

or

$$\eta_{M_C} = x_{2i}\frac{\beta_C}{s} + x_{2i+1}\frac{\alpha_C}{s}$$

respectively. If the unit force is in the bay containing C and $\alpha < \alpha_c$, we have

$$\eta_{M_C} = x_{2i-1}\frac{\beta_C}{l} + x_{2i}\frac{\alpha_C}{l} + \alpha\beta_C l$$

or

$$\eta_{M_C} = x_{2i}\frac{\beta_C}{s} + x_{2i+1}\frac{\alpha_C}{s} + \alpha\beta_C s,$$

and if $\alpha > \alpha_c$

$$\eta_{M_C} = x_{2i-1}\frac{\beta_C}{l} + x_{2i}\frac{\alpha_C}{l} + \alpha_C\beta l$$

and

$$\eta_{M_C} = x_{2i}\frac{\beta_C}{s} + x_{2i+1}\frac{\alpha_C}{s} + \alpha_C\beta s.$$

Producing these data for all load positions along the continuous girder, the influence line for moments at cross section C is completed.

5. CONCLUSION

There are obvious advantages of alternating spans for girder bridges. The first V-shaped support was applied in Hungary and this form was spread all over the world. The idea was extended to continuous superstructures applying V- or Y-shape supports. Although there are several numerical methods to produce the internal forces, it is worthwile to study the problem analytically.

To solve the problem the force method was used. The linear system of equations has been solved by simultaneously rearranging the equations and the unknowns. The coefficient matrix is a 2×2 block matrix. Each block in the main diagonal is a multiple of the identity matrix, therefore the blocks of the Green-matrix of the problem i. e. those of the inverse of the coefficient matrix can easily be expressed by the blocks of the coefficient matrix. A short calculation has shown that these blocks could be obtained by making use of the Sherman-Morrison formula for the inverse of a matrix modified by a one-rank matrix. The elements are expressed by the Chebyshev polynomials of the second kind, i. e. by applying hyperbolic functions containing the quotient of the alternating spans as a parameter. The product of the Green-matrix and the load vector yields the moments in explicit form.

Having the solution – the moments above the supports due to any unit load position – the ordinates of the moment influence line for an arbitrary cross section of the girder can be produced by elementary rules of theory of structures.

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