Which crossing number is it, anyway?

János Pach*
Courant Institute, NYU and Hungarian Academy of Sciences

Géza Tóth†
DIMACS Center, Rutgers University and Hungarian Academy of Sciences

Abstract

A drawing of a graph $G$ is a mapping which assigns to each vertex a point of the plane and to each edge a simple continuous arc connecting the corresponding two points. The crossing number of $G$ is the minimum number of crossing points in any drawing of $G$. We define two new parameters, as follows. The pairwise crossing number (resp. the odd-crossing number) of $G$ is the minimum number of pairs of edges that cross (resp. cross an odd number of times) over all drawings of $G$. We prove that the largest of these numbers (the crossing number) cannot exceed twice the square of the smallest (the odd-crossing number). Our proof is based on the following generalization of an old result of Hanani, which is of independent interest. Let $G$ be a graph and let $E_0$ be a subset of its edges such that there is a drawing of $G$, in which every edge belonging to $E_0$ crosses any other edge an even number of times. Then $G$ can be redrawn so that the elements of $E_0$ are not involved in any crossing. Finally, we show that the determination of each of these parameters is an NP-hard problem and it is NP-complete in the case of the crossing number and the odd-crossing number.

1 Introduction

The crossing number of a graph $G$ is usually defined as “the minimum number of edge crossings in any drawing of $G$ in the plane” [BL84]. However, one has to be careful with this definition, because it can be interpreted in several ways. Sometimes it is assumed that in a proper drawing no two edges cross more than once, and if two edges share an endpoint, they cannot have another point in common ([WB78], [B91]). Many authors do not make this assumption ([T70], [GJ83], [SSSV97]). If two edges are allowed to cross several times, we may count their intersections with multiplicity or without. We may also wish to impose some further restrictions on the drawings (e.g., the edges

*Supported by NSF grant CCR-94-24398 and PSC-CUNY Research Award 667339.
†Supported by OTKA-T-02994, OTKA-F-22234, and the Margaret and Herman Sokol Postdoctoral Fellowship Award.
must be straight-line segments [J71], or polygonal paths of length at most \( k \) [BD93]). No matter what definition we use, the determination of the crossing number of a graph appears to be an extremely difficult task ([GJ83], [B91]). In fact, we do not even know the asymptotic value of any of the above quantities for the complete graph \( K_n \) with \( n \) vertices and for the complete bipartite graph \( K_{n,n} \) with \( 2n \) vertices, as \( n \) tends to infinity [RT97]. The latter question, raised more than fifty years ago, is often referred to as Turán’s Brick Factory Problem [T77] or as Zarankiewicz’s problem [G69].

In the present paper, we investigate the relationship between various crossing numbers. First we agree on the terminology.

A drawing of a simple undirected graph is a mapping \( f \) that assigns to each vertex a distinct point in the plane and to each edge \( uv \) a continuous arc (i.e., a homeomorphic image of a closed interval) connecting \( f(u) \) and \( f(v) \), not passing through the image of any other vertex. For simplicity, the arc assigned to \( uv \) is called an edge of the drawing, and if this leads to no confusion, it is also denoted by \( uv \). We assume that no three edges have an interior point in common, and if two edges share an interior point \( p \), then they cross at \( p \). We also assume that any two edges of a drawing have a only a finite number of crossings (common interior points). A common endpoint of two edges does not count as a crossing.

**Definition.** Let \( G \) be a simple undirected graph.

(i) The rectilinear crossing number of \( G \), \( \text{LIN-CR}(G) \), is the minimum number of crossings in any drawing of \( G \), in which every edge is represented by a straight-line segment.

(ii) The crossing number of \( G \), \( \text{CR}(G) \), is the minimum number of edge crossings in any drawing of \( G \).

(iii) The pairwise crossing number of \( G \), \( \text{PAIR-CR}(G) \), is the minimum number of pairs of edges \((e, e')\) such that \( e \) and \( e' \) determine at least one crossing, over all drawings of \( G \). (That is, now crossings are counted without multiplicities.)

(iv) The odd-crossing number of \( G \), \( \text{ODD-CR}(G) \), is the minimum number of pairs of edges \((e, e')\) such that \( e \) and \( e' \) cross an odd number of times.

Clearly, we have

\[
\text{ODD-CR}(G) \leq \text{PAIR-CR}(G) \leq \text{CR}(G) \leq \text{LIN-CR}(G),
\]

It was shown by Bienstock and Dean [BD93] that there are graphs with crossing number 4, whose rectilinear crossing numbers are arbitrarily large. On the other hand, we cannot rule out the possibility that

\[
\text{ODD-CR}(G) = \text{PAIR-CR}(G) = \text{CR}(G)
\]

for every graph \( G \). The only result in this direction is the following remarkable theorem of Hanani and Tutte (see also [LPS97]).
Theorem A. [Ch34], [T70] If a graph $G$ can be drawn in the plane so that any two edges which do not share an endpoint cross an even number of times, then $G$ is planar.

For a generalization of this result to other surfaces, see [CN99].

In a fixed drawing of a graph $G$, an edge is called even if it crosses every other edge an even number of times. It follows from Theorem A that if all edges of $G$ are even, i.e., if $\text{odd-CR}(G) = 0$, then $\text{CR}(G) = 0$. (In this case, by Fáry’s theorem [F48], we also have $\text{LIN-CR}(G) = 0$.) In the next section, we establish the following generalization of this statement.

**Theorem 1.** For a fixed drawing of a graph $G$, let $G_0 \subseteq G$ denote the subgraph formed by all even edges.

Then $G$ can be drawn in such a way that the edges belonging to $G_0$ are not involved in any crossing.

At the end of the next section, we show how Theorem 1 implies that if the odd-crossing number of a graph is bounded, then its crossing number cannot be arbitrarily large. More precisely, we prove

**Theorem 2.** The crossing number of any graph $G$ satisfies

$$\text{CR}(G) \leq 2 \left( \text{odd-CR}(G) \right)^2.$$ 

It was discovered by Leighton [L84] that the crossing number can be used to obtain a lower bound on the chip area required for the VLSI circuit layout of a graph. For this purpose, he proved the following general lower bound for $\text{CR}(G)$, which was discovered independently by Ajtai, Chvátal, Newborn, and Szemerédi. The best known constant, $1/33.75$, in the theorem is due to Pach and Tóth.

**Theorem B.** [ACNS82], [L84], [PT97] Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$ such that $|E(G)| \geq 7.5|V(G)|$. Then we have

$$\text{CR}(G) \geq \frac{1}{33.75} \frac{|E(G)|^3}{|V(G)|^2}.$$ 

In Section 3, we prove that a similar inequality holds for the odd-crossing number.

**Theorem 3.** Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$ such that $|E(G)| \geq 4|V(G)|$. Then we have

$$\text{odd-CR}(G) \geq \frac{1}{64} \frac{|E(G)|^3}{|V(G)|^2}.$$ 

It was shown by Garey and Johnson [GJ83] that, given a graph $G$ and an integer $K$, it is an NP-complete problem to decide whether $\text{CR}(G) \leq K$. In the last section we show that the same is true for the odd-crossing number.
Theorem 4. Given a graph $G$ and an integer $K$, it is an NP-complete problem to decide whether $\text{ODD-CR}(G) \leq K$.

We can not prove the same for the pair-crossing number. (See Remark at the end of Section 4.)

2 Proofs of Theorems 1 and 2

First we establish Theorem 1. The proof somewhat resembles a proof of Kuratowski’s theorem (see [BM76]).

Suppose that Theorem 1 is false. Then there exists a graph $G$ with vertex set $V(G) = V$ and edge set $E(G) = E$, and there is a subset $E_0 \subseteq E$ such that $G$ has a drawing, in which every edge in $E_0$ is even, but there is no drawing, in which none of these edges is involved in any crossing. Let us fix a minimal counterexample to Theorem 1, i.e., a pair $(G, E_0)$ such that there exists no other pair $(\overline{G}, \overline{E_0})$, with the above property, for which the triple $(|E|, |E_0|, |V|)$ would precede $(|E|, |E_0|, |V|)$ in the lexicographic ordering. In particular, it follows from the minimality of $(G, E_0)$ that $G$ is connected.

If it leads to no confusion, throughout this section $G$ will stand both for the graph and for a particular drawing, in which all edges of $E_0$ are even. Let $G_0 = (V, E_0)$. A path (resp. cycle) in $G$ is said to be an $E_0$-path (resp. $E_0$-cycle), if all of its edges belong to $E_0$. Two edges are called independent, if they do not share an endpoint.

Claim 1. $G$ and $G_0 = (V, E_0)$ satisfy the following properties.

(i) There is no vertex of degree 1 in $G_0$.

(ii) There are no two adjacent vertices of degree 2 in $G_0$.

(iii) In any subdivision of $K_5$ or $K_{3,3}$ contained in $G$, there are two paths representing independent edges, such that neither of them is an $E_0$-path.

Proof. If $v$ has degree 1 in $G_0 = (V, E_0)$, and $uv \in E_0$, then $(G, E_0 \setminus \{uv\})$ is another counterexample, (lexicographically) smaller than $(G, E_0)$. If $u, v$ both have degree 2 in $G_0$ and $uv \in E_0$, then contract the edge $uv$ and remove all multiple edges (that is, keep only one copy of each edge), to obtain a smaller counterexample. Finally, part (iii) is an immediate corollary to Theorem A. □

Let $C$ be any $E_0$-cycle of $G$. A connected subgraph $B \subset G$ is a bridge of $C$ (in $G$) if it consists of either a single edge whose endpoints belong to $V(C)$, or of a connected component of $G - V(C)$ together with all edges connecting it to $C$. The endpoints of these edges in $C$ are called the endpoints of bridge $B$. (See also [BM76].) In the following, $P(x, y)$ will always denote a path in $G$ between two vertices, $x$ and $y$.

Claim 2. $G$ contains an $E_0$-cycle which has at least two bridges.

Proof. First we show that there is an $E_0$-cycle with a chord which is either a single $E_0$-edge or an $E_0$-path of length two.
Delete all isolated vertices of $G_0$. For every vertex $v$, which is adjacent to exactly two vertices, $u$ and $w$, in $G_0$, replace $uw, vu$, and $v$ with the single edge $uv$. Call the resulting multigraph $\tilde{G}_0$. By Claim 1, the degree of every vertex of $\tilde{G}_0$ is at least 3.

Let $P = x_0x_1 \ldots x_m$ be a longest path in $\tilde{G}_0$. Vertex $x_0$ has at least 3 neighbors, and, by the maximality of the path, all of them are on $P$. Hence, for some $1 < i < j$, $x_0x_i$ and $x_0x_j$ are edges of $\tilde{G}_0$. Then $x_0x_1 \ldots x_j$ is a cycle with chord $x_0x_i$ in $\tilde{G}_0$. Since every edge of $\tilde{G}_0$ arose from either an edge or a path of length two in $G_0$, the corresponding edges of $G_0$ form a cycle $C$ with a chord $c$ which is either a single edge or an $E_0$-path of length 2.

If $C$ has at least two bridges, then we are done. Assume it has only one bridge, $B$. Now $c$ is not a single edge, otherwise $B$ would be identical with $c$, and $G = G_0 = C \cup c$ is not a counterexample. Therefore, we can assume that $c$ is an $E_0$-path $xy$ of length 2.

The points $x$ and $y$ divide $C$ into two complementary paths (arcs). If two vertices of $C$, $a$ and $b$ (different from $x$ and $y$) do not belong to the same arc, we say that the pair $\{x, y\}$ separates $a$ from $b$ on $C$. Equivalently, the pair $\{a, b\}$ separates $x$ from $y$.

We distinguish three cases.

**Case 1.** $B$ has no two endpoints separated by the pair $\{x, y\}$.

Let $P(x, y)$ denote the arc of $C$ containing no endpoint of $B$ in its interior. Let $G'$ be the graph obtained from $G$ by replacing $P(x, y)$ with a single edge $xy$, and let $E'_0 = E_0 \cup \{xy\}$. It is easy to see that $(G', E'_0)$ is also a counterexample. By the minimality of $(G, E_0)$, we have that $G = G'$, i.e.,

$P(x, y)$ is a single edge $xy \in E_0$.

Swapping $xy$ with the chord $xy$, we obtain an $E_0$-cycle $C'$ with a chord $xy$. Therefore, $C'$ has at least two bridges, and Claim 2 is true.

**Case 2.** There is a path $P(a, b) \subset B$, not passing through $v$, which connects two points, $a$ and $b \in V(C)$, separated by the pair $\{x, y\}$.

Since $v$ and $P(a, b)$ belong to the same bridge, there is a path $P(v, q) \subset B$ connecting $v$ to an interior point $q$ of $P(a, b)$. Then $G$ contains a subdivision of $K_{3,3}$ with vertex classes $\{x, y, q\}$ and $\{a, b, v\}$. Moreover, all paths representing the edges of $K_{3,3}$ belong to $E_0$, with the possible exceptions of those adjacent to $q$. This contradicts Claim 1 (iii), which shows that this case cannot occur.

**Case 3.** Every path in $B$, whose endpoints are separated on $C$ by the pair $\{x, y\}$, passes through $v$.

Let $P_1(x, y)$ and $P_2(x, y)$ denote the two complementary arcs of $C$, and let $B_i$ be the union of all paths in $B$, which connect an internal point of $P_i(x, y)$ to $x, v$, or $y$.

Suppose first that $B = B_1 \cup B_2$. Then, by the minimality $(G, E_0)$, $G - B_i$, for $i = 1, 2$, has a drawing where no edge belonging to $E_0$ is involved in any crossing. In particular, in this drawing, $xy$ and the edges of $C$ are not crossed by any edge, so we can assume that all curves representing the edges of $B_i$ lie in the region bounded by $P_i(x, y)$ and $xy$ ($i = 1, 2$). Redrawing $G - B_2$, if necessary, so that $C$ and $xy$ are mapped to exactly the same curves as in the drawing of $G - B_1$, the
two drawings can be combined to give a drawing of $G$, contradicting our assumption that $(G, E_0)$ is a counterexample.

We are left with the case when $B \neq B_1 \cup B_2$. Then there is a vertex $s$ of $B$ which can not be reached from any internal point of $P_i(x, y)$ without passing through $x, v,$ or $y$ ($i = 1, 2$). Swapping $P_i(x, y)$ with $xvy$, we obtain an $E_0$-cycle $C'$ with a chord $P_i(x, y)$, which can be arbitrarily long. $C'$ has at least two bridges, because $P_i(x, y)$ and $s$ do not be in the same bridge. □

$$
\text{Case 1.}
\begin{array}{c}
\text{Case 2.}
\end{array}
$$

$$
\text{Case 3.}
\begin{array}{c}
\text{Figure 1.}
\end{array}
$$

In the sequel, let $C$ denote a fixed $E_0$-cycle of $G$ which has at least two bridges.

**Claim 3.** $C$ has at least three bridges.

**Proof.** Suppose there are only two bridges of $C$, $B_1$ and $B_2$. By the minimality of $G$, $G - B_1$ (resp. $G - B_2$) can be drawn in the plane so that none of its edges belonging to $E_0$ is involved in any crossing. In particular, in this drawing none of the edges of $C$ is involved in any crossing, therefore $B_2$ (resp. $B_1$) lies entirely on one side of $C$, say, in its interior (resp. exterior). But then

6
we can combine the two drawings and get a drawing of $G$. It is a contradiction since $G$ is assumed to be a counterexample. □

Let $B_1$ and $B_2$ be two bridges of $C$. By the minimality of $(G, E_0)$, the graph $C \cup B_1 \cup B_2$ can be drawn in the plane so that none of its edges belonging to $E_0$ participates in any crossing. If in all such drawings $B_1$ and $B_2$ are on different sides of $C$, then $B_1$ and $B_2$ are said to be conflicting.

**Claim 4.** $C$ has exactly three bridges, at least one of which is a single edge.

**Proof.** Construct a graph $\Gamma$ whose vertices correspond to the bridges of $C$, and two vertices are connected by an edge if and only if the corresponding bridges are conflicting. By the minimality of $(G, E_0)$, after the removal of any bridge the remaining graph can be drawn in the plane so that none of its edges belonging to $E_0$ is involved in any crossing. In other words, if we delete any vertex of $\Gamma$, it becomes two-colorable (the two colors correspond to the bridges inside and outside $C$). Therefore, any odd cycle of $\Gamma$ passes through every vertex of $\Gamma$, hence $\Gamma$ itself is an odd cycle.

Fix now any drawing of $G$, in which all edges belonging to $E_0$ are even. The closed curve representing $C$ divides the plane into connected cells. Color them with black and white so that no two cells that share a boundary arc receive the same color.

Let $B_i$ be a bridge of $C$. We need the following observation, which is an immediate consequence of the fact that every edge of $B_i$ crosses all edges of $C$ an even number of times. Assume that in a small neighborhood of one of its endpoints some edge of $B_i$ runs in the black (white) region. Then every edge of $B_i$ is black (resp. white) in a sufficiently small neighborhood of both of its endpoints. In this case, $B_i$ is said to be a black (resp. white) bridge. Every non-endpoint of a black (white) bridge must lie in the black (resp. white) region.

Since $\Gamma$ is an odd cycle, it has two consecutive vertices such that the corresponding bridges, say, $B_1$ and $B_2$, are conflicting and they are of the same color, say, black. We will specify two edges, $b_1 \in E(B_1)$ and $b_2 \in E(B_2)$. We distinguish two cases.

Suppose first that $B_1$ and $B_2$ have a common endpoint $v$. In a small neighborhood of $v$, all edges of $B_1$ and $B_2$ emanating from $v$ are disjoint and run in the black region. Therefore, we can find two consecutive edges, $b_1$ and $b_2$, in the cyclic order around $v$ such that $b_1 \in B_1$, $i = 1, 2$. In this case, set $w_1 = w_2 = v$.

Suppose next that $B_1$ and $B_2$ do not have a common endpoint. Let $v_iv_{i+1} \ldots v_j$ be a piece of $C$ such that $v_i$ is an endpoint of $B_1$, $v_j$ is an endpoint of $B_2$, and no $v_k$ ($i < k < j$) is an endpoint of either $B_1$ or $B_2$. There may be several edges of $B_1$ adjacent to $v_i$, which lie in the black region in a small neighborhood of $v_i$; let $b_1$ denote the last one in the cyclic order from the initial piece of $v_iv_{i-1}$ to that of $v_iv_{i+1}$. Similarly, let $b_2$ denote the first edge of $B_2$ emanating from $v_j$ in the cyclic order from the initial piece of $v_jv_{j-1}$ to that of $v_jv_{j+1}$. Now set $w_1 = v_i$ and $w_2 = v_j$.

Consider the drawing of $C \cup B_1 \cup B_2$ inherited from the original drawing of $G$. In this drawing, all edges belonging to $E_0 \cap (E(C) \cup E(B_1) \cup E(B_2))$ are even. We distinguish three cases depending on whether $B_1$ and $B_2$ are single edges, and in each case we slightly modify the graph $C \cup B_1 \cup B_2$.
and its drawing. The modified graph and its drawing will be denoted by $\overline{G} = (V, \overline{E})$, and we will also specify a set of edges $\overline{E}_0 \subseteq \overline{E}$.

\[ \begin{align*}
\text{Figure 2.}
\end{align*} \]

**Case 1. Both $B_1$ and $B_2$ are single edges.**

Then $E(B_i) = \{b_i\} = \{w_iu_i\}$, $i = 1, 2$. Split $b_i$ into two edges by adding an extra vertex $z_i$ very close to $w_i$, $i = 1, 2$. Connect $z_1$ and $z_2$ by an edge running very close to the path $z_1w_1...w_2z_2$, but not intersecting it (see Fig. 2), and denote the resulting graph drawing by $\overline{G}$. Since $b_1$ and $b_2$ are conflicting, at least one of them (say, $b_1$) belongs to $\overline{E}_0$. Then set $\overline{E}_0 = E(C) \cup \{w_1z_1, z_1u_1\}$.

**Case 2. $B_1$ is a single edge, $B_2$ is not.**

Then $E(B_1) = \{b_1\} = \{w_1u_1\}$, $E(B_2) \supset \{b_2\} = \{w_2z_2\}$, where $u_1 \in V(C)$ and $z_2 \notin V(C)$. Split $b_1$ into two edges by adding a vertex $z_1$ very close to $w_1$. As before, connect $z_1$ and $z_2$ by an edge running very close to the path $z_1w_1...w_2z_2$, and denote the resulting graph drawing by $\overline{G}$. If $b_1 \in \overline{E}_0$ then set $\overline{E}_0 = E(C) \cup \{w_1z_1, z_1u_1\}$. Otherwise, let $\overline{E}_0 = \overline{E}_0 \cap (E(C) \cup E(B_2))$, i.e., we leave the set of specified edges unchanged.

**Case 3. Neither $B_1$ nor $B_2$ is a single edge.**

Then $E(B_i) \supset \{b_i\} = \{w_iz_i\}$, where $z_i \notin V(C)$, for $i = 1, 2$. Connect $z_1$ and $z_2$ by an edge running very close to the path $z_1w_1...w_2z_2$, and denote the resulting graph drawing by $\overline{G}$. As in the previous case, let us leave the set of specified edges unchanged, i.e., set $\overline{E}_0 = \overline{E}_0 \cap (E(C) \cup E(B_1) \cup E(B_2))$.

It follows from the construction that in the above drawing of $\overline{G}$, every edge belonging to $\overline{E}_0$ is even. Recall that $B_1$ and $B_2$ were conflicting (see the last paragraph before Claim 4), which implies that in every drawing of $\overline{G}$ with the property that no edge in $\overline{E}_0$ is involved in any crossing, $z_1$ and $z_2$ lie on different sides of $C$. However, $z_1z_2 \in E(\overline{G}) = \overline{E}$, proving that $\overline{(G, E_0)}$ is also a counterexample to Theorem 1.

Suppose, to obtain a contradiction, that $C$ has more than three bridges in $G$. Since $\Gamma$ is an odd cycle, the number of bridges is odd, i.e., $C$ has at least five bridges. In the construction of $\overline{G}$, we kept only two of these bridges, so we deleted at least three bridges, hence at least three edges.
In Cases 1 and 2, we added at most two new edges. Thus, in these cases, $|E(\overline{G})| = |\overline{E}| < |E|$, contradicting our assumption that $(G, E_0)$ is a minimal counterexample.

The only remaining possibility is that $C$ has exactly five bridges, all of which are single edges. It follows from the structure of $\Gamma$ that at least three of these bridges (edges) belong to $E_0$. On the other hand, $\overline{G}$ has only two edges not in $C$ that belong to $E_0$. Thus, in this case, $|\overline{E}| = |E|$, but $|\overline{E}_0| < |E_0|$. This again contradicts the minimality of our counterexample.

Therefore, we can assume that $C$ has exactly three bridges in $G$, $B_1$, $B_2$, and $B_3$. If none of them is a single edge, then we can add one edge (as in Case 3) and delete a bridge, which contains more than one edge, to obtain a counterexample smaller than $(G, E_0)$. \( \Box \)

**Claim 5.** $C$ has at least two bridges which are single edges.

**Proof.** Assume, to obtain a contradiction, that $C$ has only one bridge which consists of a single edge. Take a closer look at the transformation in the proof of Claim 4. By deleting $B_3$ and adding one, two, or three edges, we obtained another counterexample $(\overline{G}, \overline{E}_0)$.

If $B_1$ or $B_2$ was the bridge consisting of a single edge, then we added two edges (cf. Case 2 in the proof of Claim 4) and deleted $B_3$, which had at least three edges. This contradicts the assumption that $(G, E_0)$ was a minimal counterexample.

Therefore, we can assume that $B_3$ consists of a single edge $xy$. Then, during the above transformation we deleted $B_3$ and added an edge that does not belong to $\overline{E}_0$ (cf. Case 3). Therefore, using the minimality of $(G, E_0)$ again, we obtain that $xy \notin E_0$.

Since $B_1$ and $B_3$ are conflicting, it follows that there is an $E_0$-path $P(a, b) \subset B_1$ whose endpoints, $a$ and $b$, separate $x$ and $y$ on $C$. Let $P_x(a, b)$ and $P_y(a, b)$ denote the two complementary arcs of $C$ between $a$ and $b$, containing $x$ and $y$, respectively.

We distinguish two cases.

**Case 1.** All endpoints of $B_2$ belong to the same arc, $P_x(a, b)$ or $P_y(a, b)$.

By symmetry, we can assume that all endpoints of $B_2$ are on $P_x(a, b)$. Then all endpoints of $B_1$ must also belong to $P_x(a, b)$. Indeed, if an endpoint of $B_1$ did not lie on this arc, then we could delete all edges of $B_1$ adjacent to it and obtain a smaller counterexample.

Consider the graph $\overline{G}$ constructed in the proof of Claim 4. In this graph, $y$ is adjacent to only two vertices, $y'$ and $y''$, both of which belong to $C$. Let $G'$ denote the graph obtained from $\overline{G}$ by deleting $y$ and replacing the $E_0$-path $y'y''$ by a single edge $y'y''$. Set $E'_0 = E_0 \setminus \{yy', yy''\} \cup \{y'y''\}$. Clearly, $(G', E'_0)$ is a counterexample to Theorem 1, which precedes $(G, E_0)$, contradicting the minimality of $(G, E_0)$.

**Case 2.** There exists a path $P(p, q) \subseteq B_2$ such that $p$ and $q$ are interior points of $P_x(a, b)$ and $P_y(a, b)$, respectively.

Consider again the graph $\overline{G}$. Clearly, $B_2$ contains a path connecting $b_1$ to some internal point $r$ of $P(a, b)$. (Note that $r$ may be an endpoint of $b_1$. Moreover, $b_1$ may belong to $P(a, b)$.) Similarly, $B_2$ contains a path connecting $b_2$ to some internal point $s$ of $P(p, q)$. However, in this case, $\overline{G}$ contains
a subdivision of $K_{3,3}$ with vertex classes $\{a, b, s\}$ and $\{p, q, r\}$. Furthermore, with the exception of the paths incident to $s$, all paths representing the edges of $K_{3,3}$ belong to $E_0$. However, this contradicts Claim 1 (iii). □

Now we can complete the proof of Theorem 1. By Claims 4 and 5, $C$ has precisely three pairwise conflicting bridges $B_i$, ($i = 1, 2, 3$) in $G$. Two of them, say, $B_1$ and $B_2$, are single edges, $xy$ and $ab$, respectively. Since $B_1$ and $B_2$ are conflicting, at least one of them, say $xy$, is in $E_0$.

Using the fact that $B_3$ is in conflict with $xy \in E_0$, we obtain that it contains a path connecting a pair of points $\{p, q\} \subseteq V(C)$ which separates $x$ from $y$. Similarly, since $B_3$ is in conflict with $ab$, it also contains a path connecting a pair of points $\{p', q'\} \subseteq V(C)$ which separates $a$ from $b$, and this path belongs to $E_0$ unless $ab \in E_0$. According to the position of these paths, we can distinguish four different cases up to symmetry (see Fig. 3). $P(p, q)$ always stands for a path connecting $p$ and $q$, whose internal vertices do not belong to $C$.

**Case 1.** $B_3$ contains a path $P(p, q)$; $p, q \in V(C)$, such that the pair $\{p, q\}$ separates $a$ from $b$ and $x$ from $y$, and $ab$ or $P(p, q)$ belongs to $E_0$.

Then $G$ has a subdivision of $K_{3,3}$ with vertex classes $\{a, p, y\}$ and $\{b, q, x\}$. Moreover, with the exception of $ab$ or $P(p, q)$, all paths representing the edges of $K_{3,3}$ belong to $E_0$. This contradicts Claim 1 (iii).

**Case 2.** $B_3$ contains three internally disjoint paths, $P(a, r)$, $P(p, r)$ and $P(q, r)$, such that $r$ does not belong to $C$; the pair $\{p, q\}$ separates $b$ from the set $\{a, x, y\}$; and $ab$ or $P(p, r) \cup P(q, r)$ belongs to $E_0$.

Then $G$ properly contains a subdivision of $K_{3,3}$ with vertex classes $\{x, r, b\}$, and $\{a, p, q\}$. It is easy to see that deleting from $G$ the arc of $C$ between $a$ and $y$ which does not contain $\{x, p, b, q\}$, we obtain a smaller counterexample. Thus, this case cannot occur.

**Case 3.** $B_3$ contains three internally disjoint paths, $P(p, r)$, $P(q, r)$, and $P(y, r)$, such that $r$ does not belong to $C$; the pair $\{p, q\}$ separates $x$ from the set $\{a, b, y\}$; and at least one of $ab$, $P(p, r) \cup P(y, r)$ and $P(q, r) \cup P(y, r)$ belongs to $E_0$.

Then $G$ properly contains a subdivision of $K_{3,3}$ with vertex classes $\{x, r, b\}$, and $\{y, p, q\}$. If $ab$ belongs to $E_0$, then deleting from $G$ the arc of $C$ between $a$ and $y$ which does not contain $\{p, x, q, b\}$, we obtain a smaller counterexample. If $ab$ does not belong to $E_0$, but, say, $P(p, r) \cup P(y, r)$ does, then, by the minimality of $(G, E_0)$, all paths depicted in Fig. 3 (3) are single edges, and $G$ has no further edges. However, this case cannot occur, because here $b$ and $q$ are two adjacent vertices of degree 2 in $G_0$, contradicting Claim 1 (ii).

**Case 4.** The endpoints of $B_3$ are $a, b, x, y$.

Since $B_2$ and $B_3$ are conflicting, $B_3$ contains two intersecting paths, $P(a, b)$ and $P(x, y)$, such that either $ab$ or $P(x, y)$ belongs to $E_0$. It follows from the minimality of our counterexample that $P(a, b)$ and $P(x, y)$ have only one vertex in common. Denoting it with $r$, we can write
$P(a,b) = P(a,r) \cup P(b,r)$ and $P(x,y) = P(x,r) \cup P(y,r)$. Then $G$ contains a subdivision of $K_5$ induced by $a, b, x, y, r$. Moreover, with the exception of $ab$, $P(a,r)$, and $P(b,r)$, all paths representing the edges of $K_{3,3}$ belong to $E_0$. This contradicts Claim 1 (iii).

In each case, we arrived at a contradiction. Thus, there exists no (minimal) counterexample $(G, E_0)$ to Theorem 1. The proof of Theorem 1 is complete. $\square$

Figure 3.

Theorem 2 is an easy corollary to Theorem 1. Let $G = (V, E)$ be a simple graph drawn in the plane with $\lambda = \text{ODD-CR}(G)$ pairs of edges that cross an odd number of times. Let $E_0 \subseteq E$ denote the set of even edges in this drawing. Since every edge not in $E_0$ crosses at least one other edge an odd number of times, we obtain that

$$|E \setminus E_0| \leq 2\lambda.$$

By Theorem 1, there exists a drawing of $G$, in which no edge of $E_0$ is involved in any crossing. Pick a drawing with this property such that the total number of crossing points between all pairs of edges not in $E_0$ is minimal. Notice that in this drawing, any two edges cross at most once. Therefore, the number of crossings is at most

$$\left(\frac{|E \setminus E_0|}{2}\right) \leq \left(\frac{2\lambda}{2}\right) \leq 2\lambda^2,$$

and Theorem 2 follows.

3 Proof of Theorem 3

The proofs of Theorem B readily generalize to this case. We include a short argument, for completeness.
First, we show that for any graph $G$,

$$\text{ODD-CR}(G) \geq |E(G)| - 3|V(G)|. \quad (1)$$

If $|E(G)| \leq 3|V(G)|$, then (1) is trivially true. Let $|E(G)| > 3|V(G)|$ and suppose that (1) holds for any graph with $|V(G)|$ vertices and less than $|E(G)|$ edges. Consider a drawing of $G$ with exactly $\text{ODD-CR}(G)$ pairs of edges crossing an odd number of times. Since $|E(G)| > 3|V(G)|$, $G$ is not planar, so by Theorem A, $\text{ODD-CR}(G) \geq 1$. Let $\overline{G}$ denote the graph obtained from $G$ by deleting one edge that crosses at least one other edge an odd number of times. Applying the induction hypothesis to $\overline{G}$, we get

$$\text{ODD-CR}(G) \geq \text{ODD-CR}(\overline{G}) + 1 \geq |E(\overline{G})| - 3|V(\overline{G})| + 1 = |E(G)| - 3|V(G)|,$$

as required.

To prove Theorem 3, fix a drawing of $G$ with exactly $\text{ODD-CR}(G)$ pairs of edges crossing an odd number of times, and suppose that $|E(G)| \geq 4|V(G)|$. Construct a random subgraph $G' \subseteq G$ by selecting each vertex of $G$ independently with probability $p$, and letting $G'$ be the subgraph induced by the selected vertices. The expected number of vertices of $G'$, $\text{Exp}[|V(G')|] = p|V(G)|$. Similarly, $\text{Exp}[|E(G')|] = p^2|E(G)|$. The expected number of pairs of edges that cross an odd number of times in the drawing of $G'$ inherited from $G$ is $p^4\text{ODD-CR}(G)$, hence the expected value of the odd-crossing number of $G'$ cannot be larger than this.

By (1), $\text{ODD-CR}(G') \geq |E(G')| - 3|V(G')|$ for every particular $G'$. Taking expectations,

$$p^4\text{ODD-CR}(G) \geq \text{Exp}[\text{ODD-CR}(G')] \geq \text{Exp}[|E(G')|] - 3\text{Exp}[|V(G')|] = p^2|E(G)| - 3p|V(G)|.$$

Setting $p = 4|V(G)|/|E(G)|$ we obtain

$$\text{ODD-CR}(G) \geq \frac{1}{64} \frac{|E(G)|^3}{|V(G)|^2}, \quad (2)$$

whenever $|E(G)| \geq 4|V(G)|$. □

Remarks. 1. In case $|E(G)| \geq 6|V(G)|$, Theorem 2 trivially follows from Theorem 3. Indeed, for any graph $G$,

$$\text{CR}(G) \leq \left( \frac{|E(G)|}{2} \right) < \frac{|E(G)|^2}{2} / 2.$$

If $|E(G)| \geq 6|V(G)|$ then Theorem 3 implies

$$2(\text{ODD-CR}(G))^2 \geq 2 \cdot \left( \frac{1}{64} \frac{|E(G)|^3}{|V(G)|^2} \right)^2 \geq \frac{|E(G)|^2}{2} > \text{CR}(G).$$
2. Using the fact that Theorem A guarantees, in any non-planar graph, the existence of two independent edges that cross an odd number of times, the above proof gives the same lower bound, \((1/64)|E(G)|^3/|V(G)|^2\), for the minimum number of pairs of independent edges that cross an odd number of times. This result is somewhat stronger than Theorem 3, because here we do not count any odd crossing between two edges that share an endpoint.

4 Proof of Theorem 4

First, we prove that the Odd Crossing Number Problem, \textsc{odd-cr}(G) \leq K, is in NP, and then we show that there is an NP-complete problem that can be reduced to it in polynomial time.

Fix a graph \(G\) with vertex set \(V = \{v_1, v_2, \ldots, v_n\}\) and edge set \(E\). Every drawing \(D\) of \(G\) can be represented by an \({\lfloor |E|/2\rfloor}\)-dimensional \((0,1)\)-vector \(\vec{X}_D(G)\), in which each coordinate is assigned to an unordered pair of edges \(\{e, f\} \subseteq E\), and is equal to 1 if and only if \(e\) and \(f\) cross an odd number of times. That is,

\[
\vec{X}_D(G) = (x_D\{e, f\})_{e\neq f, e,f \in E},
\]

where, for every \(e, f \in E\),

\[
x_D\{e, f\} = \begin{cases} 
0 & \text{if } e \text{ and } f \text{ cross an even number of times}, \\
1 & \text{if } e \text{ and } f \text{ cross an odd number of times}. 
\end{cases}
\]

We say that two drawings of \(G\), \(D\) and \(D'\) are equivalent if they are represented by the same vector, i.e., if \(\vec{X}_D(G) = \vec{X}_{D'}(G)\). An \({\lfloor |E|/2\rfloor}\)-dimensional \((0,1)\)-vector \(\vec{X}\) is said to be realizable if there exists a drawing \(D\) of \(G\) such that \(\vec{X}_D(G) = \vec{X}\).

Using an idea of Tutte [T70], it is not hard to describe the set of all realizable vectors of \(G\). We need some further notation. For any \(v \in V\), \(g \in E\), let

\[
\vec{Y}_{v,g} = (y\{e, f\})_{e\neq f, e,f \in E},
\]

where

\[
y\{e, f\} = \begin{cases} 
1 & \text{if } e = g \text{ and } f \text{ is adjacent to } v, \text{ or } f = g \text{ and } e \text{ is adjacent to } v, \\
0 & \text{otherwise}. 
\end{cases}
\]

Let \(\Phi\) denote the vector space over GF(2) generated by the vectors \(\vec{Y}_{v,g}\), i.e.,

\[
\Phi = \langle \vec{Y}_{v,g} \mid v \in V, \ g \in E \rangle_{\text{gen}} \subset \{0,1\}^{\lfloor |E|/2\rfloor}.
\]
Place the vertices $v_1, v_2, \ldots, v_n$ on a circle in this clockwise order so that they form a regular $n$-gon, and connect $v_i$ and $v_j$ ($i \neq j$) by a straight-line segment if and only if $v_i v_j \in E$. This drawing is said to be the convex drawing of $G$, and is denoted by $C$.

For any $1 \leq i \leq n$ let $d_i$ be the degree of $v_i$ and let $e_1^i, e_2^i, \ldots, e_{d_i}^i$ be the list of edges adjacent to $v_i$, in clockwise in the convex drawing of $G$. Let $\sigma_i : \{1, 2, \ldots, d_i\} \rightarrow \{1, 2, \ldots, d_i\}$ be any permutation. Define

$$\tilde{Z}_{v_i, \sigma_i} = (z\{e, f\})_{e \neq f, e, f \in E},$$

where

$$z\{e, f\} = \begin{cases} 1 & \text{if } e = e_\alpha^i, f = e_\beta^i \text{ and } (\alpha - \beta)(\sigma_i(\alpha) - \sigma_i(\beta)) < 0, \\ 0 & \text{otherwise.} \end{cases}$$

![Diagram](image)

**Figure 4.**

**Lemma 4.1.** Let $\Phi$ denote the vector space over $GF(2)$ generated by the vectors $\tilde{Y}_{v, g}$, $v \in V$, $g \in E$, let $\tilde{X}_C(G)$ be the $(0, 1)$-vector representing the convex drawing of $G$, and let

$$\Gamma = \left\{ \sum_{i=1}^{n} \tilde{Z}_{v_i, \sigma_i} \mid \sigma_i \text{ is any permutation } \{1, 2, \ldots, d_i\} \rightarrow \{1, 2, \ldots, d_i\} \right\}.$$

Then the set of all realizable vectors of $G$ is

$$\Psi = \tilde{X}_C(G) + \Gamma + \Phi,$$

14
where the sum is taken mod 2.

**Proof.** Let $D$ be any drawing of $G$, let $v \in V, g \in E$. Consider the following two operations:

(i) Choose a simple smooth arc $\gamma$ connecting any internal point $p$ of $g$ to $v$ such that it does not pass through any vertex, is not tangent to any edge, and crosses every edge a finite number of times. Replace a small piece of $g$ containing $p$ by a path going around $v$ and running extremely close to $\gamma$ (see Fig. 4). The $(0,1)$-vector representing this new drawing is

$$\bar{X}_g(G) = \bar{X}_D(G) + \bar{Y}_{v,g} \quad \text{(mod 2)}.$$

(ii) Let $\sigma_i$ be the clockwise order of $e_1, e_2, \ldots, e_d$ as they emanate from $v_i$ in drawing $D$. Change the clockwise order of edges as they emanate from $v_i$ to $e_1, e_2', \ldots, e_d'$ in a small neighborhood of $v_i$. (See Fig. 5.) The $(0,1)$-vector representing this new drawing is

$$\bar{X}_g(G) = \bar{X}_D(G) + \bar{Z}_{v_i,\sigma_i} \quad \text{(mod 2)}.$$

This shows that any vector in $\Psi$ is realizable.

![Figure 5.](image)

Next we prove that $\bar{X}_D(G) \in \Psi$, for any drawing $D$ of $G$. Using a topological transformation of the plane, if necessary, we can assume without loss of generality that the vertices of $G, v_1, v_2, \ldots, v_n$, form a regular $n$-gon, in this clockwise order. First, for every $1 \leq i \leq n$, in a small neighborhood of $v_i$, change the clockwise order of edges as they emanate from $v_i$ to $e_1, e_2', \ldots, e_d'$, such that in a very small neighborhood of $v_i$, each edge $v_iv_j$ is represented by the corresponding part of the segment $v_iv_j$.

Then, pick an edge $g = v_iv_j$, and transform it into the straight-line segment between $v_i$ and $v_j$, by continuous deformation. Performing this operation for all edges, one by one, we obtain $C$ the convex drawing of $G$.

Let $D'$ denote the drawing after the first step. Then,
\[ \tilde{\mathbf{x}}_{D'}(G) = \tilde{\mathbf{x}}_D(G) + \sum_{i=1}^{n} \tilde{z}_{v_i, \sigma_i} \pmod{2} \]

for some permutations \( \sigma^1, \sigma^2, \ldots, \sigma^n \).

During the second step, the representation vector of the drawing changes whenever a deforming edge \( g \) hits a vertex \( v \). Let \( \mathcal{E} \) and \( \mathcal{F} \) denote the drawing immediately before and after this event. Clearly,

\[ \tilde{\mathbf{x}}_{\mathcal{F}}(G) = \tilde{\mathbf{x}}_\mathcal{E}(G) + \tilde{Y}_{v, g} \pmod{2}. \]

Finally, we obtain

\[ \tilde{\mathbf{x}}_{\mathcal{C}}(G) = \tilde{\mathbf{x}}_{\mathcal{F}}(G) + \tilde{Y} \pmod{2}, \]

for some \( \tilde{Y} \in \Phi \), hence

\[ \tilde{\mathbf{x}}_{\mathcal{D}}(G) \in \tilde{\mathbf{x}}_{\mathcal{C}}(G) + \tilde{Y} = \Psi, \quad \square \]

Now we are in a position to prove that the Odd Crossing Number Problem is in NP. Suppose that odd-cr(G) \( \leq K \). Then, by Lemma 4.1, there is a realizable vector \( \tilde{Y} \in \Psi \) such that all but at most \( K \) coordinates of \( \tilde{Y} \) are 0. We can give the vector \( \tilde{Y} \) in the form

\[ \tilde{Y} = \tilde{\mathbf{x}}_{\mathcal{C}}(G) + \sum_{i=1}^{n} \tilde{z}_{v_i, \sigma_i} + \sum_{v \in V, g \in E} \alpha_{(v, g)} \tilde{Y}_{v, g} \pmod{2}, \]

where \( \alpha_{(v, g)} \in \{0, 1\} \) and \( \sigma_i : \{1, 2, \ldots, d_i\} \rightarrow \{1, 2, \ldots, d_i\} \) are permutations. Clearly, the correctness of this equation can be checked in polynomial time. Thus, the Odd Crossing Number Problem is in NP.

The Optimal Linear Arrangement Problem is the following. Given a graph \( G = (V, E) \) and an integer \( K \), is there a one-to-one function \( \sigma : V \rightarrow \{1, 2, \ldots, |V|\} \) such that \( \sum_{u \in V} |\sigma(u) - \sigma(v)| \leq K \)?

Notice that the Odd Crossing Number Problem for simple graphs is equivalent to the same problem for multigraphs, i.e., when the graph \( G \) may have multiple (parallel) edges. Indeed, we can remove all multiplicities by introducing new vertices along the edges of \( G \). For any graph \( \overline{G} \) obtained from \( G \) by subdividing one (or more) of its edges, we have

\[ \text{odd-cr}(\overline{G}) = \text{odd-cr}(G). \]

**Lemma 4.2.** The Optimal Linear Arrangement Problem can be reduced to the Odd Crossing Number Problem in polynomial time.

**Proof.** Suppose we are given an instance \( G = (V, E), K \), and we want to decide if there exists a one-to-one function \( \sigma : V \rightarrow \{1, 2, \ldots, |V|\} \) such that \( \sum_{u \in V} |\sigma(u) - \sigma(v)| \leq K \). Let \( V = \{v_1, v_2, \ldots, v_n\} \)

16
and assume without loss of generality that $G$ is connected. We construct a multigraph $G'_K$ and a number $K'$ such that the answer to our **OPTIMAL LINEAR ARRANGEMENT PROBLEM** is affirmative if and only if $\text{ODD-CR}(G'_K) \leq K'$.

Let $G'_K = (V', E')$, where $V' = V_1 \cup V_2 \cup \{u, w\}$, $E = E_1 \cup E_2 \cup E_3$,

$$V_1 = \{u_i \mid 1 \leq i \leq n\}, \quad V_2 = \{w_i \mid 1 \leq i \leq n\},$$

$$E_1 = \{|E|^2 \text{ copies of } u_iw_i \mid 1 \leq i \leq n\},$$

$$E_2 = \{u_iw_j \mid i < j \text{ and } v_iw_j \in E\},$$

$$E_3 = \{K^2|E|^2 \text{ copies of } uw, w_iu_i, w_iw_i, 1 \leq i \leq n\},$$

and let

$$K' = |E|^2(K - |E|) + |E|^2 - 1.$$

![Figure 6.](image)

Suppose first that there exists a bijection $\sigma : V \to \{1, 2, \ldots, |V|\}$ such that $\sum_{uv \in E} |\sigma(u) - \sigma(v)| \leq K$. We construct a drawing of $G'$ with at most $K'$ pairs of crossing edges. Place $u_i$ at $(1, \sigma(v_i))$, $w_i$ at $(0, \sigma(v_i))$, $u$ at $(2, 0)$, and $w$ at $(-1, 0)$. Represent all single edges by straight-line segments.
and all multiple edges by pairwise disjoint curves running very close to the corresponding straight line segment. It is easy to see that the total number of crossing pairs of edges is at most

$$\sum_{uv \in E} (|\sigma(u) - \sigma(v)| - 1)|E|^2 + |E|^2 - 1 \leq |E|^2 (K - |E|) + |E|^2 - 1 = K'.$$

Next, suppose that $\text{odd-cr}(G'_K) \leq K'$. We show, using some simple transformations, that there is another drawing of $G'$ generated by a function $\sigma$ in the way described above, which has at most $K'$ pairs of edges that cross an odd number of times. Consider a drawing of $G'_K$ with at most $K'$ pairs of edges that cross an odd number of times.

(a) We can assume that any two parallel edges, $e$ and $f$, are drawn very close to each other, so that they are openly disjoint, and any other edge crosses both of them the same number of times. Indeed, if $e$ and $f$ are drawn differently, then replacing either $e$ by an arc running very close to $f$, or $f$ by an arc running very close to $e$, we obtain a new drawing of $G$ which has at most as many pairs of edges that cross an odd number of times as the original drawing.

(b) Any two edges $e, f \in E_1 \cup E_3$ must cross an even number of times. Indeed, otherwise, by (a), we can assume that each of the at least $|E|^2$ edges parallel (or identical) to $e$ crosses each of the at least $|E|^2$ edges parallel (or identical) to $f$ an odd number of times. This implies that the number of edge pairs that cross an odd number of times is at least $|E|^4 > K'$, a contradiction.

(c) No edge of $G'_K$ can cross any edge between $u$ and $w$ an odd number of times. Otherwise, by (a), the number of pairs of edges that cross an odd number of times would be at least $K^2|E|^2 > K'$, which is impossible.

(d) Let $e$ be any edge between $u$ and $w$, and let $f_i$ (resp. $g_i$) be any edge whose endpoints are $u$ and $u_i$ (resp. $w$ and $w_i$), $1 \leq i \leq n$. If for some $i \neq j$, the edges $(e, f_i, f_j)$ emanate from $u$ in clockwise order, then $(e, g_i, g_j)$ must emanate from $v$ in counter-clockwise order.

To see this, consider a cycle $C$ formed by $f_i, e, g_i$, and any edge connecting $u_i$ and $w_i$. The closed curve representing this cycle divides the plane into connected cells. As in the proof of Theorem 1, color these cells with black and white so that no two cells that share a boundary are receive the same color. Let $P$ be a path formed by $f_j, g_j$, and any edge between $u_j$ and $w_j$. Suppose that in a small neighborhood of $u, f_j$ is in, say, the black region. Then, in a small neighborhood of $w, g_j$ must also lie in the black region, because, by (b), every edge of $P$ crosses (every edge of $C$) an even number of times.

(e) Suppose that $e, f_1, f_2, \ldots, f_n$ emanate from $u$ in the clockwise order $e, f_1, f_2, \ldots, f_n$. Then, by (d), $e, g_1, g_2, \ldots, g_n$ must emanate from $w$ in the reverse order $e, g_1, g_2, \ldots, g_n$. Let $\sigma(v_i) = \sigma^{-1}(i)$, $1 \leq i \leq n$.

We claim that for every $u_tw_j \in E_2$, there are at least $(|\sigma(v_i) - \sigma(v_j)| - 1)|E|^2$ edges in $G'_K$ that cross $u_tw_j$ an odd number of times. To see this, it is enough to show that for every $r < s < t$, if $v_{\alpha(r)}v_{\alpha(t)} \in E$, then the edge $e_{rt} := u_{\alpha(r)}w_{\alpha(t)}$ must cross the path $P_s := f_{\alpha(s)} \cup e_{\alpha(s)} \cup g_{\alpha(s)}$ an odd number of times, where $e_{\alpha(s)}$ denotes any edge between $u_{\alpha(s)}$ and $w_{\alpha(s)}$. As before, color the cells determined by the closed curve $P_s \cup e$ with black and white. It follows from (d) that if in a small
neighborhood of \( u \), \( f_{\alpha(r)} \cup e_{rt} \cup g_{\alpha(t)} \) is in the black region, then in a small neighborhood of \( w \) it is in the white region. In view of (b) and (e), this implies that \( e_{rt} \) crosses at least one of the edges \( f_{\alpha(s)} \), \( e_{\alpha(s)} \), and \( g_{\alpha(s)} \) an odd number of times. In each case, we are done, and our claim is true.

Therefore, we have

\[
\sum_{uv \in E} (|\sigma(u) - \sigma(v)| - 1)|E|^2 \leq \text{odd-cr}(G'_K) \leq |E|^2(K - |E|) + |E|^2 - 1,
\]

which implies that

\[
\sum_{uv \in E} (|\sigma(u) - \sigma(v)| \leq K,
\]

as desired. \( \square \)

With Lemma 4.2, the proof of Theorem 4 (ii) is complete, because the \textsc{Optimal Linear Arrangement Problem} is known to be NP-complete [GJS76].

\textbf{Remark.} We can prove that the \textsc{Pair Crossing Number Problem}, \( \text{pair-cr}(G) \leq K \), is NP-hard. The proof is analogous to the proofs of the corresponding results for the crossing number (see [GJ83]) and for the odd-crossing number (see Lemma 4.2). On the other hand, we are unable to prove that the \textsc{Pair Crossing Number Problem} is in NP, that is, we cannot generalize Lemma 4.1 for \( \text{pair-cr}(G) \).

A preliminary version of this paper appeared in the proceedings of FOCS 1998 [PT98].

\textbf{Acknowledgement.} We would like to thank Pavel Valtr for his valuable ideas and comments.

\textbf{References}


[BD92] D. Bienstock and N. Dean, New results on rectilinear crossing numbers and plane embeddings, \textit{J. Graph Theory} 16 (1992), 389–398.


