Multiple coverings of the plane with triangles

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Dedicated to the memory of László Fejes Tóth

Abstract

We prove that any 43-fold covering of the plane with translanslates of a triangle can be decomposed into two coverings.

1 Introduction

Let $\mathcal{C} = \{ T_i \mid i \in I \}$ be a collection of planar sets. It is a *k-fold covering* if every point in the plane is contained in at least k members of $\mathcal{C}$. A 1-fold covering is simply called a *covering*.

Definition. A planar set T is said to be cover-decomposable if the following holds. There exists a constant $k = k(T)$ such that every k-fold covering of the plane with translates of T can be decomposed into two coverings. J. Pach proposed the problem of determining all cover-decomposable sets.

Conjecture. (J. Pach) *All convex sets are cover-decomposable.*

The conjecture is verified in two special cases.

Theorem A. (i) [P86] *Every centrally symmetric open convex polygon is cover-decomposable.*

(ii) [MP89] *The open unit disc is cover-decomposable.*

Based on the ideas of Pach [P86], in this note we prove another special case.

Theorem 1. *Every open triangle is cover-decomposable.*

In fact we prove the following somewhat stronger statement that, however, requires the technical condition that the collection of translates is *locally finite*, i.e., that every compact region intersects a finite number of the translates.

Theorem 2. *Every locally finite collection $\mathcal{C}$ of translates of the same (open or closed) triangle T can be partitioned into two parts such that every point that is covered by at least 43 of the triangles in $\mathcal{C}$ is covered by a triangle in both parts.*

We believe that Theorem 2 remains true without the requirement that $\mathcal{C}$ is locally finite, however our proof uses this assumption. If Theorem 2 is true without this assumption, then, clearly, closed

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triangles are also cover-decomposible. We find it hard to immagine that closed triangles are not cover-decomposible, but a proof of this seems to be elusive.

We finish this introductory section by observing that Theorem 2 implies Theorem 1. Indeed, from every 43-fold covering of the plane by translates of the same open triangle (or by arbitrary open sets of bounded diameter) one can select a locally finite subcollection that is also a 43-fold cover. To see this it is enough to use the square grid to cut the the plane into squares and one can use the compactness of the closed squares S_i to find a finite subset $\mathcal{C}_i$ of $\mathcal{C}$ that is covering S_i 43-fold. If we delete all sets from $\mathcal{C}_i$ that are disjoint from S_i, then the union $\cup_i \mathcal{C}_i$ is locally finite and a 43-fold covering of the plane.

Notice that similar statement for closed triangles is false. It is easy to find a covering of the plane by translates of a closed triangle that does not contain a locally finite subset that also covers the plane. One can also find a 2-fold covering consisting of translates of a closed triangle such that no proper subset is a 2-fold covering and the cardinality of the collection is the continuum.

2 Proof of Theorem 2

Just like in [P86], we formulate (and solve) the problem in its dual form. Fix O, the center of gravity of the triangle T as our origin in the plane. For a planar set P and a point x in the plane we use $P(x)$ to denote the translate of P by the vector $\vec{Ox}$. Let $\bar{T}$ be the reflection through O of the triangle T. Consider any collection $\mathcal{C} = \{ T(x_i) \mid i \in I\}$ of translates of T. For any point p, $p \in T(x_i)$ if and only if $x_i \in \bar{T}(p)$. To see this, apply a reflection through the midpoint of the segment px_i. This switches $T(x_i)$ and $\bar{T}(p)$, and also switches x_i and p.

The collection $\mathcal{C}$ covers p at least 43 times if and only if $\bar{T}(p)$ contains at least 43 elements of the set $\mathcal{S} = \{ x_i \mid i \in I\}$. The required partition of $\mathcal{C}$ exists if and only if the set $\mathcal{S}$ can be colored with two colors such that every translate of $\bar{T}$ that contains at least 43 elements of $\mathcal{S}$ contains at least one element of each color.

It is easy to see that $H = T \cap \bar{T}$ is a hexagon whose vertices are those points which divide the sides of T in the ratio $1:2$. Observe that the plane has a lattice tiling with translates of this hexagon. We can assume without loss of generality that each x_i is in the interior of some hexagon. By our assumption on local finiteness each hexagon contains finitely many points x_i.

Elementary analysis shows that any translate of $\bar{T}$ intersects at most 6 of the hexagons in the tiling, so if it contains at least 43 elements of $\mathcal{S}$, then it contains at least 8 elements belonging to the same hexagon.

Let A, B and C be the vertices of $\bar{T}$. Let us consider the natural linear order on the lines of the plane that are parallel to the side BC with the line through A defined smaller than the line BC. We define the partial order $<_A$ on the points with $x <_A y$ if the line through x is smaller than the line through y. We have $A <_A B$ and thus also $A <_A C$. Similarly consider the partial ordering $<_B$ according to the lines parallel to AC with $B <_B C$, and the partial ordering $<_C$ according to the lines parallel to AB with $C <_C A$. We define W_A to be the set of points p with $O <_B p$ and $O <_C p$. Similarly, let W_B is the set of points p with $O <_C p$ and $O <_A C$ and let W_C be the set of points p with $O <_A p$ and $O <_B p$. For $X = A$, B, or C the set W_X is an open wedge with O as apex, and it the smallest such wedge such that $W_X(X)$ contains $\bar{T}$.

Since the hexagon H intersects at most two sides of any translate of $\bar{T}$, the intersection of H with any translate of $\bar{T}$ is equal to the intersection of H with a suitable translate of W_A, W_B, or W_C. This is immediate for open triangles T and $\bar{T}$. For closed triangles one should consider the closures of the cones. However, for any finite set $\mathcal{S}$ and $X = A$, B, or C the intersection of a translate of the closure of W_X with $\mathcal{S}$ can be obtained as the intersection of $\mathcal{S}$ with another translate of W_X, so we can consider the open wedges in this case too.

We finish the proof of Theorem 2 by separately coloring the subsets of $\mathcal{S}$ belonging to each hexagon in the tiling. Each such subset is colored by the coloring guaranteed by Theorem 3. This results in a coloring required by the dual form of Theorem 2. This finishes the proof of Theorem 2 provided we prove the following Theorem 3.

Theorem 3. *Any finite set $\mathcal{S}$ of points in the plane can be colored with two colors such that any translate of W_A, W_B or W_C which contains at least 8 of the points, contains at least one of each color.*

Proof: Notice first that a small enough perturbation of the points in $\mathcal{S}$ may introduce new intersections with translates of W_A, W_B, and W_C but will not result in the loss of such sets. Therefore we may and will assume without loss of generality that the points in $\mathcal{S}$ are in general position, i.e., no two points in $\mathcal{S}$ determine a line parallel to any one of the sides of the triangle $\bar{T}$. In other words we assume that $<_A$, $<_B$ and $<_C$ are linear orders on $\mathcal{S}$.

For $X = A$, B, or C we call a point $x \in \mathcal{S}$ an *X-boundary point* if $W_X(x) \cap \mathcal{S} = \emptyset$. In other words, x is an X-boundary point if translating W_X such that its apex moves to x, the translate is disjoint from $\mathcal{S}$. See Figure 1 (a). We call a point in $\mathcal{S}$ a *boundary point* of $\mathcal{S}$ if it is an X-boundary point for $X = A$, B, or C. Note that the same point can be a boundary point of more than one type, for instance it can be an A-boundary and a B-boundary point at the same time.

Notice that $<_B$ and $<_C$ give the opposite linear order on A-boundary points. When speaking of A-boundary elements we always consider them with one of these orderings. In particular we call two points of $\mathcal{S}$ *A-neighbors* if both of them are A-boundary points and there is no A-boundary point between them according to the ordering $<_B$ (or equivalently, according to $<_C$). See Figure 1 (b) for a geometric interpretation of this concept. We use B-neighbors and C-neighbors in an analogous meaning. We call two points of $\mathcal{S}$ *neighbors* if they are X-neighbors for some $X = A$, B or C.

Notice that any translate $W_A(p)$ of the wedge W_A that contains any element of $\mathcal{S}$ contains at least one A-boundary point and the A-boundary points in $W_A(x)$ form an interval of the set of all A-boundary points in the ordering $<_B$. Notice also that the smallest elements of $\mathcal{S}$ in the linear orderings $<_B$ and $<_C$ are A-boundary points, these are the two extremal A-boundary points.

Similar observation hold for B- and C-boundary points of $\mathcal{S}$. As a consequence one may notice that the minimal elements in $\mathcal{S}$ according to the orders $<_A$, $<_B$ and $<_C$ are X-boundary elements of at least two different values of X and there is at most a single element of $\mathcal{S}$ that is an X-boundary point for all three possible values of X.

Let $X = A$, B, or C. We call a point $x \in \mathcal{S}$ *X-rich* if x is an X-boundary point, there is a translate $W_X(p)$ of W_X containing x such that $|\mathcal{S} \cap W_X(p)| \geq 8$, and x is the only X-boundary point in $W_X(p)$. We call a point in $\mathcal{S}$ *rich* if it is X-rich for some $X = A$, B, or C.

We call a coloring of $\mathcal{S}$ to blue and red *acceptable* if all rich points are colored blue and between any pair of neighbors at least one is blue.

We say that a point $x \in \mathcal{S}$ is *good* for a particular acceptable coloring of $\mathcal{S}$ if it is red or it is rich or for $X = A$, B or C x has a red X-neighbor y with $y <_X x$. If a point of $\mathcal{S}$ is not good for an acceptable coloring we call it *bad*.

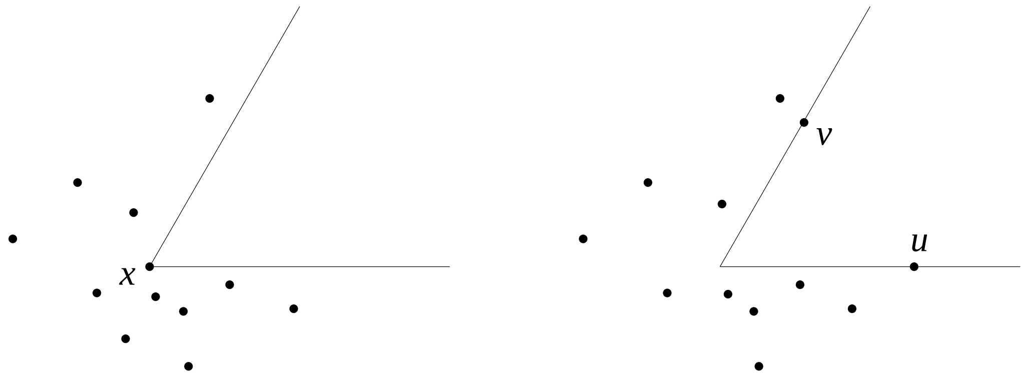

Figure 1: (a) x is an A-boundary point. (b) u, v are A-neighbors.

The following three claims finish the proof of Theorem 3 and therefore also the proof of Theorems 1 and 2. Claims 1 and 2 show that the coloring established in Claim 3 satisfies the requirements of Theorem 3. □

Claim 1. *Consider an acceptable coloring of $\mathcal{S}$. Every wedge of the form $W_X(p)$ that contains at least 8 points of S contains a blue point of $\mathcal{S}$.*

Proof: If $W_X(p)$ contains any points of $\mathcal{S}$ it contains a non-empty interval of X boundary points. If this interval consists of more than a single point, then it contains a pair of X-neighbors and one of these points is blue. If, however, $W_X(p)$ contains a single X-boundary point x but contains at least 8 points of $\mathcal{S}$, then x is rich by definition and therefore it is blue. □

Claim 2. *Consider an acceptable coloring of $\mathcal{S}$. If a wedge of the form $W_X(p)$ contains at least 8 points of $\mathcal{S}$, then it contains at least four bad points or at least one red point.*

Proof: Assume without loss of generality that $X = A$. By moving p we can shrink $W_A(p)$ so we may assume without loss of generality that $S = W_A(p) \cap \mathcal{S}$ has exactly 8 points. If S contains a red point we are done, so assume all points in S are blue. Suppose that $x \in S$ is good. We claim that either x is the first or last A-boundary point in S, or x is the $<_A$-minimal B-boundary point in S, or else x is the $<_A$-minimal C-boundary point in S.

To see this notice first that x, being blue and good, must be a boundary point, moreover, for $Y = A$, B, or C, either x is Y-rich or x has a red Y-neighbor y with $y <_Y x$. The cases of the following case analysis are ilustrated in Figure 2.

Assume first that $Y = A$. If x is not the first or last A-boundary point in S, then any translate of W_A containing x but not containing other A-boundary points must be contained in $W_A(p)$ and therefore x is not A-rich. Both A-neighbors of x are contained in $W_A(p)$, so neither is red.

In the remaining case $Y = B$ or C and we assume by symmetry that that $Y = B$. Now we assume that x is not the $<_A$-minimal B-boundary point in $W_A(p)$. Consider a translate $W_B(q)$ of W_B containing x but no other B-boundary points. It is easy to see that $W_B(q) \cap \mathcal{S}$ is contained in $W_A(p)$ so x cannot be B-rich. If x has a B-neighbor y with $y <_B x$, then another easy argument shows $y \in W_A(p)$ and therefore y must be blue. □

Claim 3. *There is an acceptable coloring of $\mathcal{S}$ for which there are at most 3 bad points.*

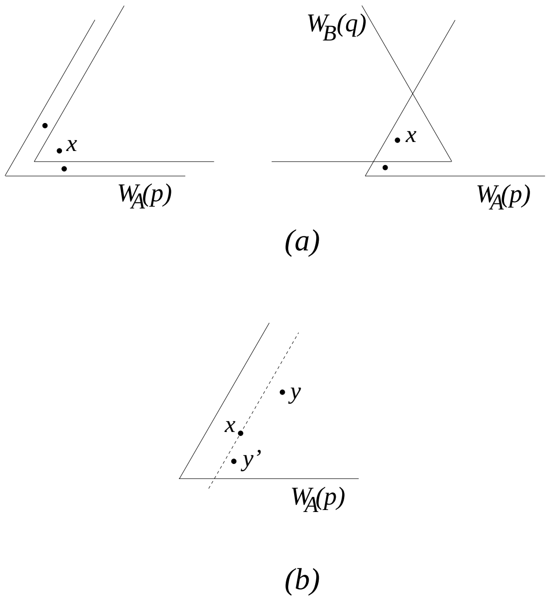

Figure 2: (a) x cannot be rich. (b) If y is a B-neighbor of x with $y <_B x$, then $y \in W_A(p)$.

Proof: We give a construction for such a coloring. Naturally, we start by coloring red all points of $\mathcal{S}$ that are not boundary points and by coloring blue all rich points.

Let z_A be the $<_A$-maximal among the points of $\mathcal{S}$ that are both B- and C-boundary points. The $<_A$-minimal point in $\mathcal{S}$ has this property, therefore z_A exists. Similarly, let z_B be the $<_B$-maximal among the points of $\mathcal{S}$ that are simultaneously A- and C-boundary points and let z_C be the $<_C$-maximal among the points of $\mathcal{S}$ that are simultaneously A- and B-boundary points. Note that these points may coincide. See Figure 3.

We partition $\mathcal{S} \setminus \{x_A, x_B, x_C\}$ into four sets as follows. For $X = A$, B or C we let $\mathcal{S}_X = \{x \in \mathcal{S} \mid x <_X z_X\}$. We let $\mathcal{S}_0 = (\mathcal{S} \setminus \{z_A, z_B, z_C\}) \setminus (\mathcal{S}_A \cup \mathcal{S}_B \cup \mathcal{S}_C) = \{x \in \mathcal{S} \mid z_A <_A x, z_B <_B x, z_C <_C x\}$. It is not hard to verify that this is indeed a partition. Note that each of these sets may be empty.

We color z_A, z_B, and z_C blue, then we color the boundary points of the four parts separately. For $X = A$, B, or C and $Y = A$, B, C, or 0 we take the X-boundary points in $\mathcal{S}_Y$ and consider them in increasing order according to $<_X$. If we get to a point that is not colored we color it red and we color every neighbor of it blue. This goes for every Z-neighbor for arbitrary Z, for example a point in $\mathcal{S}_A$ can have two B-neighbors and two C-neighbors. These neighbors may have already been colored blue (because they are rich, because they are one of z_A, z_B or z_C, or because of an earlier red neighbor) but they are not colored red as any neighbor of any red point is immediately colored blue. This same observation proves that the resulting coloring is acceptable.

Notice that the coloring processes for the four parts $\mathcal{S}_A$, $\mathcal{S}_B$, $\mathcal{S}_C$, and $\mathcal{S}_0$ are independent, no neighbor of a point in one part lies in another. The three coloring procedures for $\mathcal{S}_0$ are also independent, because boundary points here have only one type. In $\mathcal{S}_A$ there are no A-boundary points, but there may be several points that are B- and C-boundary points simultaneously. Therefore the actual coloring of $\mathcal{S}_A$ may depend on which process reaches these common points first. Synchronize the two coloring processes on $\mathcal{S}_A$ so that they reach the common points at the same time. This is possible, as their order is the same in the two ordering $<_B$ and $<_C$. We assume similar synchronizations of the coloring processes of $\mathcal{S}_B$ and $\mathcal{S}_C$.

We finish the proof by observing that with the possible exception of z_A, z_B, and z_C every point of $\mathcal{S}$ is good for the coloring obtained. To see this notice that we only colored a point x other than z_A, z_B, or z_C blue if it was rich or if one of its neighbors y were colored red. But the order in which these points were considered ensures that if x and y are X-neighbors then $y <_X x$ as needed. □

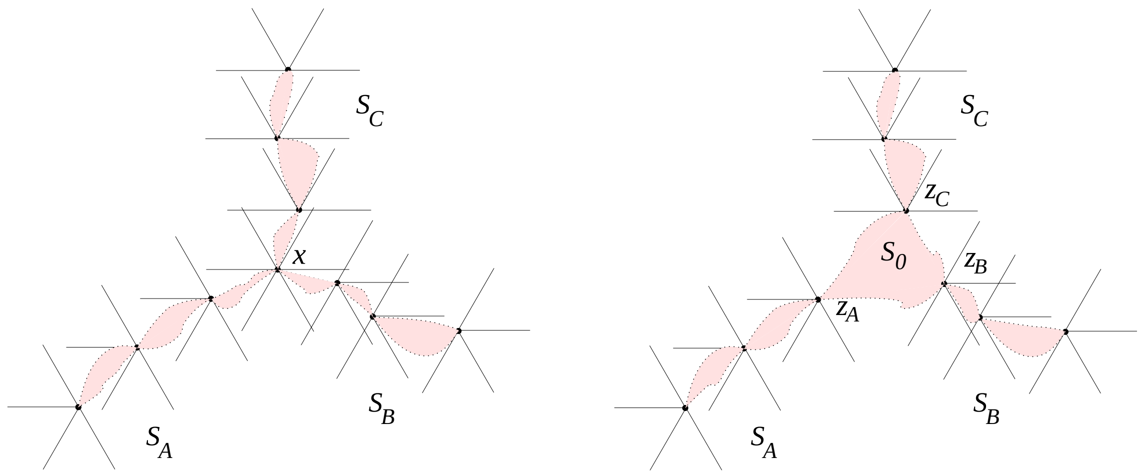

Figure 3: The boundary vertices and the partition of $\mathcal{S}$. On the left we have $x = z_A = z_B = z_C$.

Remarks. The value 43 in Theorem 2 is probably far from being optimal. We cannot even rule out the possibility that the result holds with 3 instead of 43. An obvious route for improvement is the strengthening of Claim 3 such that it allows for a single bad point only. This would improve the constant 8 in Theorem 3 to 6 and the constant 43 in Theorem 2 to 31. Note however that some sets $\mathcal{S}$ do not allow for an acceptable coloring without any bad vertices.

By hardly any modification of our argument one can prove the following generalization of Theorem 3.

Theorem 3'. *For any positive integer k any finite set $\mathcal{S}$ in the plane can be colored red and blue such that any translate of W_A, W_B or W_C which contains at least $5k+3$ of the points contains a blue point and at least k red points.*

Using Theorem 3', one can re-color the red points recursively to obtain the following generalization of Theorems 1 and 2 for more than two colors.

Theorem 4. *For every $m > 0$ the following two statements hold. Every locally finite collection $\mathcal{C}$ of translates of the same triangle T can be partitioned into m parts such that for every point that is covered by at least $\frac{21}{2}5^m - \frac{19}{2}$ of the triangles in $\mathcal{C}$ is covered by a triangle in each of the parts. Every $\frac{21}{2}5^m - \frac{19}{2}$-fold covering of the plane by translates of the same open triangle can be partitioned into m coverings.*

We belive that the exponential bound in Theorem 4 is far from being optimal. It is possible that the statement is true with a *linear* bound in m.

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