Note on an Art Gallery Problem

György Csizmadia and Géza Tóth
Courant Institute, New York University

Abstract

It is proved that for $n > 3$, $\lceil \frac{2}{5}(n-3) \rceil$ guards are enough to monitor any simply connected art gallery room of n sides if they are stationed at fixed points and their range of vision is $180°$. Furthermore, the position of the guards can be determined by an $O(n)$-time algorithm.

1 Introduction

A typical "art gallery theorem" provides combinatorial bounds on the number of guards needed to visually cover a polygonal region $\boldsymbol{P}$ (the art gallery) defined by n vertices.

J. Urrutia posed the following question: What is the minimum number $f(n)$ of guards needed to monitor any simply connected art gallery of n sides if the guards are to be stationed at fixed points and their range of vision is $180°$?

If the range of vision of the guards is $360°$ then exactly $\lfloor n/3 \rfloor$ of them are needed to monitor any simply connected art gallery room of n sides [5, 9]. Therefore, $f(n) \geq \lfloor n/3 \rfloor$ and it is conjectured [13] that $f(n) = \lfloor n/3 \rfloor$.

In 1992, H. Bunting, D. Larman, and the present authors showed that $f(n) \leq \lfloor \frac{4}{9}(n+\frac{1}{4}) \rfloor$. In this note we prove that $f(n) \leq \lceil \frac{2}{5}(n-3) \rceil$ for $n > 3$.

2 The Main Theorem

Definition. For any polygon $\boldsymbol{P}$, let $s(\boldsymbol{P})$ denote the number of sides of $\boldsymbol{P}$.

$\boldsymbol{P}$ is said to be *reducible* if there exist two numbers, n and m, and two polygons $\boldsymbol{P'}$ and $\boldsymbol{Q}$ such that

- $\boldsymbol{P} = \boldsymbol{P'} \cup \boldsymbol{Q}$,
- $\boldsymbol{P'}$ is simply connected,
- $s(\boldsymbol{P'}) = s(\boldsymbol{P}) - n$,
- $\boldsymbol{Q}$ is visible by m guards, whose range of vision is $180°$,
- $\frac{n}{m} \leq \frac{2}{5}$.

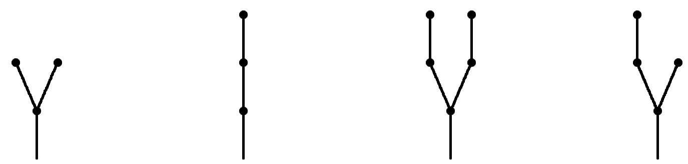
Theorem. $\lceil \frac{2}{5}(n-3) \rceil$ *guards whose range of vision is* 180° *are sufficient to monitor any simply connected art gallery with n sides* ($n > 3$)*.*

Proof of the Theorem. In the sequel we assume that the range of vision of the guards is 180°. We use three lemmas whose proof is postponed to Section 3.

Lemma 1. *Every octagon can be monitored by two guards.*

Let $\boldsymbol{P}$ (a polygon of n sides) denote the art gallery. We prove the Theorem by induction on n. It is easy to see that every pentagon is visible by one guard, so by Lemma 1 it is enough to prove that $\boldsymbol{P}$ is reducible.

Let us triangulate $\boldsymbol{P}$. Denote by $G_{\boldsymbol{P}}$ the dual graph of the triangulation, that is, a graph whose nodes are the triangles of this triangulation, and two nodes are connected by an edge whenever the corresponding triangles share a side. $G_{\boldsymbol{P}}$ is a tree since $\boldsymbol{P}$ is simply connected.



E_1 E_2 E_3 E_4

Fig. 1.

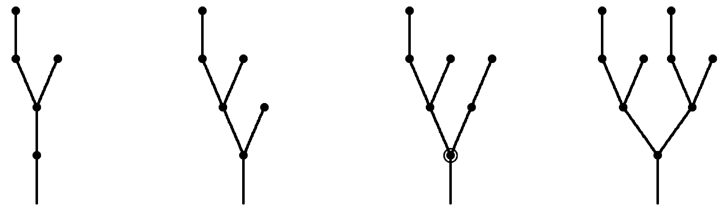
Suppose first that $G_{\boldsymbol{P}}$ has an ending isomorphic to E_1 or E_2 (Fig. 1.). Both configurations correspond to a pentagon in $\boldsymbol{P}$. These pentagons join to the rest of $\boldsymbol{P}$ by an edge of a triangle, so we can cut it off by this edge. This way we get a $\boldsymbol{P'}$ polygon of $n-3$ edges and since we used only one guard so far, $\boldsymbol{P}$ is reducible.

If $G_{\boldsymbol{P}}$ has an ending isomorphic to E_3, then it represents a heptagon which can obviously be monitored by two guards (one for the left three nodes and one for the right two). Thus again $\boldsymbol{P}$ is reducible.

So we can suppose that all the endings of $G_{\boldsymbol{P}}$ are isomorphic to E_4. Let us take a look at the last four nodes of a longest path of the tree. By the above argument there are four possibilities as shown in Fig. 2.

Lemma 2. *If* $G_{\boldsymbol{P}}$ *has an ending isomorphic to* H_2*, then* $\boldsymbol{P}$ *is reducible.*

Lemma 3. *Suppose* $G_{\boldsymbol{P}}$ *has an ending isomorphic to* H_1 *(see Fig. 2.). Then there is a triangle* ABC *(corresponding to the marked node), so that it joins to the rest of the polygon with its side* AB*, there is a hexagon joining to its side* AC *and a quadrilateral joining to its side* BC*. If* A *and* B *are not both concave vertices of the hexagon and the quadrilateral resp., then* $\boldsymbol{P}$ *is reducible (Fig. 2.; see also Fig. 11a.).*



H_1 H_2

Case (a) *Case (b)* *Case (c)* *Case (d)*

Fig. 2.

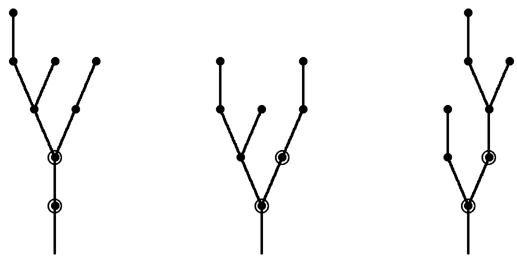
In Case (a) the ending can be seen by 2 guards (since any 3 neighboring triangles can be monitored by 1 guard), so $\boldsymbol{P}$ is reducible.

In Case (b) the ending corresponds to an octagon, so by Lemma 1 we can cut it off and proceed by induction.

In Case (d), by Lemma 2, $\boldsymbol{P}$ is reducible.

Case (c) satisfies the assumption of Lemma 3. Let us use the notation of Lemma 3. Let D be the third vertex of the triangle joined to the side AB of the triangle ABC. If $\boldsymbol{P}$ is not reducible then by Lemma 3 the quadrilateral $ADBC$ is convex. In the present triangulation $ADBC$ is divided by its diagonal AB. But since $ADBC$ is convex, we can modify the triangulation by changing the diagonal AB to CD (see Fig. 11a.). We distinguish five subcases by looking at the node representing the triangle ABC (see Fig. 3-7). For each tree, the union of the triangles represented by the two marked vertices is the convex quadrilateral $ADBC$.

Fig. 3.
Case c.1.

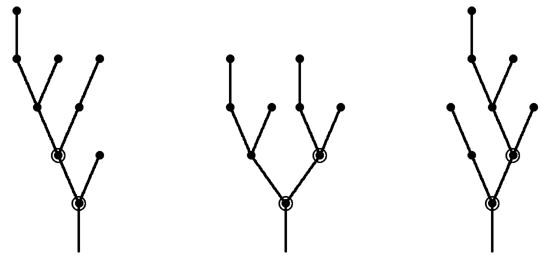


F_1 F_1' F_1''

• Case c.1. $G_{\boldsymbol{P}}$ has an ending isomorphic to F_1. After retriangulation the ending will be isomorphic to F_1' or F_1''.

The right branch of F_1' (3 nodes) represents a pentagon, so it can be cut off using one guard. The right branch of F_1'' (5 nodes) represents a heptagon, so it can be cut off using two guards. Therefore, in both cases $\boldsymbol{P}$ is reducible.

Fig. 4.
Case c.2.

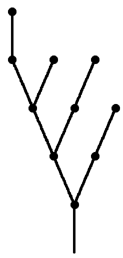


F_2 F_2' F_2''

• Case c.2. $G_{\boldsymbol{P}}$ has an ending isomorphic to F_2. After retriangulation the ending will be isomorphic to F_2' or F_2''.

F_2' corresponds to Case (d). The right branch of F_2'' (6 nodes) represents an octagon, so by Lemma 1 it can be cut off using two guards. Therefore, in both cases, $\boldsymbol{P}$ is reducible.

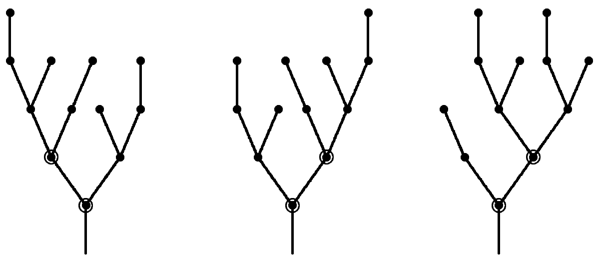
Fig. 5.
Case c.3.



F_3

• Case c.3. $G_{\boldsymbol{P}}$ has an ending isomorphic to F_3. Then the lowest branch of three nodes of F_3 are visible by one guard, the middle three by another one and the top four by two more guards. So we can cut off the ten nodes of F_3 using four guards, therefore $\boldsymbol{P}$ is reducible.

Fig. 6.
Case c.4.



F_4 F_4' F_4''

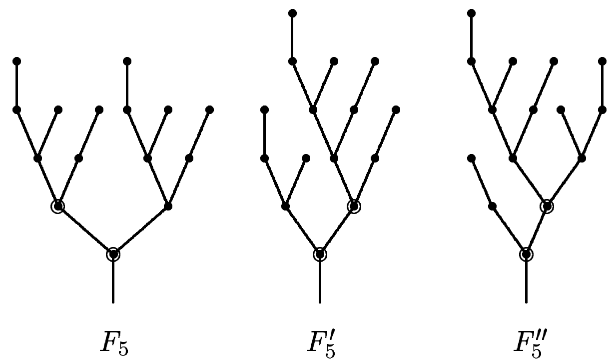
• Case c.4. $G_{\boldsymbol{P}}$ has an ending isomorphic to F_4. After retriangulation the ending will be isomorphic to F_4' or F_4''.

Suppose we got F_4'. Then in F_4 the two marked nodes represent a convex quadrilateral $ABCD$ joining to the rest of $\boldsymbol{P}$ by its AD side. There are two hexagons joining to the

AC and BD sides and a quadrilateral joining to the BC side. Applying Lemma 3 for the triangles ABC and BCD, we get that if $\boldsymbol{P}$ is not reducible, the quadrilateral has a concave vertex at both B and C. It is a contradiction, so $\boldsymbol{P}$ is reducible.

The right branch of F_4'' corresponds to Case (d), so in both cases, $\boldsymbol{P}$ is reducible.

Fig. 7.
Case c.5.



• Case c.5. $G_{\boldsymbol{P}}$ has an ending isomorphic to F_5. After retriangulation the ending will be isomorphic to F_5' or F_5''. The right branch of F_5' is isomorphic to F_3, the right branch of F_5'' is isomorphic to F_4, so in both cases it is already shown that $\boldsymbol{P}$ is reducible.

We proved that $\boldsymbol{P}$ is always reducible, which completes the proof of the theorem. □

Based on [4], our proof can easily be turned into an $O(n)$-time algorithm for determining the positions of the guards.

3 Proof of the Lemmas

Definition. Let $ABCDEF$ be a hexagonal part of $\boldsymbol{P}$, joined to the rest of $\boldsymbol{P}$ by its side AB. It is called a *NR-ending* (non-reducible ending) of $\boldsymbol{P}$ if there does not exist any triangle ABX in $ABCDEF$ such that $ABCDEF - ABX$ is visible by one guard.

Let $ABCDEF$ be a NR-ending. It is easy to see that one of A and E (and one of B and D) have to be a convex vertex, the other one is a concave vertex, so considering only A there are two possibilities, A can be a convex or concave vertex.

Proof of Lemma 1. Let $\boldsymbol{P}$ denote the octagon. Clearly if $G_{\boldsymbol{P}}$, the dual graph of $\boldsymbol{P}$, has either E_1 or E_2 (Fig. 1.) as an ending, then we are done. Otherwise $G_{\boldsymbol{P}}$ should have the unique form as shown in Fig. 8a.

Definition. A *U-ending* of a polygon is a configuration of four triangles connected to each other as in Fig. 8b., and joined to the rest of the polygon by its BC side.

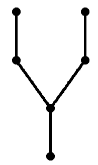


Fig. 8a.

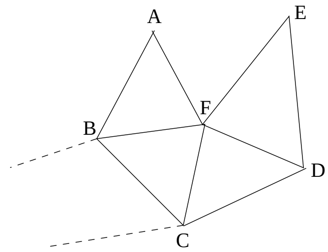


Fig. 8b. A U-ending

It is easy to see the following:

Claim. *A U-ending is not a NR-ending.*

Corollary. *If an n-gon* $\boldsymbol{P}$ *has a U-ending, then* $\boldsymbol{P}$ *is reducible.*

By the corollary we should consider only the case when $\boldsymbol{P}$ does not have a U-ending. Then its triangulation should be like the one in Fig. 9a.

If the quadrilateral $BHEC$ is convex, then we can modify the triangulation by changing the diagonal CH to BE and get a U-ending ($BAHGFE$). So we can suppose that $BHEC$ is concave. If the concave vertex is H, then a guard at H can monitor 3 triangles, so $\boldsymbol{P}$ is reducible. So BCE and similarly CEF should be concave angles.

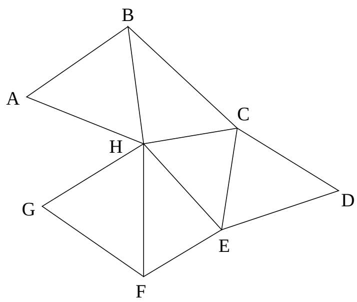


Fig. 9a.

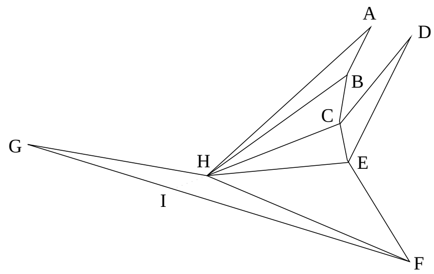


Fig. 9b.

It is easy to see that quadrilaterals $AHCB$ and $GFEH$ should be concave too, by similar reasons. It is impossible that both quadrilaterals have a concave vertex at H, so we can suppose without loss of generality that ABC is concave. Now if GFE is concave too, then a guard at H can monitor all the heptagon $ABCEFGH$. So GHE should be concave, therefore the octagon should look like the one in Fig. 9b.

Now the line HC crosses the GF segment at I. So a guard at H can monitor the (degenerate) hexagon $GICBAH$ and the rest of the octagon is a pentagon, which also can be seen by one guard. □

Proof of Lemma 2. H_2 corresponds to a triangle OXY which joins to the rest of the polygon with the XY side and has a hexagon on both of its OX and OY sides. We may suppose that both hexagons are NR-endings.

Fig. 10a. **Fig. 10b.**

If $\angle AOB \geq 180°$, a guard at O can monitor both triangles AOA' and $BB'O$, and another one can monitor the rest of one of the NR-endings (Fig. 10a.). So we could remove 5 triangles using 2 guards.

If $\angle AOB < 180°$, then either the segment AB or the intersection of the lines of $A'A$ and $B'B$ is in the polygon and we can place a guard either to a suitable point of AB or to the intersection point of $A'A$ and $B'B$ to monitor the triangles AOA' and $BB'O$ (Fig. 10b.). Then we proceed as above. □

Proof of Lemma 3. We will prove that if $\boldsymbol{P}$ is not reducible, then A and B are concave vertices of the hexagon and the quadrilateral, respectively (see Fig. 11a.).

Assume without loss of generality that the line AB is horizontal and C lies above it.

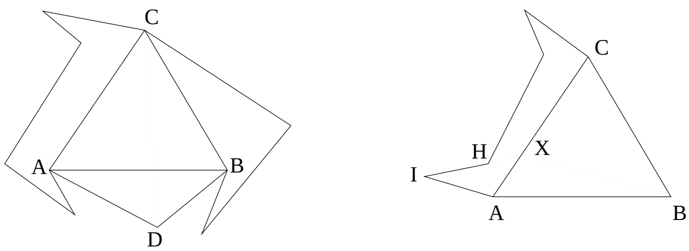


Fig. 11a. **Fig. 11b.**

Suppose that $\boldsymbol{P}$ is not reducible. Then the hexagon, joining to the side AC, is not reducible, therefore it is a NR-ending. If the hexagon has a convex angle at A, then denote by X the intersection of the segment AC and the line IH (see Fig. 11b.). Now if we make a cut along the segments BX and HX, then we cut off an octagon which can be seen by 2 guards, so $\boldsymbol{P}$ is reducible.

So the hexagon has a concave angle at A.

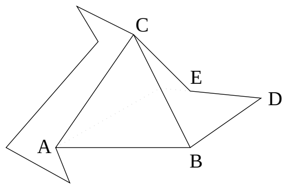


Fig. 12a.

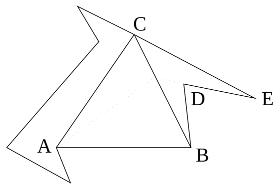


Fig. 12b.

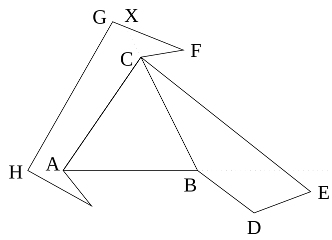


Fig. 12c.

Now look at the quadrilateral. If it has a concave angle at D or E, then we can cut off an octagon and go on by induction as shown on Fig. 12a. and 12b.

If it has a concave angle at C, then we can proceed as in the proof of Lemma 2.

So we can suppose that the quadrilateral $BDEC$ is convex. Now there are three subcases.

a) Either D or E can be seen from A. In this case we are done by induction (we can cut off an octagon or a heptagon by AD or AE resp.).

b) D is below the AB line (and E is either above or below).

- If the hexagon has a concave angle at C, then the BC line intersects the FG (or GH) segment at X (see Fig. 12c.). Now a guard at C can monitor both the quadrilateral $BDEC$ and the triangle CFX (or quadrilateral $CFGX$). So we can make a cut along BX and go on by induction.
- If the hexagon has a convex angle at C, then we have three subcases:

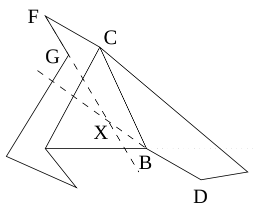


Fig. 13a.

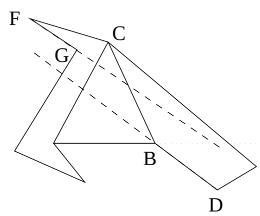


Fig. 13b.

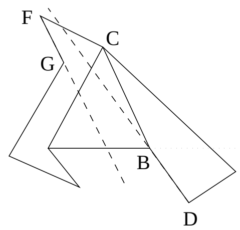


Fig. 13c.

If the line FG intersects the line BD inside $\boldsymbol{P}$ at X (see Fig. 13a.), then we can cut along GXB.

If FG is above BD then we cut along DG (Fig. 13b.).

If FG is below BD then we cut along BF (Fig. 13c.). In all three cases we cut off a pentagon which can be seen by one guard, so we can go on by induction.

c) Both D and E are above the line AB.

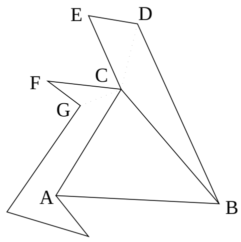


Fig. 14.

In this case D and E should be above the line AC. Thus the triangles FGC and CDE can be monitored by one guard, and the rest of the hexagon can be monitored by another one, so we can cut off 5 triangles by using only 2 guards (Fig. 14.). □

4 Remarks

Let $f_\alpha(n)$ denote the minimum number of guards needed to monitor a simply connected art gallery $\boldsymbol{P}$ of n sides if their range of vision is restricted to α degrees.

By the above results $f_\alpha(n) \leq 2\lceil\frac{2}{5}(n-3)\rceil$ for $\alpha \geq 90°$ and $f_\alpha(n) \leq 3\lceil\frac{2}{5}(n-3)\rceil$ for $\alpha \geq 60°$.

For $\alpha \geq 60°$ we can give a better bound. Let us triangulate $\boldsymbol{P}$. Then we get $n-2$ triangles, and each of them has an angle less than or equal to α. A guard placed at the corresponding vertex of each triangle can monitor the whole triangle, therefore $n-2$ guards are enough, so $f_\alpha(n) \leq n-2$.

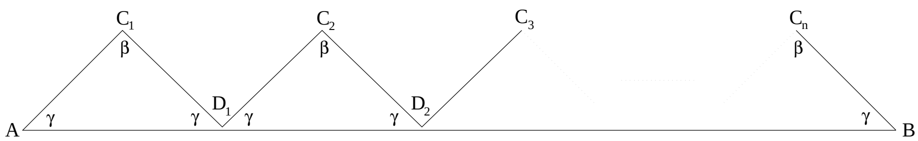


Fig. 15.

The construction in Fig. 15 gives a lower bound for all $\alpha < 180°$. It shows that whenever $\alpha < \beta < 180$, $f_\alpha(n) \geq \frac{n-1}{2}$. Observe that if the D_i's are close enough to the line AB, then a guard with a range of vision less than β can monitor only one of the C_i's.

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