

# New bounds on crossing numbers

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## Abstract

The *crossing number*,  $\text{cr}(G)$ , of a graph  $G$  is the least number of crossing points in any drawing of  $G$  in the plane. Denote by  $\kappa(n, e)$  the minimum of  $\text{cr}(G)$  taken over all graphs with  $n$  vertices and at least  $e$  edges. We prove a conjecture of P. Erdős and R. Guy by showing that  $\kappa(n, e)n^2/e^3$  tends to a positive constant as  $n \rightarrow \infty$  and  $n \ll e \ll n^2$ . Similar results hold for graph drawings on any other surface of fixed genus.

We prove better bounds for graphs satisfying some monotone properties. In particular, we show that if  $G$  is a graph with  $n$  vertices and  $e \geq 4n$  edges, which does not contain a cycle of length *four* (resp. *six*), then its crossing number is at least  $ce^4/n^3$  (resp.  $ce^5/n^4$ ), where  $c > 0$  is a suitable constant. These results cannot be improved, apart from the value of the constant. This settles a question of M. Simonovits.

## 1 Introduction

Let  $G$  be a simple undirected graph with  $n(G)$  nodes (vertices) and  $e(G)$  edges. A *drawing* of  $G$  in the *plane* is a mapping  $f$  that assigns to each vertex of  $G$  a distinct point in the plane and to each edge  $uv$  a continuous arc connecting  $f(u)$  and  $f(v)$ , not passing through the image of any other vertex. For simplicity, the arc assigned to  $uv$  is also called an *edge*, and if this leads to no confusion, it is also denoted by  $uv$ . We assume that no three edges have an interior point in common. The *crossing number*,  $\text{cr}(G)$ , of  $G$  is the minimum number of crossing points in any drawing of  $G$ .

The determination of  $\text{cr}(G)$  is an NP-complete problem [GJ83]. It was discovered by Leighton [L84] that the crossing number can be used to estimate the chip area required for the VLSI circuit

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layout of a graph. He proved the following general lower bound for  $\text{cr}(G)$ , which was discovered independently by Ajtai, Chvátal, Newborn, and Szemerédi. The best known constant,  $1/33.75$ , in the theorem is due to Pach and Tóth.

**Theorem A.** [ACNS82], [L84], [PT97] *Let  $G$  be a graph with  $n(G) = n$  nodes and  $e(G) = e$  edges,  $e \geq 7.5n$ . Then we have*

$$\text{cr}(G) \geq \frac{1}{33.75} \frac{e^3}{n^2}.$$

Theorem A can be used to deduce the best known upper bounds for the number of unit distances determined by  $n$  points in the plane [S98], for the number of different ways how a line can split a set of  $n$  points into two equal parts [D98], and it has some other interesting corollaries [PS98].

It is easy to see that the bound in Theorem A is tight, apart from the value of the constant. However, as it was suggested by Miklós Simonovits [S97], it may be possible to strengthen the theorem for some special classes of graphs, e.g., for graphs not containing some fixed, so-called *forbidden* subgraph. In Sections 2 and 3 of the present paper we verify this conjecture.

A graph property  $\mathcal{P}$  is said to be *monotone* if

- whenever a graph  $G$  satisfies  $\mathcal{P}$ , then every subgraph of  $G$  also satisfies  $\mathcal{P}$ ;
- whenever  $G_1$  and  $G_2$  satisfy  $\mathcal{P}$ , then their disjoint union also satisfies  $\mathcal{P}$ .

For any monotone property  $\mathcal{P}$ , let  $\text{ex}(n, \mathcal{P})$  denote the maximum number of edges that a graph of  $n$  vertices can have if it satisfies  $\mathcal{P}$ . In the special case when  $\mathcal{P}$  is the property that the graph does not contain a subgraph isomorphic to a fixed forbidden subgraph  $H$ , we write  $\text{ex}(n, H)$  for  $\text{ex}(n, \mathcal{P})$ .

**Theorem 1.** *Let  $\mathcal{P}$  be a monotone graph property with  $\text{ex}(n, \mathcal{P}) = O(n^{1+\alpha})$  for some  $\alpha > 0$ .*

*Then there exist two constants  $c, c' > 0$  such that the crossing number of any graph  $G$  with property  $\mathcal{P}$ , which has  $n$  vertices and  $e \geq cn \log^2 n$  edges, satisfies*

$$\text{cr}(G) \geq c' \frac{e^{2+1/\alpha}}{n^{1+1/\alpha}}.$$

*If  $\text{ex}(n, \mathcal{P}) = \Theta(n^{1+\alpha})$ , then this bound is asymptotically tight, up to a constant factor.*

In some interesting special cases when we know the precise order of magnitude of the function  $\text{ex}(n, \mathcal{P})$ , we obtain some slightly stronger results. The *girth* of a graph is the length of its shortest cycle.

**Theorem 2.** *Let  $G$  be a graph with  $n$  vertices and  $e \geq 4n$  edges, whose girth is larger than  $2r$ , for some  $r > 0$  integer. Then the crossing number of  $G$  satisfies*

$$\text{cr}(G) \geq c_r \frac{e^{r+2}}{n^{r+1}},$$

where  $c_r > 0$  is a suitable constant. For  $r = 2, 3$ , and  $5$ , these bounds are asymptotically tight, up to a constant factor.

What happens if the girth of  $G$  is larger than  $2r + 1$ ? Since one can destroy every odd cycle of a graph by deleting at most half of its edges, even in this case we cannot expect an asymptotically better lower bound for the crossing number of  $G$  than the bound given in Theorem 2.

**Theorem 3.** *Let  $G$  be a graph with  $n$  vertices and  $e \geq 4n$  edges, which does not contain a complete bipartite subgraph  $K_{r,s}$  with  $r$  and  $s$  vertices in its classes,  $s \geq r$ .*

*Then the crossing number of  $G$  satisfies*

$$\text{cr}(G) \geq c_{r,s} \frac{e^{3+1/(r-1)}}{n^{2+1/(r-1)}},$$

where  $c_{r,s} > 0$  is a suitable constant. These bounds are tight up to a constant factor if  $r = 2, 3$ , or if  $r$  is arbitrary and  $s > (r - 1)!$ .

The *bisection width*,  $b(G)$ , of a graph  $G$  is defined as the minimum number of edges whose removal splits the graph into two roughly equal subgraphs. More precisely,  $b(G)$  is the minimum number of edges running between  $V_1$  and  $V_2$ , over all partitions of the vertex set of  $G$  into two parts  $V_1 \cup V_2$  such that  $|V_1|, |V_2| \geq n(G)/3$ .

Leighton [L83] observed that there is an intimate relationship between the bisection width and the crossing number of a graph, which is based on the Lipton–Tarjan separator theorem for planar graphs [LT79]. The proofs of Theorems 1–3 are based on repeated application of the following version of this relationship.

**Theorem B.** [PSS96] *Let  $G$  be a graph of  $n$  vertices, whose degrees are  $d_1, d_2, \dots, d_n$ . Then*

$$b(G) \leq 10\sqrt{\text{cr}(G)} + 2\sqrt{\sum_{i=1}^n d_i^2}.$$

Let  $\kappa(n, e)$  denote the minimum crossing number of a graph  $G$  with  $n$  vertices and at least  $e$  edges. That is,

$$\kappa(n, e) = \min_{\substack{n(G) = n \\ e(G) \geq e}} \text{cr}(G).$$

It follows from Theorem A that, for  $e \geq 4n$ ,  $\kappa(n, e)n^2/e^3$  is bounded from below and from above by two positive constants. Paul Erdős and Richard K. Guy [EG73] conjectured that if  $e \gg n$  then  $\lim \kappa(n, e)n^2/e^3$  exists. (We use the notation  $f(n) \gg g(n)$  to express that  $\lim_{n \rightarrow \infty} f(n)/g(n) = \infty$ .) In Section 4, we settle this problem.

**Theorem 4.** *If  $n \ll e \ll n^2$ , then*

$$\lim_{n \rightarrow \infty} \kappa(n, e) \frac{n^2}{e^3} = C > 0$$

*exists.*

We call the constant  $C > 0$  in Theorem 4 the *midrange crossing constant*. It is necessary to limit the range of  $e$  from below and from above. (See the Remark at the end of Section 4.)

All of the above problems can be reformulated for graph drawings on other surfaces. Let  $S_g$  denote a torus with  $g$  holes, i.e., a compact oriented surface of *genus*  $g$  with no boundary. Define  $\text{cr}_g(G)$ , the crossing number of  $G$  on  $S_g$ , as the minimum number of crossing points in any drawing of  $G$  on  $S_g$ . Let

$$\kappa_g(n, e) = \min_{\substack{n(G) = n \\ e(G) \geq e}} \text{cr}_g(G).$$

With this notation,  $\text{cr}_0(G)$  is the planar crossing number and  $\kappa_0(n, e) = \kappa(n, e)$ .

In Section 5, we prove that there is a midrange crossing constant for graph drawings on any surface  $S_g$  of fixed genus  $g \geq 0$ .

**Theorem 5.** *For every  $g \geq 0$ , if  $n \ll e \ll n^2$  then the limit*

$$\lim_{n \rightarrow \infty} \kappa_g(n, e) \frac{n^2}{e^3}$$

*exists and is equal to the constant  $C > 0$  in Theorem 4.*

To prove this result, we have to generalize Theorem B.

**Theorem 6.** *Let  $G$  be a graph with  $n$  vertices, whose degrees are  $d_1, d_2, \dots, d_n$ . Then*

$$b(G) \leq 300(1 + g^{3/4}) \sqrt{\text{cr}_g(G) + \sum_{i=1}^n d_i^2}.$$

## 2 Crossing numbers and monotone properties

### – Proof of Theorem 1

Let  $\mathcal{P}$  be a monotone graph property with  $\text{ex}(n, \mathcal{P}) \leq An^{1+\alpha}$ , for some  $A, \alpha > 0$ . Let  $G$  be a graph with vertex set  $V(G)$  and edge set  $E(G)$ , where  $|V(G)| = n(G) = n$  and  $|E(G)| = e(G) = e$ . Suppose that  $G$  satisfies property  $\mathcal{P}$  and  $e \geq cn \log^2 n$ . To prove Theorem 1, we assume that

$$\text{cr}(G) < c' \frac{e^{2+1/\alpha}}{n^{1+1/\alpha}},$$

and, if  $c$  and  $c'$  are suitable constant, we will obtain a contradiction.

We break  $G$  into smaller components, according to the following procedure.

#### DECOMPOSITION ALGORITHM

**STEP 0.** Let  $G^0 = G$ ,  $G_1^0 = G$ ,  $M_0 = 1$ ,  $m_0 = 1$ .

Suppose that we have already executed **STEP**  $i$ , and that the resulting graph,  $G^i$ , consists of  $M_i$  components,  $G_1^i, G_2^i, \dots, G_{M_i}^i$ , each of at most  $(2/3)^i n$  vertices. Assume, without loss of generality, that the first  $m_i$  components of  $G^i$  have at least  $(2/3)^{i+1} n$  vertices and the remaining  $M_i - m_i$  have fewer. Then

$$(2/3)^{i+1} n(G) \leq n(G_j^i) \leq (2/3)^i n(G) \quad (j = 1, 2, \dots, m_i).$$

Thus, we have that  $m_i \leq (3/2)^{i+1}$ .

**STEP**  $i + 1$ . **If**

$$(2/3)^i < \frac{1}{(2A)^{1/\alpha}} \cdot \frac{e^{1/\alpha}}{n^{1+1/\alpha}}, \quad (1)$$

**then STOP.** (1) is called the *stopping rule*.

**Else**, for  $j = 1, 2, \dots, m_i$ , delete  $b(G_j^i)$  edges from  $G_j^i$  such that  $G_j^i$  falls into two components, each of at most  $(2/3)n(G_j^i)$  vertices. Let  $G^{i+1}$  denote the resulting graph on the original set of  $n$  vertices. Clearly, each component of  $G^{i+1}$  has at most  $(2/3)^{i+1} n$  vertices.

Suppose that the **DECOMPOSITION ALGORITHM** terminates in **STEP**  $k + 1$ . If  $k > 0$ , then

$$(2/3)^k < \frac{1}{(2A)^{1/\alpha}} \cdot \frac{e^{1/\alpha}}{n^{1+1/\alpha}} \leq (2/3)^{k-1}.$$

First, we give an upper bound on the total number of edges deleted from  $G$ .

Using that, for any non-negative reals  $a_1, a_2, \dots, a_m$ ,

$$\sum_{j=1}^m \sqrt{a_j} \leq \sqrt{m \sum_{j=1}^m a_j}, \quad (2)$$

we obtain that, for any  $0 \leq i < k$ ,

$$\sum_{j=1}^{m_i} \sqrt{\text{cr}(G_j^i)} \leq \sqrt{m_i \sum_{j=1}^{m_i} \text{cr}(G_j^i)} \leq \sqrt{(3/2)^{i+1}} \sqrt{\text{cr}(G)} < \sqrt{(3/2)^{i+1}} \sqrt{\frac{c'e^{2+1/\alpha}}{n^{1+1/\alpha}}}.$$

Denoting by  $d(v, G_j^i)$  the degree of vertex  $v$  in  $G_j^i$ , we have

$$\sum_{j=1}^{m_i} \sqrt{\sum_{v \in V(G_j^i)} d^2(v, G_j^i)} \leq \sqrt{(3/2)^{i+1}} \sqrt{\sum_{v \in V(G^i)} d^2(v, G^i)}$$

$$\leq \sqrt{(3/2)^{i+1}} \sqrt{\max_{v \in V(G^i)} d(v, G^i) \sum_{v \in V(G^i)} d(v, G^i)} \leq \sqrt{(3/2)^{i+1}} \sqrt{(2/3)^i n(2e)} = \sqrt{3en}.$$

In view of Theorem B in the Introduction, the total number of edges deleted during the procedure is

$$\begin{aligned} \sum_{i=0}^{k-1} \sum_{j=1}^{m_i} b(G_j^i) &\leq 10 \sum_{i=0}^{k-1} \sum_{j=1}^{m_i} \sqrt{\text{cr}(G_j^i)} + 2 \sum_{i=0}^{k-1} \sum_{j=1}^{m_i} \sqrt{\sum_{v \in V(G_j^i)} d^2(v, G_j^i)} \\ &< 10\sqrt{c'} \sqrt{\frac{e^{2+1/\alpha}}{n^{1+1/\alpha}}} \sum_{i=0}^{k-1} \sqrt{(3/2)^i} + 2k\sqrt{3en} \leq 250\sqrt{c'} \sqrt{\frac{e^{2+1/\alpha}}{n^{1+1/\alpha}}} \sqrt{(2A)^{1/\alpha} \frac{n^{1+1/\alpha}}{e^{1/\alpha}}} + 2k\sqrt{3en} \leq \frac{e}{2}, \end{aligned}$$

provided that  $c'$  is sufficiently small and  $c$  is sufficiently large.

Therefore, the number of edges of the graph  $G^k$  obtained in the final STEP of the algorithm satisfies

$$e(G^k) \geq \frac{e}{2}.$$

(Note that this inequality trivially holds if the algorithm terminates in the very first STEP, i.e., when  $k = 0$ .)

Next we give a lower bound on  $e(G^k)$ . The number of vertices of each connected component of  $G^k$  satisfies

$$n(G_j^k) \leq (2/3)^k n < \frac{1}{(2A)^{1/\alpha}} \cdot \frac{e^{1/\alpha}}{n^{1+1/\alpha}} n = \left( \frac{e}{2An} \right)^{1/\alpha} \quad (j = 1, 2, \dots, M_k).$$

Since each  $G_j^k$  has property  $\mathcal{P}$ , it follows that

$$e(G_j^k) \leq An^{1+\alpha}(G_j^k) < An(G_j^k) \cdot \frac{e}{2An}.$$

Therefore, for the total number of edges of  $G_k$ , we have

$$e(G^k) = \sum_{j=1}^{M_k} e(G_j^k) < A \frac{e}{2An} \sum_{j=1}^{M_k} n(G_j^k) = \frac{e}{2},$$

the desired contradiction. This proves the bound of Theorem 1.

It remains to show that the bound is tight up to a constant factor. Suppose that  $\text{ex}(n, \mathcal{P}) \geq A'n^{1+\alpha}$ . For every  $e$  ( $cn < e \leq An^{1+\alpha}$ ), we construct a graph  $G$  of at most  $n$  vertices and at least  $e$  edges, which has property  $\mathcal{P}$  and crossing number

$$\text{cr}(G) \leq c' \frac{e^{2+1/\alpha}}{n^{1+1/\alpha}},$$

for a suitable constant  $c'' = c''(A', \alpha)$ .

Let

$$k = \left\lceil \frac{2e}{A'n} \right\rceil^{\frac{1}{\alpha}},$$

and let  $G_k$  denote a graph of  $k$  vertices and at least  $A'k^{1+\alpha}$  edges, which has property  $\mathcal{P}$ . Clearly,

$$\text{cr}(G_k) \leq e^2(G_k) \leq (A'k^{1+\alpha})^2 = A^2k^{2+2\alpha}.$$

Let  $G$  be the union of  $\lfloor n/k \rfloor$  disjoint copies of  $G_k$ . Then  $n(G) = \lfloor n/k \rfloor k \leq n$ ,

$$e(G) = \left\lfloor \frac{n}{k} \right\rfloor e(G_k) \geq \frac{n}{2k} A'k k^\alpha \geq e,$$

$$\text{cr}(G) = \left\lfloor \frac{n}{k} \right\rfloor \text{cr}(G_k) \leq \frac{n}{k} A^2 k^{2+2\alpha} \leq A^2 n \left( 2 \left( \frac{2e}{A'n} \right)^{\frac{1}{\alpha}} \right)^{1+2\alpha} = \frac{2^{3+2\alpha+1/\alpha} A^2}{(A')^{2+1/\alpha}} \cdot \frac{e^{2+1/\alpha}}{n^{1+1/\alpha}},$$

as required.  $\square$

### 3 Forbidden subgraphs – Proofs of Theorems 2 and 3

In Section 1, we established Theorem 1 under the assumption  $e \geq cn \log^2 n$ , where  $c$  is a suitable constant depending on property  $\mathcal{P}$ . It seems very likely that the same result is true for every  $e \geq cn$ . The appearance of the  $\log^2 n$  factor was due to the fact that to estimate the total number of edges deleted during the DECOMPOSITION ALGORITHM, we applied Theorem B. We used a poor upper bound on the term  $\sum d_i^2$ , because some of the degrees  $d_i$  may be very large. However, in some interesting special cases, this difficulty can be avoided by a simple trick. We can split each vertex of high degree into vertices of ‘average degree,’ unless the new graph ceases to have property  $\mathcal{P}$ .

We illustrate this technique by proving the following result, which is the  $r = s = 2$  special case of Theorem 3 and a slight modification of Theorem 2 for  $r = 2$ .

**Theorem 3.1.** *Let  $G$  be a  $K_{2,2}$ -free ( $C_4$ -free) graph with  $n(G) = n$  vertices and  $e(G) = e$  edges,  $e \geq 1000n$ . Then*

$$\text{cr}(G) \geq \frac{1}{10^8} \frac{e^4}{n^3}.$$

*This bound is tight up to a constant factor.*

**Proof.** Let  $G$  be a graph with  $n$  vertices and  $e \geq 1000n$  edges, which does not contain  $K_{2,2}$  as a subgraph. Suppose, in order to obtain a contradiction, that

$$\text{cr}(G) < \frac{1}{10^8} \frac{e^4}{n^3},$$

and  $G$  is drawn in the plane with  $\text{cr}(G)$  crossings.

First, we split every vertex of  $G$  whose degree exceeds  $\bar{d} := 2e/n$  into vertices of degree at most  $\bar{d}$ , as follows. Let  $v$  be a vertex of  $G$  with degree  $d(v, G) = d(v) = d > \bar{d}$ , and let  $vw_1, vw_2, \dots, vw_d$  be the edges incident to  $v$ , listed in clockwise order. Replace  $v$  by  $\lceil d/\bar{d} \rceil$  new vertices,  $v_1, v_2, \dots, v_{\lceil d/\bar{d} \rceil}$ , placed in clockwise order on a very small circle around  $v$ . Without introducing any new crossings, connect  $w_j$  to  $v_i$  if and only if  $\bar{d}(i-1) < j \leq \bar{d}i$  ( $1 \leq j \leq d, 1 \leq i \leq \lceil d/\bar{d} \rceil$ ). Repeat this procedure for every vertex whose degree exceeds  $\bar{d}$ , and denote the resulting graph by  $G'$ .

Obviously,  $G'$  is also  $K_{2,2}$ -free,  $e(G') = e(G) = e$ , and

$$\text{cr}(G') \leq \text{cr}(G) < \frac{1}{10^8} \frac{e^4(G)}{n^3(G)}.$$

Since all but at most  $n$  vertices of  $G'$  have degree  $\bar{d}$ , we have  $n(G') < 2n(G) = 2n$ .

Apply the DECOMPOSITION ALGORITHM described in the previous section to the graph  $G'$  with the difference that, instead of (1), use the following stopping rule: STOP in STEP  $i+1$  if

$$\left(\frac{2}{3}\right)^i < \frac{e^2(G')}{16n^3(G')}.$$

Suppose that the algorithm terminates in STEP  $k+1$ . If  $k > 0$ , then

$$(2/3)^k < \frac{e^2(G')}{16n^3(G')} \leq (2/3)^{k-1}.$$

Just like in the proof of Theorem 1, for every  $i < k$ , we have that

$$\sum_{j=1}^{m_i} \sqrt{\text{cr}(G_j^i)} \leq \sqrt{(3/2)^{i+1}} \sqrt{\text{cr}(G)} < \frac{1}{10^4} \sqrt{(3/2)^{i+1}} \frac{e^2}{n^{3/2}}$$

and, using the fact that the maximum degree in  $G'$  is at most  $\bar{d}$ ,

$$\sum_{j=1}^{m_i} \sqrt{\sum_{v \in V(G_j^i)} d^2(v, G_j^i)} \leq \sqrt{(3/2)^{i+1}} \sqrt{\sum_{v \in V(G')} d^2(v, G')} \leq \sqrt{(3/2)^{i+1}} \sqrt{\bar{d}2e(G')} \leq 2\sqrt{(3/2)^{i+1}} \frac{e}{\sqrt{n}}.$$

Hence, by Theorem B, the total number of edges deleted during the algorithm is

$$\begin{aligned} & \sum_{i=0}^{k-1} \sum_{j=1}^{m_i} b(G_j^i) \leq 10 \sum_{i=0}^{k-1} \sum_{j=1}^{m_i} \sqrt{\text{cr}(G_j^i)} + 2 \sum_{i=0}^{k-1} \sum_{j=1}^{m_i} \sqrt{\sum_{v \in V(G_j^i)} d^2(v, G_j^i)} \\ & < \frac{1}{1000} \frac{e^2}{n^{3/2}} \sum_{i=0}^{k-1} \sqrt{(3/2)^{i+1}} + 4 \frac{e}{\sqrt{n}} \sum_{i=0}^{k-1} \sqrt{(3/2)^{i+1}} = \sqrt{3/2} \frac{\sqrt{(3/2)^k} - 1}{\sqrt{3/2} - 1} \left( \frac{e^2}{1000n^{3/2}} + \frac{4e}{\sqrt{n}} \right) \end{aligned}$$



$$< 100 \frac{n^{3/2}}{e} \left( \frac{e^2}{1000n^{3/2}} + \frac{4e}{\sqrt{n}} \right) < \frac{e}{10} + 400n < \frac{e}{2}.$$

Therefore, for the resulting graph,

$$e(G^k) \geq \frac{e}{2}.$$

On the other hand, each component of  $G^k$  has relatively few vertices:

$$n(G_j^k) < (2/3)^k n(G^k) < \frac{e^2}{16n^2(G^k)} = \frac{e^2}{16n^2(G^k)} \quad (j = 1, 2, \dots, M_k).$$

**Claim C.** [R58] *Let  $\text{ex}(n, K_{2,2})$  denote the maximum number of edges that a  $K_{2,2}$ -free graph with  $n$  vertices can have. Then*

$$\text{ex}(n, K_{2,2}) \leq \frac{n(1 + \sqrt{4n-3})}{4} \leq n^{3/2}.$$

Applying the Claim to each  $G_k^j$ , we obtain

$$e(G_j^k) \leq n^{3/2}(G_j^k) < n(G_j^k) \cdot \sqrt{\frac{e^2}{16n^2(G^k)}},$$

therefore,

$$e(G^k) = \sum_{j=1}^{M_k} e(G_j^k) < \frac{e}{4n(G^k)} \sum_{j=1}^{M_k} n(G_j^k) = \frac{e}{4},$$

the desired contradiction. The tightness of Theorem 3.1 immediately follows from the fact that Theorem 1 was tight.  $\square$

Theorems 2 and 3 can be proved similarly. It is enough to notice that splitting a vertex of high degree does not decrease the girth of a graph  $G$  and does not create a subgraph isomorphic to  $K_{r,s}$ . Instead of Claim C, now we need

**Claim C'.** [BS74], [B66], [Be66], [S66], [W91] *For a fixed positive integer  $r$ , let  $\mathcal{G}_{2r}$  denote the property that the girth of a graph is larger than  $2r$ .*

*Then the maximum number of edges of a graph with  $n$  vertices, which has property  $\mathcal{G}_{2r}$ , satisfies*

$$\text{ex}(n, \mathcal{G}_{2r}) = O(n^{1+1/r}).$$

*For  $r = 2, 3$  and  $5$ , this bound is tight.*

**Claim C''.** [KST54], [F96], [ER62], [B66], [ARS98] *For any integers  $s \geq r \geq 2$ , the maximum number of edges of a  $K_{r,s}$ -free graph of  $n$  vertices, satisfies*

$$\text{ex}(n, K_{r,s}) = O(n^{2-1/r}).$$

This bound is tight for  $s > (r - 1)!$ .

In case  $r = 3$ , we obtain the following slight generalization of Theorem 2.

**Theorem 3.2.** *Let  $G$  be a graph of  $n$  vertices and  $e \geq 4n$  edges, which contains no cycle  $C_6$  of length 6.*

*Then, for a suitable constant  $c'_6 > 0$ , we have*

$$\text{cr}(G) \geq c'_6 \frac{e^5}{n^4}.$$

To establish Theorem 3.2, it is enough to modify the proof of Theorem 2 at one point. Before splitting the high-degree vertices of  $G$  and running the DECOMPOSITION ALGORITHM, we have to turn  $G$  into a bipartite graph, by deleting at most half of its edges. After that, splitting a vertex cannot create a  $C_6$ , and the rest of the above argument shows that the crossing number of the remaining graph still exceeds  $c'_6 \frac{e^5}{n^4}$ .

We do not see, however, how to obtain the analogous generalization of Theorem 2 for  $r > 3$ .

## 4 Midrange crossing constant in the plane – Proof of Theorem 4

**Lemma 4.1.** (i) *For any  $a > 0$ , the limit*

$$\gamma[a] = \lim_{n \rightarrow \infty} \frac{\kappa(n, na)}{n}$$

*exists and is finite.*

(ii)  *$\gamma[a]$  is a convex continuous function.*

(iii) *For any  $a \geq 4$ ,  $1 > \delta > 0$ ,*

$$\gamma[a] - \gamma[a(1 - \delta)] \leq \gamma[a(1 + \delta)] - \gamma[a] \leq 10^3 \delta \gamma[a].$$

**Proof.** Clearly, any two graphs,  $G_1$  and  $G_2$ , can be drawn in the plane so that the edges of  $G_1$  do not intersect the edges of  $G_2$ . Therefore,

$$\kappa(n_1 + n_2, e_1 + e_2) \leq \kappa(n_1, e_1) + \kappa(n_2, e_2). \quad (3)$$

In particular, the function  $f_a(n) = \kappa(n, na)$  is subadditive and hence the limit

$$\gamma[a] = \lim_{n \rightarrow \infty} \frac{\kappa(n, na)}{n}$$

exists and is finite for every fixed  $a > 0$ . It also follows from (3) that for any  $a, b > 0$  and  $1 > \alpha > 0$ , if  $n$  and  $\alpha n$  are both integers,

$$\kappa(n, (\alpha a + (1 - \alpha)b)n) \leq \kappa(\alpha n, \alpha a n) + \kappa((1 - \alpha)n, (1 - \alpha)b n),$$

so for any  $1 > \alpha > 0$  rational,

$$\gamma[\alpha a + (1 - \alpha)b] \leq \alpha \gamma[a] + (1 - \alpha) \gamma[b].$$

But since the function  $\gamma[a]$  is monotone increasing, it follows that for any  $1 > \alpha > 0$ ,

$$\gamma[\alpha a + (1 - \alpha)b] \leq \alpha \gamma[a] + (1 - \alpha) \gamma[b]. \quad (4)$$

That is, the function  $\gamma[a]$  is *convex*. In particular, for every  $1 > \delta > 0$ , we have

$$\gamma[a] - \gamma[a(1 - \delta)] \leq \gamma[a(1 + \delta)] - \gamma[a].$$

It is known that for any  $a \geq 4$ ,

$$\frac{a^3 n}{100} \leq \kappa(n, a n) \leq a^3 n \Rightarrow \frac{a^3}{100} \leq \gamma[a] \leq a^3 \quad (5)$$

(see e.g. [PT97]). Let  $a \geq 4$ ,  $1 > \delta > 0$ . By (4),

$$\gamma[a(1 + \delta)] \leq (1 - \delta) \gamma[a] + \delta \gamma[2a].$$

Therefore, using (5),

$$\gamma[a(1 + \delta)] - \gamma[a] \leq \delta \gamma[2a] \leq \delta 8a^3 < 10^3 \delta \gamma[a]. \quad \square$$

Set

$$C := \limsup_{a \rightarrow \infty} \frac{\gamma[a]}{a^3}.$$

By (5), we have that  $C < 1$ .

**Lemma 4.2.** *For any  $0 < \varepsilon < 1$ , there exists  $N = N(\varepsilon)$  such that  $\kappa(n, e) > C \frac{e^3}{n^2} (1 - \varepsilon)$ , whenever  $\min\{n, e/n, n^2/e\} > N$ .*

**Proof.** Let  $A > \frac{10^9}{\varepsilon^3}$  be a rational number satisfying

$$\frac{\gamma[A]}{A^3} > C \left(1 - \frac{\varepsilon}{10}\right). \quad (6)$$

Let  $N = N(\varepsilon) \geq A$  such that, if  $n > N$ ,  $e = nA'$ , and  $|A - A'| \leq A\varepsilon$ , then

$$\kappa(n, e) > \gamma[A'](1 - \frac{\varepsilon}{10})n. \quad (7)$$

Let  $n$  and  $e$  be fixed,  $\min\{n, e/n, n^2/e\} > N$  and let  $G = (V, E)$  be a graph with  $|V| = n$  vertices and  $|E| = e$  edges, drawn in the plane with  $\kappa(n, e)$  crossings. Set  $p = An/e$ . Let  $U$  be a randomly chosen subset of  $V$  with  $\Pr[v \in U] = p$ , independently for all  $v \in V$ . Let  $\nu = |U|$ , and let  $\eta$  (resp.  $\xi$ ) be the number of edges (resp. crossings) in the (drawing of the) subgraph of  $G$  induced by the elements of  $U$ .

$\nu$  has mean  $pn$  and variance  $p(1-p)n \leq pn$ , so, by the Chebyshev Inequality,

$$\Pr \left[ |\nu - pn| > \frac{\varepsilon}{10^4} pn \right] < \frac{\varepsilon}{10}.$$

Write  $\eta = \sum I_{uv}$ , where the sum is taken over all edges  $uv = vu \in E$ , and  $I_{uv}$  denotes the indicator for the event  $u, v \in U$ . Obviously,  $E[\eta] = \sum_{uv \in E} E[I_{uv}] = ep^2$ . We decompose

$$\text{Var}[\eta] = \sum_{uv \in E} \text{Var}[I_{uv}] + \sum_{uv, uw \in E} \text{Cov}[I_{uv}, I_{uw}],$$

as  $\text{Cov}[I_{uv}, I_{wz}] = 0$  when all four indices are distinct. As always with indicators, we have

$$\sum_{uv \in E} \text{Var}[I_{uv}] \leq \sum_{uv \in E} E[I_{uv}] = E[\eta] = ep^2.$$

Using the bound  $\text{Cov}[I_{uv}, I_{uw}] \leq E[I_{uv}I_{uw}] = p^3$ , we obtain

$$\text{Var}[\eta] \leq p^2e + p^3 \sum_{v \in V} \binom{d(v)}{2},$$

where  $d(v)$  is the degree of vertex  $v$  in  $G$ . But  $\sum_{v \in V} d(v) = 2e$  and all  $d(v) < n$ , so

$$\sum_{v \in V} \binom{d(v)}{2} \leq \frac{1}{2} \sum_{v \in V} d^2(v) \leq en.$$

Thus, we have

$$\text{Var}[\eta] \leq p^2e + p^3en \leq 2p^3en,$$

as  $pn = An^2/e \geq 1$ . Again, by the Chebyshev Inequality,

$$\Pr \left[ |\eta - p^2e| > \frac{\varepsilon}{10^4} p^2e \right] < \frac{\varepsilon}{10}.$$

With probability at least  $1 - \frac{\varepsilon}{5}$ ,

$$pn(1 - \frac{\varepsilon}{10^4}) < \nu < pn(1 + \frac{\varepsilon}{10^4}) \quad \text{and} \quad p^2e(1 - \frac{\varepsilon}{10^4}) < \eta < p^2e(1 + \frac{\varepsilon}{10^4}),$$

so with probability at least  $1 - \frac{\varepsilon}{5}$ ,

$$A(1 - \frac{3\varepsilon}{10^4}) < \frac{\eta}{\nu} = A' < A(1 + \frac{3\varepsilon}{10^4}).$$

Therefore, in view of (7), with probability at least  $1 - \frac{\varepsilon}{5}$ , the subgraph of  $G$  induced by  $U$  has at least  $pn(1 - \frac{\varepsilon}{10})\gamma[A'](1 - \frac{\varepsilon}{10})$  crossings. But then, we have

$$\begin{aligned} E[\xi] &\geq (1 - \frac{\varepsilon}{5})pn(1 - \frac{\varepsilon}{10})\gamma[A'](1 - \frac{\varepsilon}{10}) \geq (1 - \frac{\varepsilon}{5})pn(1 - \frac{\varepsilon}{10})\gamma[A](1 - \frac{3\varepsilon}{10})(1 - \frac{\varepsilon}{10}) \\ &\geq (1 - \frac{\varepsilon}{5})pn(1 - \frac{\varepsilon}{10})CA^3(1 - \frac{\varepsilon}{10})(1 - \frac{3\varepsilon}{10})(1 - \frac{\varepsilon}{10}) \geq (1 - \varepsilon)CA^3pn, \end{aligned}$$

where the second and third inequalities follow from Lemma 4.1 (iii) and from the choice of  $A$ , respectively.

On the other hand,

$$E[\xi] = p^4\kappa(n, e),$$

as every crossing lies in  $U$  with probability  $p^4$ . Thus

$$\kappa(n, e) \geq (1 - \varepsilon)\frac{pnCA^3}{p^4} = C\frac{e^3}{n^2}(1 - \varepsilon)$$

as desired.  $\square$

To complete the proof of Theorem 4, we have to establish the ‘‘counterpart’’ of Lemma 4.2.

**Lemma 4.3.** *For any  $1 > \varepsilon > 0$ , there exists  $M = M(\varepsilon)$  such that  $\kappa(n, e) < C\frac{e^3}{n^2}(1 + \varepsilon)$ , whenever  $\min\{n, e/n, n^2/e\} > M$ .*

**Proof.** Let  $A > \frac{10^4}{\varepsilon^2}$  be a rational number satisfying

$$C(1 - \frac{\varepsilon}{10}) < \frac{\gamma[A]}{A^3} < C(1 + \frac{\varepsilon}{10}).$$

Let  $M_1 = M_1(\varepsilon) \geq A$  such that, if  $n > M_1$  and  $e = nA$ , then

$$CA^3n(1 - \frac{\varepsilon}{5}) < \kappa(n, e) < CA^3n(1 + \frac{\varepsilon}{5}).$$

Let  $G_1 = G_1(n_1, e_1)$  be a graph with  $n_1 > M_1$  vertices,  $e_1 = An_1$  edges, and suppose that  $G_1$  is drawn in the plane with  $\kappa(n_1, e_1)$  crossings, where  $CA^3n_1(1 - \frac{\varepsilon}{5}) < \kappa(n_1, e_1) < CA^3n_1(1 + \frac{\varepsilon}{5})$ .

For each vertex  $v$  of  $G_1$  with degree  $d(v) > A^{3/2}$ , we do the following. Let  $d(v) = rA^{3/2} + s$ , where  $0 \leq s < A^{3/2}$ . Substitute  $v$  with  $r + 1$  vertices, each of degree  $A^{3/2}$ , except one which has degree  $s$ , each drawn very close to the original position of  $v$ . Clearly, this can be done without creating any additional crossing. We obtain a graph  $G_2(n_2, e_2)$  such that

$$n_1 \leq n_2 \leq n_1 \left(1 + \frac{2}{\sqrt{A}}\right) \leq n_1 \left(1 + \frac{\varepsilon}{10}\right),$$

$e_2 = e_1$ , and  $G_2$  is drawn in the plane with  $\kappa(n_1, e_1)$  crossings.

Suppose that  $n$  and  $e$  are fixed,  $\min\{n, e/n, n^2/e\} > M(\varepsilon) = \frac{10M_1}{\varepsilon}$ . Let

$$L = \frac{e/n}{e_2/n_2} \quad \text{and} \quad K = \frac{n^2/e}{n_2^2/e_2},$$

so that

$$n = KLn_2 \quad \text{and} \quad e = KL^2e_2.$$

Let

$$\tilde{L} = \left\lfloor L \left(1 + \frac{\varepsilon}{10}\right) \right\rfloor \quad \text{and} \quad \tilde{K} = \left\lfloor K \left(1 - \frac{\varepsilon}{10}\right) \right\rfloor$$

and let

$$\tilde{n} = \tilde{K}\tilde{L}n_2 \quad \text{and} \quad \tilde{e} = \tilde{K}\tilde{L}^2e_2.$$

Then  $n(1 - \frac{\varepsilon}{5}) < \tilde{n} < n$  and  $e_2 < \tilde{e} \leq e_2(1 + \frac{\varepsilon}{4})$ , so we have  $\kappa(n, e) < \kappa(\tilde{n}, \tilde{e})$ .

Substitute each vertex of  $G_2$  with  $\tilde{L}$  very close vertices, and substitute each edge of  $G_2$  with the corresponding  $\tilde{L}^2$  edges, all running very close to the original edge. Make  $\tilde{K}$  copies of this drawing, each separated from the others. This way we got a graph  $\tilde{G}(\tilde{n}, \tilde{e})$  drawn in the plane. We estimate the number of crossings  $X$  in this drawing.

A crossing in the original drawing of  $G_2$  corresponds to  $\tilde{K}\tilde{L}^4$  crossings in the present drawing of  $\tilde{G}$ . For any two edges of  $G_2$  with common endpoint,  $uv$  and  $uw$ , the edges arise from them have at most  $\tilde{K}\tilde{L}^4$  crossings with each other. So

$$X \leq \tilde{K}\tilde{L}^4 \left( \kappa(n_1, e_1) + \sum_{v \in V(G_2)} \binom{d(v)}{2} \right)$$

But  $\sum_{v \in V(G_2)} d(v) = 2e_2$  and  $d(v) \leq A^{3/2}$ , so

$$\sum_{v \in V(G_2)} \binom{d(v)}{2} < 3A^{5/2}n_2.$$

Therefore,

$$\begin{aligned}
\kappa(n, e) &< \kappa(\tilde{n}, \tilde{e}) \leq c < \tilde{K}\tilde{L}^4\kappa(n_1, e_1) + \tilde{K}\tilde{L}^43A^{5/2}n_2 < \tilde{K}\tilde{L}^4\kappa(n_1, e_1)\left(1 + \frac{\varepsilon}{10}\right) \\
&< \tilde{K}\tilde{L}^4CA^3n_1\left(1 + \frac{\varepsilon}{5}\right)\left(1 + \frac{\varepsilon}{10}\right) = \tilde{K}\tilde{L}^4C\frac{e_1^3}{n_1^2}\left(1 + \frac{\varepsilon}{5}\right)\left(1 + \frac{\varepsilon}{10}\right) \\
&< KL^4C\frac{e_2^3}{n_2^2}\left(1 + \frac{\varepsilon}{10}\right)^6\left(1 + \frac{\varepsilon}{5}\right)\left(1 + \frac{\varepsilon}{10}\right) < C(1 + \varepsilon)\frac{e^3}{n^2}. \quad \square
\end{aligned}$$

**Remark 4.4.** It was shown in [PT97] that  $.06 \geq C \geq .029$ .

We cannot decide whether Theorem 4 remains true under the weaker condition that  $C_1n \leq e \leq C_2n^2$  for suitable positive constants  $C_1$  and  $C_2$ . If the answer were in the affirmative, then, clearly,  $C_1 > 3$ . We would also have that  $C_2 < 1/2$ , because, by [G72], for  $e = \binom{n}{2}$ ,  $\text{cr}(K_n) > \left(\frac{1}{10} - \varepsilon\right)\frac{e^3}{n^2}$  for any  $\varepsilon > 0$  if  $n$  is large enough.

## 5 Midrange crossing constants on other surfaces – Proof of Theorem 5

**Lemma 5.1.** *For any integer  $g \geq 0$  and for any  $1 > \varepsilon > 0$ , there exists  $N = N(g, \varepsilon)$  such that  $\kappa_g(n, e) > C\frac{e^3}{n^2}(1 - \varepsilon)$ , whenever  $\min\{n, e/n, n^{3/2}/e\} > N$ .*

**Proof.** For  $g = 0$ , the assertion follows from Lemma 4.2. Suppose that  $g > 0$  is fixed and we have already proved the lemma for  $g - 1$ . For any  $\varepsilon > 0$ , let  $N(g, \varepsilon) = \frac{10^5}{\varepsilon^2}gN(g - 1, \varepsilon/10)$ . Suppose, in order to get a contradiction, that  $\min\{n, e/n, n^{3/2}/e\} > N$ , and let  $G(n, e)$  be a graph drawn on  $S_g$  with  $\text{cr}_g(G) = \kappa_g(n, e) < C\frac{e^3}{n^2}(1 - \varepsilon)$  crossings.

As long as there is an edge with at least  $4C\frac{e^2}{n^2}$  crossings, delete it. Let the resulting graph be  $G_1(n_1, e_1)$ . Suppose that we deleted  $e'$  edges. Then  $G_1$  has  $n_1 = n$  vertices,  $e_1 = e - e'$  edges, and the number of crossings in the resulting drawing of  $G_1$  is at most  $\text{cr}_g(G) - 4C\frac{e^2}{n^2}e'$ . Therefore,  $e' < e/4$ , so  $e \geq e_1 \geq 3e/4$ . It is not hard to check that  $\text{cr}_g(G_1) < C\frac{e_1^3}{n_1^2}(1 - \varepsilon)$  and  $G_1$  contains no edge with more than  $4C\frac{e^2}{n^2} < 8C\frac{e_1^2}{n_1^2}$  crossings.

Consider all cycles of  $G_1$ , as they are drawn on  $S_g$ . If each cycle is *trivial*, i.e., each cycle is contractible to a point of  $S_g$ , then every connected component of  $G$  is contractible to a point. That is, in this case, our drawing of  $G$  on  $S_g$  is equivalent to a drawing of  $G_1$  on the plane. Consequently,  $\text{cr}_{g-1}(G_1) \leq \text{cr}_0(G_1) < C\frac{e_1^3}{n_1^2}(1 - \varepsilon)$  contradicting the induction hypothesis.

Suppose that there is a non-trivial (i.e., non-contractible) cycle  $\mathcal{C}$  of  $G_1$  with at most  $\frac{\varepsilon}{80C}\frac{n_1^2}{e_1}$  edges. Clearly,  $\mathcal{C}$  contains a non-trivial closed curve,  $\mathcal{C}'$ , which does not intersect itself. The total

number of crossings along  $\mathcal{C}'$  is at most

$$\frac{\varepsilon}{80C} \frac{n_1^2}{e_1} 8C \frac{e_1^2}{n_1^2} = \frac{\varepsilon}{10} e_1.$$

Delete all edges that cross  $\mathcal{C}'$ . Cut  $S_g$  along  $\mathcal{C}'$ . Replace every vertex (resp. edge)  $\mathcal{C}'$  by two vertices, one on each side of the cut. Every edge of  $G$  arriving at a vertex  $v$  of  $\mathcal{C}'$  from a given side of the cut will be connected to the copy of  $v$  lying on the same side. Thus, we obtain a graph  $G_2(n_2, e_2)$ , drawn with fewer than  $\text{cr}_g(G_1)$  crossings. Attaching a half-sphere to each side of the cut, we obtain either a surface of genus  $g - 1$  or two surfaces whose genres are smaller than  $g$ . We discuss only the former case (the calculation in the latter one is very similar). Since we doubled at most  $\frac{\varepsilon}{80C} \frac{n_1^2}{e_1} = \varepsilon n_1 \frac{n_1}{e_1} \frac{1}{80C} < \varepsilon n_1 \frac{1}{N} < n_1 \frac{\varepsilon}{10}$  vertices and deleted at most  $\frac{\varepsilon}{10} e$  edges, we have  $n_2 \leq n_1(1 + \frac{\varepsilon}{10})$  and  $e_2 \geq e_1(1 - \frac{\varepsilon}{10})$ . In the resulting drawing there are fewer than  $\text{cr}_g(G_1)$  crossings, therefore

$$\text{cr}_{g-1}(G_2) < \text{cr}_g(G_1) < C \frac{e_1^3}{n_1^2} (1 - \varepsilon) \leq C \frac{e_2^3}{n_2^2} (1 - \varepsilon) (1 - \frac{\varepsilon}{10})^{-3} (1 + \frac{\varepsilon}{10})^2 \leq C \frac{e_2^3}{n_2^2} (1 - \frac{\varepsilon}{10}),$$

contradicting the induction hypothesis.

Thus, we can assume that every non-trivial cycle of  $G_1$  contains at least  $\frac{\varepsilon}{80C} \frac{n_1^2}{e_1}$  edges. For each vertex  $v$  of  $G_1$  with degree  $d(v) > \frac{10e_1}{\varepsilon n_1}$ , we do the following. Let  $d(v) = r \frac{10e_1}{\varepsilon n_1} + s$ , where  $0 \leq s < \frac{10e_1}{\varepsilon n_1}$ . Without creating any new crossing, replace  $v$  by  $r + 1$  nearby vertices, each of degree  $\frac{10e_1}{\varepsilon n}$ , except one, whose degree is  $s$ . We obtain a graph  $G_3(n_3, e_3)$  drawn on  $S_g$  with  $n_1 \leq n_3 \leq n_1(1 + \frac{\varepsilon}{5})$ ,  $e_3 = e_1$ , and with the same number of crossings as  $G_1$ . Hence,

$$\text{cr}_g(G_3) \leq \text{cr}_g(G_1) \leq C \frac{e_1^3}{n_1^2} (1 - \varepsilon) \leq C \frac{e_3^3}{n_3^2} (1 - \varepsilon) (1 + \frac{\varepsilon}{5})^2 \leq C \frac{e_3^3}{n_3^2} (1 - \frac{\varepsilon}{2}).$$

The maximum degree  $D$  in  $G_3$  cannot exceed  $\frac{10e_1}{\varepsilon n_1} < \frac{18e_3}{\varepsilon n_3}$ , and the length of each non-trivial cycle is at least  $\frac{\varepsilon}{80C} \frac{n_1^2}{e_1} \geq \frac{\varepsilon}{100C} \frac{n_3^2}{e_3}$ . Apply to  $G_3$  the DECOMPOSITION ALGORITHM described in Section 2 with the difference that, instead of (1), use the following stopping rule: STOP in STEP  $i + 1$  if

$$(2/3)^i < \frac{\varepsilon}{100C} \frac{n_3}{e_3}.$$

Suppose that the algorithm terminates in STEP  $k + 1$ . Then

$$(2/3)^k < \frac{\varepsilon}{100C} \frac{n_3}{e_3} \leq (2/3)^{k-1}.$$



First, we give an upper bound on the total number of edges deleted from  $G_3$ . Let  $G^0 = G_1^0 = G_3$  and  $m_0 = 1$ . Using (2), we obtain that, for every  $0 \leq i < k$ ,

$$\sum_{j=1}^{m_i} \sqrt{\text{cr}_g(G_j^i)} \leq \sqrt{m_i \sum_{j=1}^{m_i} \text{cr}_g(G_j^i)} \leq \sqrt{(3/2)^{i+1}} \sqrt{\text{cr}_g(G_3)} \leq \sqrt{(3/2)^{i+1}} \sqrt{C \frac{e_3^3}{n_3^2} (1 - \frac{\varepsilon}{2})}.$$

Denoting by  $d(v, G_j^i)$  the degree of vertex  $v$  in  $G_j^i$ , we have

$$\begin{aligned} & \sum_{j=1}^{m_i} \sqrt{\sum_{v \in V(G_j^i)} d^2(v, G_j^i)} \leq \sqrt{(3/2)^{i+1}} \sqrt{\sum_{v \in V(G^i)} d^2(v, G^i)} \\ & \leq \sqrt{(3/2)^{i+1}} \sqrt{\max_{v \in V(G^i)} d(v, G^i) \sum_{v \in V(G^i)} d(v, G^i)} \leq \sqrt{(3/2)^{i+1}} \sqrt{\frac{18e_3^3}{\varepsilon n_3^2} (2e_3)} = 12 \sqrt{(3/2)^{i+1}} \frac{e_3}{\sqrt{\varepsilon n_3}}. \end{aligned}$$

By Theorem 6 (proved in the last section), the total number of edges deleted during the algorithm is

$$\begin{aligned} & \sum_{i=0}^{k-1} \sum_{j=1}^{m_i} b(G_j^i) \leq 300(1 + g^{3/4}) \sum_{i=0}^{k-1} \sum_{j=1}^{m_i} \sqrt{\text{cr}_g(G_j^i) + \sum_{v \in V(G_j^i)} d^2(v, G_j^i)} \\ & \leq 300(1 + g^{3/4}) \sum_{i=0}^{k-1} \sum_{j=1}^{m_i} \sqrt{\text{cr}_g(G_j^i)} + 300(1 + g^{3/4}) \sum_{i=0}^{k-1} \sum_{j=1}^{m_i} \sqrt{\sum_{v \in V(G_j^i)} d^2(v, G_j^i)} \\ & \leq 300(1 + g^{3/4}) \sum_{i=0}^{k-1} \sqrt{(3/2)^{i+1}} \left( \sqrt{C \frac{e_3^3}{n_3^2} (1 - \frac{\varepsilon}{2})} + 6 \frac{e_3}{\sqrt{\varepsilon n_3}} \right) \\ & \leq 300(1 + g^{3/4}) \sqrt{3/2} \frac{\sqrt{(3/2)^k} - 1}{\sqrt{3/2} - 1} \left( \sqrt{C \frac{e_3^3}{n_3^2} (1 - \frac{\varepsilon}{2})} + 6 \frac{e_3}{\sqrt{\varepsilon n_3}} \right) \\ & \leq 2000(1 + g^{3/4}) \sqrt{\frac{C}{\varepsilon}} \sqrt{\frac{e}{n}} \left( \sqrt{C \frac{e_3^3}{n_3^2} (1 - \frac{\varepsilon}{2})} + 6 \frac{e_3}{\sqrt{\varepsilon n_3}} \right) \leq e_3 \frac{\varepsilon}{10}. \end{aligned}$$

Therefore, the number of edges  $e(G^k)$  of the graph  $G^k$  obtained in the final STEP of the algorithm satisfies  $e(G^k) \geq e_3(1 - \frac{\varepsilon}{10})$ . Consider the drawing of  $G^k$  on  $S_g$  inherited from the drawing of  $G_3$ . Each connected component of  $G^k$  has fewer than  $\frac{\varepsilon}{100C} \frac{n_3^2}{e_3}$  vertices, therefore, each cycle of  $G^k$ , as drawn on  $S_g$ , is contractible to a point. Consequently, this drawing is equivalent to a planar drawing of  $G^k$ . Hence,

$$\text{cr}_{g-1}(G^k) \leq \text{cr}_0(G^k) \leq \text{cr}_g(G_3) \leq C \frac{e_3^3}{n_3^2} (1 - \frac{\varepsilon}{2}) \leq C \frac{e^3(G^k)}{n^2(G^k)} (1 - \frac{\varepsilon}{2}) (1 - \frac{\varepsilon}{10})^{-3} < C \frac{e^3(G^k)}{n^2(G^k)} (1 - \frac{\varepsilon}{10}),$$

a contradiction. This concludes the proof of Lemma 5.1.  $\square$

**Lemma 5.2.** *For any integer  $g \geq 0$  and for any  $\varepsilon > 0$ , there exists  $N' = N'(g, \varepsilon)$  such that  $\kappa_g(n, e) > C \frac{e^3}{n^2} (1 - \varepsilon)$ , whenever  $\min\{n, e/n, n^2/e\} > N'$ .*

**Proof.** The proof is analogous to that of Lemma 4.2.  $\square$

**Lemma 5.3.** *For any integer  $g \geq 0$  and for any  $\varepsilon > 0$ , there exists  $M = M(g, \varepsilon)$  such that  $\kappa_g(n, e) < C \frac{e^3}{n^2} (1 + \varepsilon)$ , whenever  $\min\{n, e/n, n^2/e\} > M$ .*

**Proof.** Clearly, for any graph  $G$  and for any  $g \geq 0$ , we have  $\text{cr}_0(G) \geq \text{cr}_g(G)$ . Therefore, Lemma 5.3 is a direct consequence of Lemma 4.3.  $\square$

Theorem 5 now readily follows from Lemmas 5.2 and 5.3.

## 6 A separator theorem

### – Proof of Theorem 6

For the proof of Theorem 6, we need a slight variation of the notion of bisection width. The *weak bisection width*,  $\bar{b}(G)$ , of a graph  $G$  is defined as the minimum number of edges whose removal splits the graph into two components, each of size at least  $|V(G)|/5$ . That is,

$$\bar{b}(G) = \min_{|V_A|, |V_B| \geq n/5} |E(V_A, V_B)|,$$

where  $E(V_A, V_B)$  denotes the number of edges between  $V_A$  and  $V_B$ , and the minimum is taken over all partitions  $V(G) = V_A \cup V_B$  with  $|V_A|, |V_B| \geq |V(G)|/5$ .

**Lemma 6.1** *For any graph  $G$ , we have*

$$\bar{b}(G) \leq b(G) \leq 2 \max_{H \subset G} \bar{b}(H).$$

**Proof.** The first inequality is obviously true. To prove the second one, let  $|V(G)| = n$  and consider a partition  $V(G) = V_A \cup V_B$  such that  $n/5 \leq |V_A|, |V_B| \leq 4n/5$  and  $|E(V_A, V_B)| = \bar{b}(G)$ . Suppose that  $|V_A| \leq |V_B|$ . If  $n/3 \leq |V_A|$ , then  $b(G) = \bar{b}(G)$  and we are done. So we can assume that  $n/5 \leq |V_A| \leq n/3$  and  $2n/3 \leq |V_B| \leq 4n/5$ .

Let  $H$  be the subgraph of  $G$  induced by  $V_B$ . By definition, there is a partition  $V_B = V_B' \cup V_B''$  such that  $|V_B|/5 \leq |V_B'|, |V_B''| \leq 4|V_B|/5$  and  $|E(V_B', V_B'')| = \bar{b}(H)$ . We can assume that  $|V_B'| \leq |V_B''|$ . Then

$$\frac{n}{3} \leq \frac{|V_B|}{2} \leq |V_B''| \leq \frac{4|V_B|}{5} \leq \frac{16n}{25} < \frac{2n}{3}.$$

Letting  $V_1 = V_A \cup V_B'$  and  $V_2 = V_B''$ , we have  $V(G) = V_1 \cup V_2$ ,  $n/3 \leq |V_1|, |V_2| \leq 2n/3$ ,

$$|E(V_1, V_2)| \leq |E(V_A, V_B)| + |E(V_B', V_B'')| \leq \bar{b}(G) + \bar{b}(H),$$

and the result follows.  $\square$

Theorem 6 is an immediate consequence of Lemma 6.1 and the following statement.

**Theorem 6.2.** *Let  $G$  be a graph with  $n$  vertices of degrees  $d_1, d_2, \dots, d_n$ . Then*

$$\bar{b}(G) \leq 150(1 + g^{3/4}) \sqrt{cr_g(G) + \sum_{i=1}^n d_i^2}.$$

**Proof.** Clearly, we can assume that  $G$  contains no isolated vertices, that is,  $d_i > 0$  for all  $1 \leq i \leq n$ . Consider a drawing of  $G$  on  $S_g$  with exactly  $cr_g(G)$  crossings. Let  $v_1, v_2, \dots, v_n$  be the vertices of  $G$  with degrees  $d_1, d_2, \dots, d_n$ , respectively. Introduce a new vertex at each crossing. Denote the set of these vertices by  $V_0$ . Replace each  $v_i \in V(G)$  ( $i = 1, 2, \dots, n$ ) by a set  $V_i$  of vertices forming a  $d_i \times d_i$  piece of a square grid, in which each vertex is connected to its horizontal and vertical neighbors. Let each edge incident to  $v_i$  be hooked up to distinct vertices along one side of the boundary of  $V_i$  without creating any crossing. These  $d_i$  vertices will be called the *special boundary vertices* of  $V_i$ .

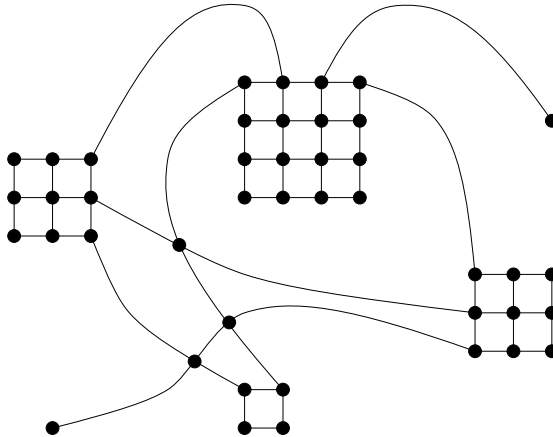


Figure 1.

Thus, we obtain a graph  $H$  of  $\sum_{i=0}^n |V_i| = cr_g(G) + \sum_{i=1}^n d_i^2$  vertices and no crossing (see Fig. 1.). For each  $1 \leq i \leq n$ , assign weight  $1/d_i$  to each special boundary vertex of  $V_i$ . Assign weight 0 to all other vertices of  $H$ . For any subset  $\nu$  of the vertex set of  $H$ , let  $w(\nu)$  denote the total weight of the vertices belonging to  $\nu$ . With this notation,  $w(V_i) = 1$  for each  $1 \leq i \leq n$ . Consequently,  $w(V(H)) = n$ .

Since  $H$  is drawn on  $S_g$  without crossing,  $H$  does not contain  $K_\alpha$  as a minor, where  $\alpha \equiv \lfloor 4 + 4\sqrt{g} \rfloor$  [RY68]. Then, by a result of Alon, Seymour, and Thomas [AST90], the vertices of  $H$  can be partitioned into three sets,  $A$ ,  $B$  and  $C$ , such that  $w(A), w(B) \geq n/3$  and  $|C| \leq 25(1 +$

$g^{3/4})\sqrt{\text{cr}_g(G) + \sum_{i=1}^n d_i^2}$ , and there is no edge from  $A$  to  $B$ . Let  $A_i = A \cap V_i$ ,  $B_i = B \cap V_i$ ,  $C_i = C \cap V_i$  ( $i = 0, 1, \dots, n$ ).

For any  $1 \leq i \leq n$ , we say that  $V_i$  is of *type A* (resp. *type B*) if  $w(A_i) \geq 5/6$  (resp.  $w(B_i) \geq 5/6$ ), and it is of *type C*, otherwise.

Define a partition  $V(G) = V_A \cup V_B$  of the vertex set of  $G$ , as follows. For any  $1 \leq i \leq n$ , let  $v_i \in V_A$  (resp.  $v_i \in V_B$ ) if  $V_i$  is of type A (resp. type B). The remaining vertices,  $\{v_i \mid V_i \text{ is of type C}\}$  are assigned either to  $V_A$  or to  $V_B$  so as to minimize  $\|V_A\| - \|V_B\|$ .

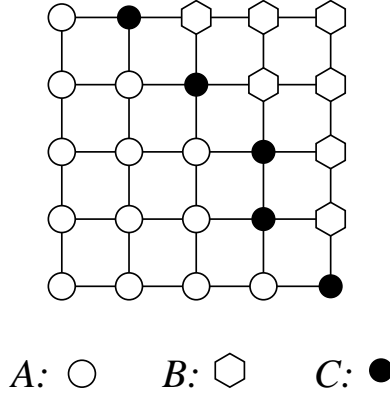
**Claim 1.**  $n/5 \leq |V_A|, |V_B| \leq 4n/5$

To prove the claim, define another partition  $V(H) = \bar{A} \cup \bar{B} \cup \bar{C}$  such that  $\bar{A} \cap V_i = A \cap V_i$  and  $\bar{B} \cap V_i = B \cap V_i$ , for  $i = 0$  and for every  $V_i$  of type C. If  $V_i$  is of type A (resp. type B), then let  $V_i = \bar{A}_i \subset \bar{A}$  (resp.  $V_i = \bar{B}_i \subset \bar{B}$ ), finally, let  $\bar{C} = V(H) - \bar{A} - \bar{B}$ .

For any  $V_i$  of type A,  $w(\bar{A}_i) - w(A_i) \leq w(A_i)/5$ . Similarly, for any  $V_i$  of type B,  $w(\bar{B}_i) - w(B_i) \leq w(B_i)/5$ . Therefore,

$$|w(\bar{A}) - w(A)| \leq \max\{w(A), w(B)\}/5 \leq 2n/15.$$

Hence,  $n/5 \leq w(\bar{A}) \leq 4n/5$  and, analogously,  $n/5 \leq w(\bar{B}) \leq 4n/5$ . In particular,  $|w(\bar{A}) - w(\bar{B})| \leq 3n/5$ . Using the minimality of  $\|V_A\| - \|V_B\|$ , we obtain that  $\|V_A\| - \|V_B\| \leq 3n/5$ , which implies Claim 1.



**Figure 2.**

**Claim 2.** For any  $1 \leq i \leq n$ ,

- (i) if  $V_i$  is of type A (resp. of type B), then  $w(B_i)d_i \leq |C_i|$  (resp.  $w(A_i)d_i \leq |C_i|$ );
- (ii) if  $V_i$  is of type C, then  $d_i/6 \leq |C_i|$ .

In  $V_i$ , every connected component belonging to  $A_i$  is separated from every connected component belonging to  $B_i$  by vertices in  $C_i$ . There are  $w(A_i)d_i$  (resp.  $w(B_i)d_i$ ) special boundary vertices in

$V_i$ , which belong to  $A_i$  (resp.  $B_i$ ). It can be shown by an easy case analysis that the number of separating points  $|C_i| \geq \min\{w(A_i), w(B_i)\}d_i$ , and Claim 2 follows (see Fig. 2).

In order to establish Theorem 6.2 (and hence Theorem 6), it remains to prove the following statement.

**Claim 3.** *The total number of edges between  $V_A$  to  $V_B$  satisfies*

$$|E(V_A, V_B)| \leq 150(1 + g^{3/4}) \sqrt{\text{cr}_g(G) + \sum_{i=1}^n d_i^2}.$$

To see this, denote by  $E_0$  the set of all edges of  $H$  adjacent to at least one element of  $C_0$ . For any  $1 \leq i \leq n$ , define  $E_i \subset E(H)$  as follows. If  $V_i$  is of type  $A$  (resp. type  $B$ ), let  $E_i$  consist of all edges leaving  $V_i$  and adjacent to a special boundary vertex belonging to  $B_i$  (resp.  $A_i$ ). If  $V_i$  is of type  $C$ , let all edges leaving  $V_i$  belong to  $E_i$ .

For any  $1 \leq i \leq n$ , let  $E'_i$  denote the set of edges of  $G$  corresponding to the elements of  $E_i$  ( $0 \leq i \leq n$ ). Clearly, we have  $|E'_i| \leq |E_i|$ , because distinct edges of  $G$  give rise to distinct edges of  $H$ . It is easy to see that every edge between  $V_A$  and  $V_B$  belongs to  $\cup_{i=0}^n E'_i$ .

Obviously,  $|E'_0| \leq |E_0| \leq 4|C_0|$ . By Claim 2, if  $V_i$  is of type  $A$  or of type  $B$ , then  $|E'_i| \leq |E_i| \leq |C_i|$ . If  $V_i$  is of type  $C$ , then  $|E'_i| \leq |E_i| = d_i \leq 6|C_i|$ . Therefore,

$$|E(V_A, V_B)| \leq |\cup_{i=0}^n E'_i| \leq \sum_{i=0}^n |E_i| \leq 6|C| \leq 150(1 + g^{3/4}) \sqrt{\text{cr}_g(G) + \sum_{i=1}^n d_i^2}.$$

This concludes the proof of Claim 3 and hence Theorem 6.2 and Theorem 6.  $\square$

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## References

- [ACNS82] M. Ajtai, V. Chvátal, M. Newborn, and E. Szemerédi, Crossing-free subgraphs, in: *Theory and Practice of Combinatorics, North-Holland Math. Stud.* **60**, North-Holland, Amsterdam-New York, 1982, 9–12.
- [ARS98] N. Alon, L. Rónyai, and T. Szabó, Norm-graphs: variations and applications, to appear.
- [AST90] N. Alon, P. Seymour, and R. Thomas, A separator theorem for nonplanar graphs, *J. Amer. Math. Soc.* **3** (1990), 801–808.
- [AST94] N. Alon, P. Seymour, and R. Thomas, Planar separators, *SIAM J. of Discrete Math.* **7** (1994), 184–193.

- [Be66] C. Benson, Minimal regular graphs of girth eight and twelve, *Canadian J. Mathematics* **18** (1966), 1091–1094.
- [BS74] J. A. Bondy and M. Simonovits, Cycles of even length in graphs, *J. Combinatorial Theory, Ser. B* **16** (1974), 97–105.
- [B66] W. G. Brown, On graphs that do not contain a Thomsen graph, *Canadian Mathematical Bulletin* **9** (1966), 281–285.
- [D98] T. K. Dey, Improved bounds for planar  $k$ -sets and related problems, *Discrete and Computational Geometry* **19** (1998), 373–382.
- [EG73] P. Erdős and R. K. Guy, Crossing number problems, *American Mathematical Monthly* **80** (1973), 52–58.
- [ER62] P. Erdős and A. Rényi, On a problem in the theory of graphs, (in Hungarian), *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **7** (1962), 623–641.
- [F96] Z. Füredi, New asymptotics for bipartite Turán numbers, *J. Combinatorial Theory, Ser. A* **75** (1996), 141–144.
- [GJ83] M. R. Garey and D. S. Johnson, Crossing number is NP-complete, *SIAM J. Alg. Disc. Meth.* **4** (1983), 312–316.
- [G72] R. K. Guy, Crossing numbers of graphs, in: *Graph theory and applications (Proc. Conf. Western Michigan Univ., Kalamazoo, Mich., 1972)* Lecture Notes in Mathematics **303**, Springer, Berlin, 111–124.
- [KST54] T. Kővári, V. T. Sós, and P. Turán, On a problem of K. Zarankiewicz, *Colloquium Mathematicum* **3** (1954), 50–57.
- [L83] F. T. Leighton, *Complexity Issues in VLSI*, MIT Press, Cambridge, 1983.
- [L84] F. T. Leighton, New lower bound techniques for VLSI, *Math. Systems Theory* **17** (1984), 47–70.
- [LT79] R. Lipton and R. Tarjan, A separator theorem for planar graphs, *SIAM J. Applied Mathematics* **36** (1979), 177–189.
- [PSS96] J. Pach, F. Shahrokhi, and M. Szegedy, Applications of the crossing number, *Algorithmica* **16** (1996), 111–117.
- [PS98] J. Pach and M. Sharir, On the number of incidences between points and curves, *Combinatorics, Probability and Computing* **7** (1998), 121–127.
- [PT97] J. Pach and G. Tóth, Graphs drawn with few crossings per edge, *Combinatorica* **17** (1997), 427–439.
- [R58] I. Reiman, Über ein Problem von K. Zarankiewicz, *Acta Math. Acad. Sci. Hungar.* **9** (1958), 269–279.
- [RT97] R. B. Richter and C. Thomassen, Relations between crossing numbers of complete and complete bipartite graphs, *American Mathematical Monthly*, **104** (1997), 131–137.

- [RY68] G. Ringel and J. W. T. Youngs, Solution of the Heawood map-coloring problem, *Proc. Nat. Acad. Sci. U.S.A.* **60** (1968), 438–445.
- [S97] M. Simonovits, personal communication.
- [S66] R. Singleton, On minimal graphs of maximum even girth, *J. Combinatorial Theory* **1** (1966), 306–332.
- [S98] L. A. Székely, Crossing numbers and hard Erdős problems in discrete geometry, *Combinatorics, Probability and Computing* **6** (1998), 353–358.
- [W91] R. Wenger, Extremal graphs with no  $C_4$ 's,  $C_6$ 's, or  $C_{10}$ 's, *J. Combinatorial Theory, Ser. B* **52** (1991), 113–116.
- [WB78] A. T. White and L. W. Beineke, Topological graph theory, in: *Selected Topics in Graph Theory* (L. W. Beineke and R. J. Wilson., eds.), Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1983, 15–49.