A generalization of quasi-planarity

János Pach, Radoš Radoičić, and Géza Tóth

ABSTRACT. A topological graph is a graph drawn in the plane by simple Jordan arcs. Suppose that a topological graph G has no $k+2$ edges such that the first two cross each other and the remaining k edges. Then G has at most $C_k n$ edges, for a suitable constant C_k.

1. Introduction

A *topological graph* is a graph drawn in the plane so that its vertices are represented by points in the plane and its edges by simple (non-selfintersecting) Jordan arcs connecting the corresponding points and not passing through any vertex other than its endpoints. Throughout this paper, we assume that if two edges of a topological graph G share an interior point, then they properly cross at this point. We also assume, for simplicity, that no three edges cross at the same point and that any two edges cross only a finite number of times. Let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. We will make no notational distinction between the vertices (edges) of the underlying abstract graph, and the points (arcs) representing them in the plane. If the edges of G are represented by straight-line segments, then G is called a *geometric graph.* If, in addition, the vertices are in convex position, then G is said to be a *convex* geometric graph.

It follows from Euler's Polyhedral Formula that if a topological graph with n vertices has no pair of crossing edges, then its number of edges cannot exceed $3n-6$. It is conjectured that for every fixed k the maximum number of edges that a topological graph of n vertices can have without containing k pairwise crossing edges is $O(n)$, where the constant hidden in the O-notation depends on k. For $k=3$, this conjecture has been verified in [**AAPPS97**] for geometric graphs and in [**PRT03**] in full generality. For larger values of k, the best upper bound known for the number of edges is n times a polylogarithmic factor [**V97**], [**PRT03**]. For convex geometric graphs G with $n \geq 2k$ vertices, Capoyleas and Pach [**CP92**] found

2000 *Mathematics Subject Classification.* Primary 05C10, 52C45; Secondary 05C35.

János Pach has been supported by NSF Grant CCR-00-98246, by PSC-CUNY Research Award 65392-0034, and by OTKA T-032452.

Radoš Radoičić and Géza Tóth has been supported by OTKA-T-038397 and by an award from the New York University Research Challenge Fund.

an exact formula: if G has no k pairwise crossing edges, then

$$|E(G)| \leq 2(k-1)n - \binom{2k-1}{2},$$

and this bound can be attained.

In [**PPST03**], it was shown that if a topological graph G with n vertices has no $k \times k$ grid-like pattern, i.e., it has no $2k$ edges so that each of the first k edges crosses the remaining ones, then $|E(G)| = O(n)$.

The aim of the present note is to make another, albeit small, step towards the solution of the above problem, by proving

THEOREM. *For any fixed positive integer k, there exists a constant C_k with the property that in every topological graph with n vertices and more than $C_k n$ edges there are $k+2$ edges such that the first 2 cross each other and the remaining k edges.*

In the literature, topological graphs with no k pairwise crossing edges are often called *k-quasi-planar*.

2. Proof of Theorem.

Let G be a topological graph with n vertices, containing no $k+2$ edges such that the first 2 cross each other and the remaining k edges, $k > 1$. We may assume without loss of generality that the underlying abstract graph of G is connected, because otherwise the theorem follows by induction on the number of vertices. Redraw G, if necessary, without creating a forbidden $(k+2)$-tuple of edges, so that the number of crossings in the resulting topological graph $\tilde{G}$ is as small as possible. Obviously, no edge of $\tilde{G}$ can cross itself, otherwise we could reduce the number of crossings by removing the loop. Suppose that $\tilde{G}$ has two distinct edges that have at least two points in common. A region enclosed by two pieces of the participating edges is called a *lens*.

CLAIM 1. *Every lens of $\tilde{G}$ contains a vertex.*

PROOF. Suppose there is a lens ℓ that contains no vertex of $\tilde{G}$. Consider a *minimal* lens $\ell' \subseteq \ell$, by containment. Notice that by swapping the two sides of ℓ', we could reduce the number of crossings without creating any new pair of crossing edges. $\square$

The following property is a direct consequence of a result of Schaefer and Stefankovič on string graphs [**SS01**].

CLAIM 2. *For any edge e of $\tilde{G}$ and for any $m > 0$, every set of 2^m consecutive crossings along e involves at least m distinct edges other than e.* $\square$

Let $e_1, e_2, \ldots, e_{n-1} \in E(G)$ be a sequence of edges such that $e_1, e_2, \ldots, e_i$ form a tree $T_i \subseteq G$ for every $1 \leq i \leq n-1$. In particular, $e_1, e_2, \ldots, e_{n-1}$ form a spanning tree $T := T_{n-1}$ of G.

First, we construct a sequence of crossing-free topological graphs (trees), $\tilde{T}_1$, $\tilde{T}_2$, …, $\tilde{T}_{n-1}$, as follows. Let $\tilde{T}_1$ be defined as a topological graph of two vertices, consisting of the single edge e_1 (as was drawn in $\tilde{G}$). Suppose that $\tilde{T}_i$ has already been defined for some $i \geq 1$, and let v denote the endpoint of e_{i+1} that does not belong to T_i. Now add to $\tilde{T}_i$ the piece of e_{i+1} between v and its first crossing with $\tilde{T}_i$. More precisely, follow the edge e_{i+1} from v up to the point v' where it hits $\tilde{T}_i$

for the first time, and denote this piece of e_{i+1} by $\tilde{e}_{i+1}$. If v' is a vertex of $\tilde{T}_i$, then add v and $\tilde{e}_{i+1}$ to $\tilde{T}_i$ and let $\tilde{T}_{i+1}$ be the resulting topological graph. If v' is in the interior of an edge e of $\tilde{T}_i$, then introduce a new vertex at v'. It divides e into two edges, e' and e''. Add both of them to $\tilde{T}_i$, and delete e. Also add v and $\tilde{e}_{i+1}$, and let $\tilde{T}_{i+1}$ be the resulting topological graph.

After $n-2$ steps, we obtain a topological tree $\tilde{T} := \tilde{T}_{n-1}$, which (1) is crossing-free, (2) has fewer than $2n$ vertices, (3) contains each vertex of $\tilde{G}$, and (4) has the property that each of its edges is either a *full edge*, or a *piece of an edge* of $\tilde{G}$.

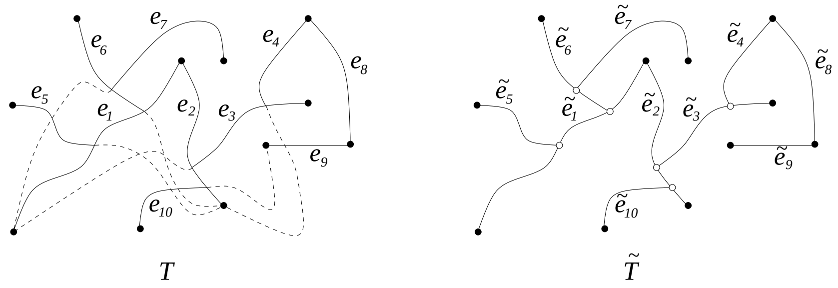

FIGURE 1. Constructing $\tilde{T}$ from T

Let D denote the open region obtained by removing from the plane every point belonging to $\tilde{T}$. Define a *directed convex* geometric graph H, as follows. Traveling around the boundary of D in clockwise direction, we encounter two kinds of different "features": vertices and edges of $\tilde{T}$. Represent each such feature by a different vertex x_i of H, in clockwise order in convex position. Note that the same feature will be represented by several x_i's: every edge will be represented twice, because we visit both of its sides, and every vertex will be represented as many times as its degree in $\tilde{T}$. It is not hard to see that the number of vertices $x_i \in V(H)$ does not exceed $8n$.

Next, we define the edges of H. Let $E = E(\tilde{G} \setminus T)$ be the set of edges of $\tilde{G} \setminus T$. Direct the elements of E arbitrarily. Every edge $e \in E$ may cross $\tilde{T}$ at several points. These crossing points divide e into several pieces, called *segments*. Let $\mathcal{S}$ denote the set of all directed segments of all edges $e \in E$. With the exception of its endpoints, every segment $s \in \mathcal{S}$ runs in the region D. Both the starting point and endpoint of s belong to a feature along the boundary of D, represented by two vertices of H, x_i and x_j, respectively. Connect x_i and x_j by a straight-line edge $\overrightarrow{x_i x_j} \in E(H)$, directed from x_i to x_j.

Notice that H has no loops, because if $x_i = x_j$, then, using the fact that $\tilde{T}$ is connected, one can easily conclude that the lens enclosed by s and by the edge of $\tilde{T}$ corresponding to x_i has no vertex of $\tilde{G}$ in its interior. This contradicts Claim 1.

Of course, several different segments may give rise to the same directed edge $\overrightarrow{x_i x_j} \in E(H)$. Two such segments are said to be of the *same type.*

CLAIM 3. *(i) H has no $k+2$ pairwise crossing edges.*

(ii) $|E(H)| < 32(k+1)n$, i.e., the number of different types of segments is smaller than $32(k+1)n$.

PROOF. To prove part (i), observe that if two edges of H cross each other, then the "features" of $\tilde{T}$ corresponding to their endpoints alternate in the clockwise order around the boundary of D. Therefore, if H had $k+2$ pairwise crossing edges, they would correspond to $k+2$ pairwise crossing edges in E, which is a contradiction.

Part (ii) immediately follows from part (i) and the Capoyleas-Pach theorem [**CP92**] quoted in the introduction, because we have

$$|E(H)| \leq 2\left(2(k+1)|V(H)| - \binom{2k+3}{2}\right) < 32(k+1)n.$$

Note that each undirected edge corresponds to two directed edges. $\square$

For any two oriented edges, a and b, crossing at some point X, we say that *a crosses b from left to right* if the direction of a at X can be obtained from the direction of b at X by a clockwise turn of less than π.

Consider a directed edge $\overrightarrow{x_i x_j} \in E(H)$, where x_i, the tail, represents a feature f_i (edge or vertex) along the boundary of $\mathbf{R}^2 \setminus \tilde{T}$. Consider all segments of type $\overrightarrow{x_i x_j}$. If f_i is a vertex v, then order all segments according to the counterclockwise order as they emanate from v. If f_i is an edge, take an orientation of f_i such that f_i crosses each of the segments (more precisely, the corresponding edges) from left to right, and order the segments according to the order they cross f_i.

This order is called the *tail order* of the segments of a given type.

A segment $s \in \mathcal{S}$ is said to be *shielded* if there are at least $5k4^k$ segments of the same type, belonging to different edges of E, *preceding* s and at least $5k4^k$ such edges coming *after* s in the tail order. Otherwise, s is called *exposed*. An *edge* $e \in E$ is called *exposed* if at least one of its segments is exposed. Otherwise, it is *shielded*.

In view of Claim 3 (ii), there are fewer than $32(k+1)n$ different types of segments. The maximum number of exposed segments of a given type which belong to different edges is $10k4^k + 2$. Thus, we have obtained

Claim 4. *The number of exposed edges of $E = E(\tilde{G} \setminus T)$ is at most $320(k+1)^2 4^k n$.* $\square$

In order to prove the theorem, it is sufficient to show that there are no shielded edges (Claim 6).

Consider a shielded edge $e = \overrightarrow{uv} \in E$. Let $e_1, e_2, \ldots, e_{m-1}$ denote the edges of $\tilde{T}$ cutting e, listed according to the orientation of e. They cut e into m segments, $s_1, s_2, \ldots, s_m$, of types $\tau_1, \tau_2, \ldots, \tau_m$, respectively.

For a fixed $1 \leq i \leq m$, take a segment r of type $\mathrm{type}(r) = \tau_i$ that belongs to a directed edge $f \in E$. Consider all segments of type τ_i strictly between s_i and r in the tail order, and assume that they altogether belong to $d(r)$ different edges of E. We need a simple relation between the values of d for two consecutive segments along the same edge. If $i < m$ ($i > 1$, resp.), then let r^+ (resp., r^-) denote the segment immediately following (resp., immediately preceding) r along the directed edge f.

Claim 5. *Suppose that $d(r) \leq 4k4^k$.*
(i) If $i < m$, then $\mathrm{type}(r^+) = \tau_{i+1}$ *and $d(r^+) \leq d(r) + 2k$.*
(ii) If $i > 1$, then $\mathrm{type}(r^-) = \tau_{i-1}$ *and $d(r^-) \leq d(r) + 2k$.*

Proof. By symmetry, it is sufficient to prove part (i). We can assume without loss of generality that s_i precedes r in the tail order of all segments of type τ_i.

Let α and β denote the heads of r and s_i, respectively. Assume without loss of generality that α comes after β in the tail order along e_i. (The other case can be treated similarly.) Let $\mathcal{B}$ and $\mathcal{A}$ denote the sets of segments of type τ_{i+1} that come *before* s_{i+1} and *after* s_{i+1}, resp., in the tail order. Furthermore, let $\mathcal{A}_1$ (and $\mathcal{A}_2$) consist of all elements of $\mathcal{A}$ whose tails lie strictly between α and β (come after α, respectively).

Suppose, in order to obtain a contradiction, that type$(r^+) \neq \tau_{i+1}$. It is easy to see, using Claim 1, that now r^+ must cross either all elements of $\mathcal{B}$ or all elements of $\mathcal{A}_2$ (see Fig. 2).

In the first case, note that, since e (and therefore s_{i+1}) is shielded, the segments in $\mathcal{B}$ belong altogether to at least $5k4^k > k$ different edges of E. These edges, together with e_i and f, would form a forbidden configuration.

So we are left with the case when r^+ intersects all segments in $\mathcal{A}_2$. If these segments belong to at least k different edges of E, then again we are done. If they belong to fewer than k edges, then the elements of $\mathcal{A}_1$ must belong altogether to more than $5k4^k - k$ different edges. All of these edges leave the quadrilateral enclosed by e_{i-1}, e_i, r, and s_i, through its side lying on e_i. If at least k of them cross f or at least k of them cross e, then they, together with e_i and f (resp., e), form a forbidden configuration. Therefore, all but at most $2k-2$ of them must once enter the quadrilateral through e_{i-1}. However, in this case, there are at least $5k4^k - k + 1 - (2k-2) > 4k4^k$ different edges containing a segment of type τ_i that lies between s_i and r. That is, we have $d(r) > 4k4^k$, contradicting our assumption.

Thus, we have shown that type$(r^+) = \tau_{i+1}$. Recall that $\mathcal{A}_1$ denotes the set of all segments of this type that lie strictly between s_{i+1} and r^+ in the tail order of all edges of this type. Then $d(r^+)$ is equal to the number of different edges that contribute at least one segment to $\mathcal{A}_1$. Suppose, in order to obtain a contradiction, that $d(r^+) > d(r) + 2k$, i.e., there are more than $d(r) + 2k$ different edges in E that leave the quadrilateral enclosed by e_{i-1}, e_i, r, and s_i, through its side that belongs to e_i. Just like before, we find that at most $k-1$ of them can cross r and at most $k-1$ can cross s_i. Therefore, more than $d(r) + 2k - 2(k-1) > d(r)$ edges must cross the side belonging to e_{i-1}. This contradicts the definition of $d(r)$. $\square$

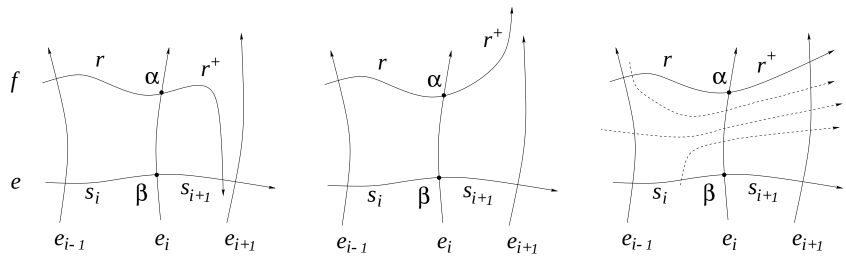

FIGURE 2. type$(r^+) = \tau_{i+1}$ and $d(r^+) \leq d(r) + 2k$.

Now we are in a position to complete the proof of the theorem by proving the following assertion.

CLAIM 6. *There are no shielded edges in E.*

PROOF. As before, suppose that there exists a shielded edge $e = \overrightarrow{uv}$ with segments s_i of type τ_i $(1 \leq i \leq m)$. Consider all segments of type τ_1. The tail of each of them is u, so they cannot cross one another, by Claim 1.

Let t_1 be the segment that follows immediately after s_1 in the tail order of all segments of type τ_1, so that we have $d(t_1) = 0$. Denote by g the edge of E

that contains t_1, and let $t_2, \ldots, t_\nu$ denote the other segments of g. By repeated application of Claim 5 (i), we obtain that t_i is of type τ_i for all $1 \le i \le 4^k$. Consequently, we have $4^k < m$, $4^k < \nu$, and $d(t_{4^k}) \le 2k4^k$.

The graph G has no parallel edges, so the endpoint (head) of g is different from v, the endpoint of e. Thus, there exists a smallest integer μ ($4^k < \mu$), for which the type of t_μ is *not* τ_μ. By Claim 5 (i), we have $d(t_{\mu-1}) > 4k4^k$. Choose an integer λ ($4^k \le \lambda < \mu$) such that $d(t_\lambda) \le 2k4^k$ and $d(t_\lambda) < d(t_{\lambda+1})$. Consider all edges that have a segment of type $\tau_{\lambda+1}$ between $s_{\lambda+1}$ and $t_{\lambda+1}$ in the tail order. All of these edges leave the quadrilateral Q enclosed by $e_{\lambda-1}$, e_λ, s_λ, and t_λ through its side belonging to e_λ. Since $d(t_\lambda) < d(t_{\lambda+1})$, at least one of them must have entered Q either through s_λ or through t_λ. Suppose, for instance, that there is such an edge $f \in E$ intersecting t_λ (the other case can be handled similarly). Let $r_1, r_2, \ldots$ denote the segments of f. Then f intersects g and, for some κ, the segment $r_{\kappa+1}$ is of type $\tau_{\lambda+1}$ (see Fig. 3).

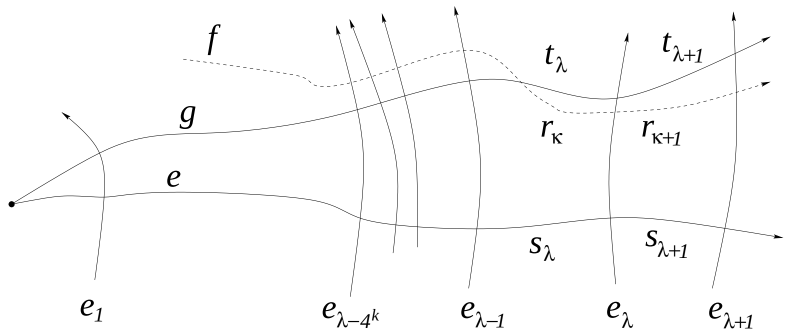

FIGURE 3. f and g cross each other and both of them cross $e_\lambda, e_{\lambda-1}, \ldots, e_{\lambda-4^k}$.

Since

$$d(r_{\kappa+1}) \le d(t_{\lambda+1}) \le 2k4^k + 2k,$$

by repeated application of Claim 5 (ii), we can conclude that $\kappa \ge 4^k$ and for $0 \le i < 4^k$, type$(r_{\kappa-i}) = \tau_{\lambda-i}$. In particular, we obtain that f and g cross each other and both of them cross $e_\lambda, e_{\lambda-1}, \ldots, e_{\lambda-4^k}$. Let

$$\bar{g} = \bigcup_{i=1}^{4^k-1} t_{\lambda-i}.$$

We show that any edge that crosses $\bar{g}$, must also cross either f, or $e_{\lambda-4^k}$. For $0 < i < 4^k$, let Q_i be the quadrilateral bounded by $e_{\lambda-i}$, $t_{\lambda-i}$, $e_{\lambda-i-1}$, and $r_{\kappa-i}$. Let Q_0 be the triangular region bounded by $e_{\lambda-1}$, t_λ, and r_κ (recall that, by assumption, t_λ and r_κ cross each other.) Suppose that there is an edge h that intersects $\bar{g}$ but does not intersect f. Then, for some j, $0 < j < 4^k$, h crosses $t_{\lambda-j}$, so, depending on its orientation, h enters or leaves Q_j, through its side $t_{\lambda-j}$. Suppose that it *enters* Q_j through $t_{\lambda-j}$, the other case is analogous. Since there is no vertex in Q_j, h must leave it through one of its sides. It cannot leave through $t_{\lambda-j}$, because then h and $t_{\lambda-j}$ would form an empty lens, contradicting Claim 1. It cannot leave through $r_{\kappa-j}$, since h does not cross f. So it leaves either through $e_{\lambda-j-1}$, or through $e_{\lambda-j}$.

Case 1: h leaves Q_j through $e_{\lambda-j}$. Then h enters Q_{j-1}. It cannot leave Q_{j-1} through $e_{\lambda-j}$ or through $t_{\lambda-j+1}$, because then it would create an empty lens, and it cannot leave through $r_{\kappa-j+1}$, since h does not cross f. Therefore, h must leave Q_{j-1} through $e_{\lambda-j+1}$, and then it must enter Q_{j-2}. By repeated application of the above argument, we conclude that h enters Q_0 through $e_{\lambda-1}$. However, it cannot leave Q_0 through any of its sides, which is a contradiction.

Case 2: h leaves Q_j through $e_{\lambda-j-1}$. Then it enters Q_{j+1}. We can argue exactly as in Case 1 that h crosses $e_{\lambda-j-2}$, $e_{\lambda-j-3}$, and eventually it must cross $e_{\lambda-4^k}$.

We know that there are at least 4^k crossings on $\bar{g}$, so by Claim 2 they correspond to at least $2k$ different edges. Each of them also crosses either f or $e_{\lambda-4^k}$. Thus, either at least k of them cross f or at least k of them cross $e_{\lambda-4^k}$. These k edges, together with g, and either with f or with $e_{\lambda-4^k}$, would form a forbidden configuration. □

References

[AAPPS97] P. K. Agarwal, B. Aronov, J. Pach, R. Pollack, and M. Sharir: Quasi-planar graphs have a linear number of edges, *Combinatorica* **17** (1997), 1–9.

[CP92] V. Capoyleas and J. Pach: A Turán-type theorem on chords of a convex polygon, *Journal of Combinatorial Theory, Series B* **56** (1992), 9–15.

[PPST03] J. Pach, R. Pinchasi, M. Sharir, and G. Tóth: Topological graphs with no large grids, to appear.

[PPTT02] J. Pach, R. Pinchasi, G. Tardos, and G. Tóth: Geometric graphs with no self-intersecting path of length three, in: Graph Drawing (M. T. Goodrich, S. G. Kobourov, eds.), Lecture Notes in Computer Science **2528**, Springer-Verlag, Berlin, 2002, 295–311.

[PRT03] J. Pach, R. Radoičić, and G. Tóth: Relaxing planarity for topological graphs, to appear.

[SS01] M. Schaefer and D. Stefankovič, Decidability of string graphs, in: *Proceedings of the 33rd Annual Symposium on the Theory of Computing (STOC 2001)*, 2001, 241–246.

[V97] P. Valtr: Graph drawing with no k pairwise crossing edges, in: *Graph Drawing (Rome, 1997), Lecture Notes in Comput. Sci.* **1353**, Springer, Berlin, 1997, 205–218.

City College, CUNY and Courant Institute of Mathematical Sciences, New York University, New York, NY, USA

E-mail address: pach@cims.nyu.edu

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA, USA

E-mail address: rados@math.mit.edu

Rényi Institute, Hungarian Academy of Sciences, Budapest, HUNGARY

E-mail address: geza@renyi.hu