A better bound for the pair-crossing number

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Abstract

The crossing number $\text{CR}(G)$ of a graph G is the minimum possible number of edge-crossings in a drawing of G, the pair-crossing number $\text{PAIR-CR}(G)$ is the minimum possible number of crossing pairs of edges in a drawing of G. Clearly, $\text{PAIR-CR}(G) \leq \text{CR}(G)$. We show that for any graph G, $\text{CR}(G) = O(\text{PAIR-CR}(G)^{7/4} \log^{3/2}(\text{PAIR-CR}(G)))$.

1 Introduction

In a *drawing* of a graph G vertices are represented by points and edges are represented by Jordan curves, in a plane, connecting the corresponding points. We assume that the edges do not pass through vertices, any two edges have finitely many common points and each of them is either a common endpoint, or a proper crossing. We also assume that no three edges cross at the same point.

The *crossing number* $\text{CR}(G)$ is the minimum number of edge-crossings (i. e. crossing points) over all drawings of G. The *pair-crossing number* $\text{PAIR-CR}(G)$ is the minimum number of crossing pairs of edges over all drawings of G. Clearly, for any graph G we have

$$\text{PAIR-CR}(G) \leq \text{CR}(G).$$

It is still an exciting open question whether $\text{CR}(G) = \text{PAIR-CR}(G)$ holds for all graphs G.

Pach and Tóth [PT00a] proved that $\text{CR}(G)$ cannot be arbitrarily large if $\text{PAIR-CR}(G)$ is bounded, namely, for any G, if $\text{PAIR-CR}(G) = k$, then $\text{CR}(G) \leq 2k^2$. Valtr [V05] managed to improve this bound to $\text{CR}(G) \leq 2k^2/\log k$. Based on the ideas of Valtr, the present author [T08] improved it to $\text{CR}(G) \leq 9k^2/\log^2 k$.

In this note, using a different approach, we obtain a further improvement.

Theorem. *For any graph G, if $\text{PAIR-CR}(G) = k$, then $\text{CR}(G) = O(k^{7/4} \log^{3/2} k)$.*

For the proof we need some results about *string graphs*. These are introduced in Section 2. In Section 3 we give the short proof of the Theorem. There are many other versions of the crossing number, for a survey see [BMP05], [PSS10] and [PT00b].

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2 String graphs

A string graph is the intersection graph of continuous arcs in the plane. More precisely, vertices of the graph correspond to continous curves (strings) in the plane such that two vertices are connected by an edge if and only if the corresponding strings intersect each other.

Suppose that $G(V, E)$ is a graph of n vertices. A *separator* in a graph G is subset $S \subset V$ for which there is a partition $V = S \cup A \cup B$, $|A|, |B| \leq 2n/3$, and there is no edge between A and B. According to the Lipton-Tarjan separator theorem, [LT79], every planar graph has a separator of size $O(\sqrt{n})$. This result has been generalized in several directions, for graphs drawn on a surface of bounded genus, graphs with a forbidden minor, intersection graphs of balls in the d dimensional space, intersection graphs of Jordan regions, intersection graphs of convex sets in the plane, and finally, for string graphs [FP08], [FP10].

Theorem A. [FP10] *There is a constant c such that for any string graph G with m edges, there is a separator of size at most $cm^{3/4}\sqrt{\log m}$.*

3 Proof of Theorem

Let c be the constant in Theorem A. In a drawing $\mathcal{D}$ of a graph G in the plane, call those edges which participate in a crossing *crossing edges*, and those which do not participate in a crossing *empty edges*.

Lemma. *Suppose that $\mathcal{D}$ is a drawing of a graph G in the plane with $l > 0$ crossing edges and $k > 0$ crossing pairs of edges. Then G can be redrawn such that (i) empty edges are drawn the same way as before, (ii) crossing edges are drawn in the neighborhood of the original crossing edges, and (iii) there are at most $6ck^{7/4}\log^{3/2} l$ edge crossings.*

Proof of Lemma. The proof is by induction on l. For $l = 1$ the statement is trivial. Suppose that the statement has been proved for all pairs (l', k'), where $l' < l$ and consider a drawing of G with k crossing pairs of edges, such that l edges participate in a crossing. Obviously, $\binom{l}{2} \geq k$, and $2k \geq l$, therefore, $2k \geq l > \sqrt{k}$.

Let V denote the vertex set of G and let E resp. F denote the set of empty resp. crossing edges of G. We define a string graph H as follows. The vertex set $\overline{F}$ of H corresponds to the crossing edges of G. Two vertices are connected by an edge if the corresponding edges cross each other. Note that the endpoints do not count; if two edges do not cross, the correspondig vertices are not connected even if the edges have a common endpoint. The graph H is a string graph, it can be represented by the crossing edges of G, as strings, with their endpoints removed. It has l vertices, and k edges. By Theorem A, H has a separator of size $ck^{3/4}\sqrt{\log k}$ that is, the vertices can be decomposed into three sets, $\overline{F}_0$, $\overline{F}_1$, $\overline{F}_2$, such that (i) $|\overline{F}_0| \leq ck^{3/4}\sqrt{\log k}$, (ii) $|\overline{F}_1|, |\overline{F}_2| \leq 2l/3$, (iii) there is no edge of H between $\overline{F}_1$ and $\overline{F}_2$.

This corresponds to a decomposition of the set of crossing edges F into three sets, F_0, F_1, and F_2 such that (i) $|F_0| \leq ck^{3/4}\sqrt{\log k}$, (ii) $|F_1|, |F_2| \leq 2l/3$, (iii) in drawing $\mathcal{D}$, edges in F_1 and in F_2 do not cross each other.

For $i = 0, 1, 2$, let $|F_i| = l_i$. Let $G_1 = G(V, E \cup F_1)$ and $G_2 = G(V, E \cup F_2)$, then in the drawing $\mathcal{D}$ of the graph G_i has l_i crossing edges. Denote by k_i the number of crossing pairs of edges of G_i in drawing $\mathcal{D}$. Then we have $k_1 + k_2 \leq k$, $l_1, l_2 \leq 2l/3$, $l_1 + l_2 + l_0 = l$.

For $i = 1, 2$, apply the induction hypothesis for G_i and drawing $\mathcal{D}$. We obtain a drawing $\mathcal{D}_i$ satisfying the conditions of the Lemma: (i) empty edges drawn the same way as before, (ii) crossing edges are drawn in the neighborhood of the original crossing edges, and (iii) there are at most $6ck_i^{7/4}\log^{3/2} l_i$ edge crossings.

Consider the following drawing $\mathcal{D}_3$ of G. (i) Empty edges are drawn the same way as in $\mathcal{D}$, $\mathcal{D}_1$, and $\mathcal{D}_2$, (ii) For $i = 1, 2$, edges in F_i are drawn as in $\mathcal{D}_i$, (iii) Edges in F_0 are drawn as in $\mathcal{D}$. Now count the number of edge crossings (crossing points) in the drawing $\mathcal{D}_3$. Edges in E are empty, edges in F_1 and in F_2 do not cross each other, there are at most $2ck_i^{7/4}\log^{3/2} l_i$ crossings among edges in F_i. The only problem is that edges in F_0 might cross edges in $F_1 \cup F_2$ and each other several times, so we can not give a reasonable upper bound for the number of crossings of this type. Color edges in F_1 and F_2 blue, edges in F_0 red. For any piece p of an edge of G, let BLUE(p) (resp. RED(p)) denote the number of crossings on p with *blue* (resp. *red*) edges of G. We will apply the following transformations.

ReduceCrossings(e, f) Suppose that two crossing edges, e and f cross twice, say, in X and Y. Let e' (resp. f') be the piece of e (resp. f) between X and Y. If BLUE(e') $<$ BLUE(f'), or BLUE(e') $=$ BLUE(f') and RED$(e') \leq$ RED(f'), then redraw f' along e' from X to Y. Otherwise, redraw e' along f' from X to Y. See Figure 1.

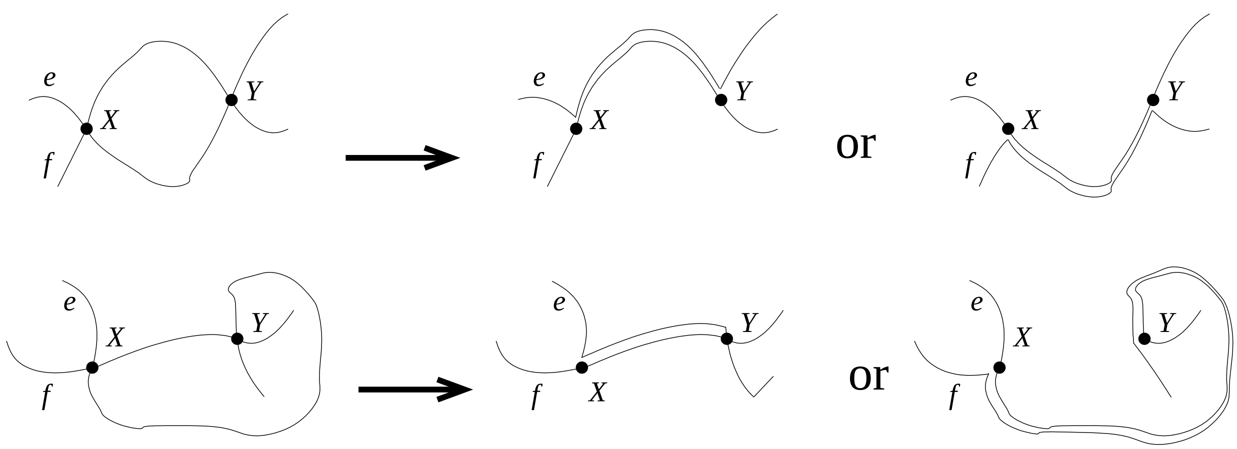

Figure 1: ReduceCrossings(e, f)

Observe that ReduceCrossings might create self-crossing edges, so we need another transformation.

RemoveSelfCrossings(e) Suppose that an edge e crosses itself in X. Then X appears twice on e. Remove the part of e between the first and last appearance of X.

Start with drawing $\mathcal{D}_3$ of G, and apply ReduceCrossings and RemoveSelfCrossings recursively, as long as there are two crossing edges that cross at least twice, or there is a self-crossing edge.

Let BB, (resp. BR, RR) denote the number of blue-blue (resp. blue-red, red-red) crossings in the current drawing of G. Observe, that the triple (BB, BR, RR) lexicographically decreases with each of the transformations. Indeed,

- if e and f are both blue edges then ReduceCrossings(e, f) decreases BB,

- if e is blue and f is red then either BB decreases, or if it stays the same then BR decreases,
- if e and f are both red edges then BB stays the same, and either BR decreases, or if it also stays the same then RR decreases,
- if e is blue then RemoveSelfCrossings(e) decreases BB,
- and finally, if e is red then BB does not change, BR does not increase, and RR decreases.

Therefore, after finitely many steps we arrive to a drawing $\mathcal{D}_4$ of G, where any two edges cross at most once, and (BB, BR, RR) is lexicograhically not larger than originally. That is, in the drawing $\mathcal{D}_4$, $BB \leq 2ck_1^{7/4}\log l_1 + 2ck_2^{7/4}\log l_2$, and any two edges cross at most once, therefore, $BR + RR \leq l_0 l$. So, for the total number of crossings we have

$$6ck_1^{7/4}\log^{3/2} l_1 + 6ck_2^{7/4}\log^{3/2} l_2 + l_0 l$$

$$\leq 6ck_1^{7/4}\sqrt{\log l}\log(2l/3) + 6ck_2^{7/4}\sqrt{\log l}\log(2l/3) + l_0 l$$

$$\leq 6c(k_1^{7/4} + k_2^{7/4})\sqrt{\log l}(\log l + \log(2/3)) + l_0 l$$

$$\leq 6ck^{7/4}\log^{3/2} l - 3ck^{7/4}\sqrt{\log l} + l_0 l$$

$$\leq 6ck^{7/4}\log^{3/2} l - 3ck^{7/4}\sqrt{\log l} + clk^{3/4}\sqrt{\log k}$$

$$\leq 6ck^{7/4}\log^{3/2} l - 3ck^{7/4}\sqrt{\log l} + 2ck^{7/4}\sqrt{\log k}$$

$$\leq 6ck^{7/4}\log^{3/2} l - 3ck^{7/4}\sqrt{\log l} + 3ck^{7/4}\sqrt{\log l}$$

$$= 6ck^{7/4}\log^{3/2} l.$$

□

Now consider a graph G and let $\text{PAIR-CR}(G) = k$. Take a drawing of G with exactly k crossing pairs of edges. Let l be the total number of crossing edges. By the Lemma, G can be redrawn with at most $6ck^{7/4}\log^{3/2} l$ crossings. Since $2k \geq l$, $\text{CR}(G) \leq 6ck^{7/4}\log^{3/2} l < 18ck^{7/4}\log^{3/2} k$. This concludes the proof of the Theorem. □

Acknowledgement. I am very grateful to the anonymous referee for pointing out several typos, errors in the calculation, and for some other useful remarks.

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