Note on a Ramsey-type problem in geometry

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Abstract

There exists a 2-coloring of the plane with red and blue and a configuration K of eight points (a regular heptagon plus center) such that there are no two red points at distance 1 from each other, and every configuration congruent to K has at least one red point. But in this 2-coloring, for every five-point configuration K, there is a translate of K all of whose points are blue.

The investigation of Ramsey-type problems in the Euclidean space was initiated in a series of articles by Erdős, Graham, Montgomery, Rothschild, Spencer and Straus in 1973 [2]. Solving a problem of Erdős (see [3 p.535]), Rozália Juhász proved that given any coloring of the plane by two colors (red and blue), and a four-point configuration K, one can find either two red points at distance 1 from each other or a congruent copy of K all of whose points are blue. However, Juhász also proved that this theorem does not remain true for all configurations K with at least 12 points.

The aim of this note is to find a counterexample with only eight points.

Theorem 1. *There exists a 2-coloring of the plane with red and blue and a configuration K of eight points such that* (i) *there are no two red points at distance 1 from each other;* (ii) *every configuration congruent to K has at least one red point.*

We will use the following 2-coloring of the plane.

Definition. (Standard 2-Coloring). Consider a (fixed) regular triangular-lattice where the minimum distance between two lattice points is 2. A point $P \in R^2$ will be colored red if and only if there is a lattice point whose distance from P is smaller than $1/2$. Every other point will be colored blue.

Lemma. *Given a regular triangular lattice with minimum distance 2, any closed disc of radius $2/\sqrt{3}$ necessarily contains at least one lattice point.*

Proof. The radius of the circumscribed circle of the regular triangle of side 2 is $2/\sqrt{3}$.

Proof of Theorem 1. Consider the standard 2-coloring of the plane. It is clear that there are no two red points at distance 1 from each other. Let $A_1A_2...A_7$ form a regular heptagon with center O of circumscribed radius 0.9. Let $K = \{A_1, A_2, ..., A_7, O\}$.

Assume now, in order to obtain a contradiction, that there is a congruent copy K' of K, all of whose points are colored blue. Without the danger of confusion let us denote the vertices of K' also by $A_1, A_2, ...A_7, O$.

By the definition of the standard 2-coloring, there can be no lattice points in the open discs of radius 1/2 around the elements of K'. The circles of radius 1/2 around $A_1, A_2, ..., A_7$ cover the entire circumference of the circle around O, because $0.9 < \cos(\pi/7)$. Hence these eight discs around the elements of K' all together cover the heptagon $\text{conv}(K')$. On the other hand, by the Lemma, the closed disc of radius $2/\sqrt{3}$ centered at O contains at least one lattice point Z. Hence Z must lie in one of the seven congruent shaded moonlike regions shown in Fig 1 and Fig. 2, say, in the closed region bounded by the circular arcs PR, RS and PS.

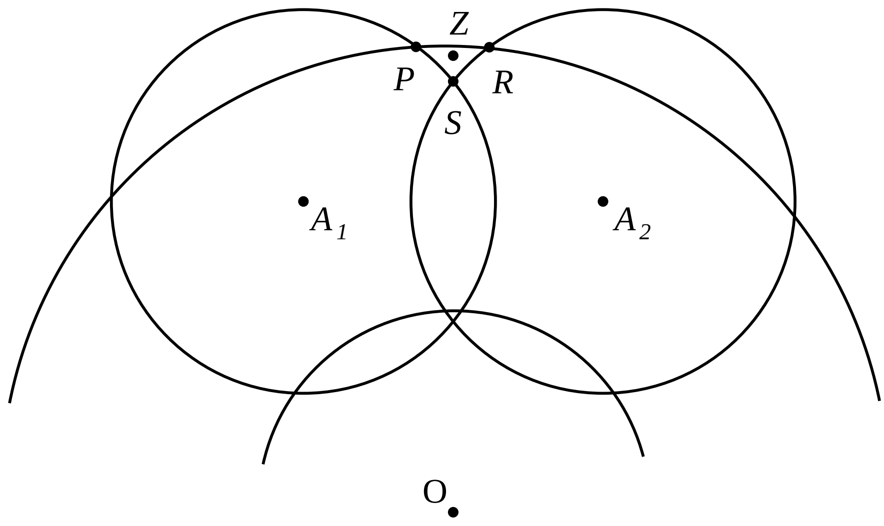

Figure 1

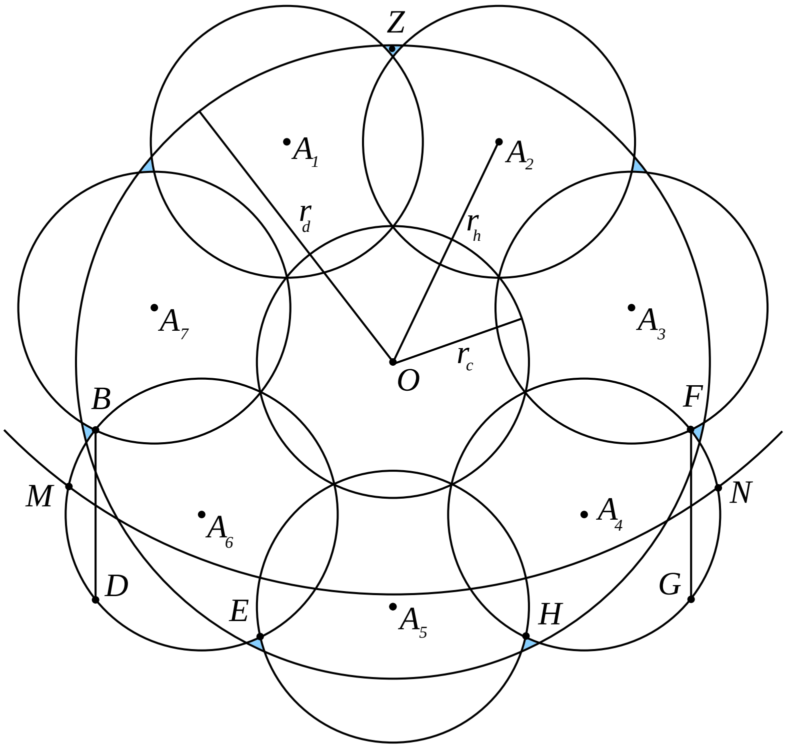

Figure 2

It is easy to see that in this region there is no point whose distance from S is larger than $SP = SR$. Denote the intersection points of the circles around A_7 and A_6, A_6 and A_5, A_5 and A_4, A_4 and A_3 by B, E, H, and F (See Fig.2).

Let D (and G) denote the intersection point of the circle of radius $1/2$ around A_6 (resp. A_4) and the line through B (resp. F) parallel to OS. Some straightforward calculations show that (with $r_h = 0.9$, $r_c = 1/2$, $r_d = 2/\sqrt{3}$)

$$SO = r_h \cos\frac{\pi}{7} + \sqrt{r_c^2 - r_h^2 \sin^2\frac{\pi}{7}} \approx 1.123$$

$$\angle PA_1S = \angle PA_1O - (\pi - \angle OSA_1 - \angle A_1OS)$$

$$= \arccos\frac{r_h^2 + r_c^2 - r_d^2}{2r_h r_c} - \pi + \arcsin\left(\frac{r_h}{r_c}\sin\frac{\pi}{7}\right) + \frac{\pi}{7}$$

$$= \arccos\frac{-41}{135} + \arcsin\left(\frac{9}{5}\sin\frac{\pi}{7}\right) - \frac{6\pi}{7} \approx 0.083,$$

$$SP = SR = 2r_c \sin\frac{\angle PA_1S}{2} \approx 0.041,$$

$$SB = SF = 2SO\sin\frac{2\pi}{7} \approx 1.756,$$

$$SE = SH = 2SO\sin\frac{3\pi}{7} \approx 2.190.$$

Using that $\angle SA_4D = 3\pi/7$ and $\angle SDA_4 = \angle DSA_4 = 2\pi/7$ we get:

$$SD = 2\cos\frac{2\pi}{7}\sqrt{SO^2 + 2r_hSO\cos\frac{2\pi}{7} + r_h^2} \approx 2.276.$$

It is easy to see that $BF = SE$ because we get BF by a rotation around O from SE. Since $BD\|FG$ and $DG = BF = SE$, the arcs BD and FG are separated by the parallel strip between the lines BD and FG whose width is BF. Thus, the minimum distance between the arcs BD and GF, is BF. It is not hard to compute that

$$ZF \le SF + SZ \le SF + SR \approx 1.797 < 2.$$

Similarly, $$ZB \le 1.797 < 2.$$

$$ZD \ge SD - SZ \ge SD - SR \approx 2.234 > 2.$$

Similarly, $$ZG \ge 2.234 > 2.$$

$$ZE \ge SE - SZ \ge SE - SR \approx 2.148 > 2.$$

Similarly, $$ZH \ge 2.148 > 2.$$

Therefore, the circle of radius 2 around Z intersects the arcs BD and FG. Let M and N denote the corresponding intersection points (See Fig.2). The arc MN of this circle is completely covered by the discs of radius 1/2 around the elements of K'. Otherwise MN would intersect one of the arcs ME, EH or HN; however, the nearest points of these arcs to Z are M, E, H and N, and we have already seen that ZE, ZH, ZD and ZG are greater than 2, a contradiction. Since $MN \ge BF > 2$, the union of the discs of radius 1/2 around the elements of K' cover an arc of the circle of radius 2 around Z, whose angle is greater than $\pi/3$. So there is at least one lattice point on this arc (because the circle of radius 2 around Z contains exactly six lattice points). Thus one of the eight open discs of radius 1/2 around the elements of K' contains a lattice point, and the center of this disc must be red. This contradiction completes the proof of Theorem 1.

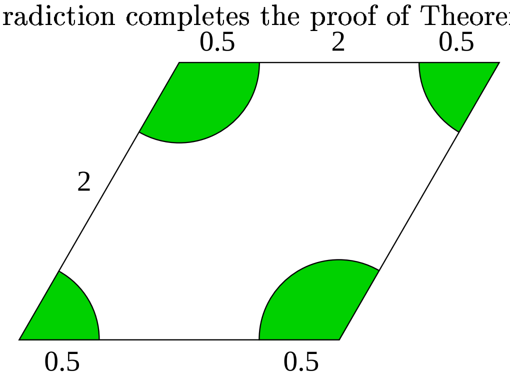

Figure 3

Proposition 2. *Given any five-point configuration $K = (ABCDE)$ in the plane, one can find a translate of K all of whose vertices are blue in the standard 2-coloring.*

Proof of Proposition 2. Suppose that every translate of $ABCDE$ has at least one red point in the standard 2-coloring. Denote the set of the red points by T. Let T_B,T_C, T_D and T_E denote congruent copies of T translated by the vectors BA, CA, DA and EA respectively. We claim that the set $T \cup T_B \cup ... \cup T_E$ covers the whole plane. Let O be any point of the plane. Translate the configuration $ABCDE$ so that A moves into O. According to our assumption, this translate has at least one red point, say $B(= O + AB)$. However, in this case T_B covers O. The set T is periodic, hence it has a density. The density of T (see Fig.3.) is the shaded (red) area divided by the area of the parallelogramm. That is $\pi/8\sqrt{3}$. Of course, $T_B, ..., T_E$ have the same density. Thus the density of the covering $T^* = T \cup T_B \cup ... \cup T_E$ is $5\pi/8\sqrt{3}$. The set T^* consists of congruent circles and covers the plane. It is well-known (see e.g. [6,p. 172]) that if we cover the plane with congruent circles, the density of this covering is at least $2\pi/\sqrt{27}$. But $2\pi/\sqrt{27} > 5\pi/8\sqrt{3}$ a contradiction. This completes the proof. This supports our conjecture that for any coloring and for any 5-point configuration K, one can find either 2 red points at distance 1 from each other or an isometric copy of K all of whose points are blue.

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