Monotone paths in line arrangements

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Abstract

We show that for any n there is an arrangement of n lines which contain an x-monotone path of length $\Omega(n^{7/4})$.

1 Introduction

Properties of line arrangements in the plane (see [PA95]) have been intensively studied, partly because of their importance in the construction and analysis of geometric algorithms (see [E87]). One of the most important and studied such problem is the k-level problem [D98], [T99]. The k-level of an arrangement of n lines is the closure of the the set of points of the lines with the property that there are exactly k lines pass below them. The k-level of a line arrangement is an x-monotone polygon (path) which has a turn in each of the line intersections on it. Its length is the number of turns plus one, which is called the complexity of the k-level. The k-level problem asks for the maximum complexity of the k-level in an arrangement of n lines.

In this note we consider a generalization of this problem, when the polygon does not necessarily have a turn in each of the intersections on it. In other words, we want to find the maximum length of an x-monotone path in an arrangement of n lines in the plane. The length of the path is the number of turns plus one. Sharir (see [EG89], [E87]) established an $\Omega(n^{3/2})$ lower bound. Matoušek [M91] improved it to $\Omega(n^{5/3})$. Yamamoto et. al. [YKII88] found an interesting application of this problem.

Theorem. *For any $n > 0$ there exists an arrangement of n lines which contain a monotone path of length $\Omega(n^{7/4})$.*

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Obviously, there are at most $\binom{n}{2}$ intersection points in any arrangement of n lines, so a monotone path has length at most $\binom{n}{2}+1$. We very slightly improve this trivial upper bound (see Remarks).

2 Proof of Theorem

We construct an arrangement of at most n lines which contain a monotone path of length $\Omega(n^{7/4})$. We define it in three steps. For any arrangement $\mathcal{A}$ of lines, $|\mathcal{A}|$ denotes the number of lines in $\mathcal{A}$.

STEP 1. For any $m > 0$, let $\mathcal{A}_m^1$ be an arrangement of $2m$ lines, arranged into two bundles of m parallel lines, called the row bundle $\mathcal{R}_m^1$ and the column bundle $\mathcal{C}_m^1$. More precisely, let

$$\mathcal{R}_m^1 = \{(y = i) \mid i = 1, 2, \dots m\},$$
$$\mathcal{C}_m^1 = \{(x - y = i) \mid i = 1, 2, \dots m\},$$

and let $\mathcal{A}_m^1 = \mathcal{R}_m^1 \cup \mathcal{C}_m^1$. Clearly, there is a monotone path of length $2m$ in this arrangement (see Fig. 1).

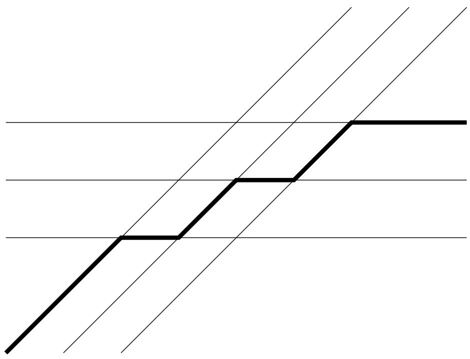

Figure 1. $\mathcal{A}_m^1$

STEP 2. Suppose for simplicity that $\sqrt[4]{m}$ is an integer. We define $\mathcal{A}_m^2$, an arrangement of $3m-1$ lines, arranged into four bundles of parallel lines. Let $\varepsilon > 0$ very small. $\mathcal{A}_m^2 = \mathcal{R}_m^2 \cup \mathcal{C}_m^2 \cup \mathcal{U}_m^2 \cup \mathcal{V}_m^2$. $\mathcal{R}_m^2$ and $\mathcal{C}_m^2$ are further subdivided into sub-bundles. $\mathcal{R}_m^2 = \bigcup_{j=1}^{\sqrt{m}} \mathcal{R}_m^2(j)$ where

$$\mathcal{R}_m^2(j) = \left\{(y = \varepsilon j + \varepsilon^2 i) \mid i = 1, 2, \dots \sqrt{m}\right\}.$$

$\mathcal{R}^2_m(j)$ is called the *j-th row.*

Similarly, $\mathcal{C}^2_m = \bigcup_{j=1}^{\sqrt{m}} \mathcal{C}^2_m(j)$ where

$$\mathcal{C}^2_m(j) = \left\{(x - y = j + \varepsilon^2 i) \mid i = 1, 2, \dots \sqrt{m}\right\}.$$

$\mathcal{C}^2_m(j)$ is called the *j-th column.*

Clearly, any row $\mathcal{R}^2_m(j)$ and column $\mathcal{C}^2_m(j')$ forms an arrangement isomorphic to $\mathcal{A}^1_{\sqrt{m}}$, so in the intersection of any row and column we have a monotone path of length $2\sqrt{m}$. The lines in $\mathcal{U}^2_m$ and $\mathcal{V}^2_m$ allow us to link all these monotone paths.

$$\mathcal{U}^2_m = \{\ell_{i,j} \mid i = 1, 2, \dots \sqrt{m}, j = 1, 2, \dots \sqrt{m} - 1\}$$

where

$$\ell_{i,j} = (2x - y = 2(i + (j + \frac{1}{2})\varepsilon) - (j + \frac{1}{2})\varepsilon = 2i + (j + \frac{1}{2})\varepsilon).$$

$$\mathcal{V}^2_m = \{\ell'_i \mid i = 1, 2, \dots \sqrt{m} - 1\}$$

where

$$\ell'_i = (2x + y = 2i + 1).$$

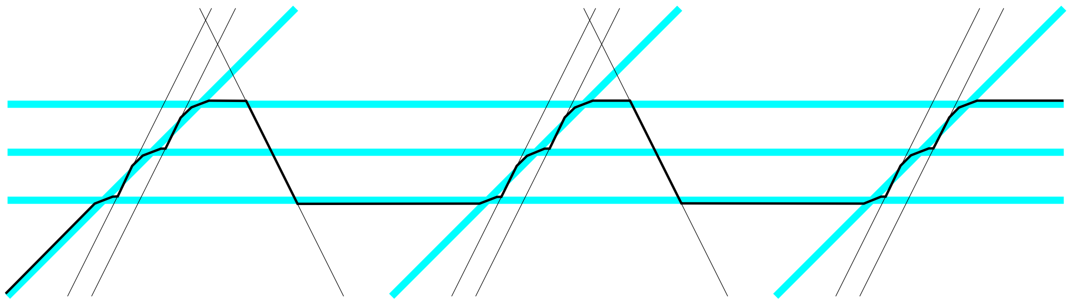

Figure 2. $\mathcal{A}^2_m$

Now we have the following monotone path. Start with a monotone path of length $2\sqrt{m}$ in the intersection of the first row and first column. We leave the intersection on the highest line in the first row. Then we use $\ell_{1,1}$ to go up to the highest line in the first column, and then we go along the monotone path of length $2\sqrt{m}$ in the intersection of the second row and first column. After leaving the intersection, we use $\ell_{1,2}$ to reach again the highest line in the first column, and we continue analogously, until leaving the intersection of the last row and first column. Then we go down on ℓ_1' to the lowest line of the first row, and proceed similarly along the second column, then the third column, until the last column. This path includes a monotone path of length $2\sqrt{m}$ in the intersection of each row and column. Therefore, the length is at least $2m\sqrt{m} > m^{3/2}$ (see Fig. 2).

STEP 3. First we define $\mathcal{A}_m^3 = \mathcal{R}_m^3 \cup \mathcal{C}_m^3 \cup \mathcal{U}_m^3 \cup \mathcal{V}_m^3$, $|\mathcal{A}_m^3| < 6m$.

$\mathcal{R}_m^3$ is divided into $\sqrt{m}$ bundles of $\sqrt{m}$ parallel lines, called the *rows*, and each row is further subdivided into $\sqrt[4]{m}$ sub-bundles of $\sqrt[4]{m}$ parallel lines. More precisely, $\mathcal{R}_m^3 = \bigcup_{i=1}^{\sqrt{m}} \mathcal{R}_m^3(i)$ and $\mathcal{R}_m^3(i)$ is called the *i-th row*, $\mathcal{R}_m^3(i) = \bigcup_{j=1}^{\sqrt[4]{m}} \mathcal{R}_m^3(i,j)$ where

$$\mathcal{R}_m^3(i,j) = \left\{(y = i + \varepsilon^2 j + \varepsilon^3 k) \mid k = 1, 2, \ldots \sqrt[4]{m}\right\},$$

so

$$\mathcal{R}_m^3(i) = \left\{(y = i + \varepsilon^2 j + \varepsilon^3 k) \mid j = 1, 2, \ldots, \sqrt[4]{m},\ k = 1, 2, \ldots \sqrt[4]{m}\right\}.$$

Similarly, $\mathcal{C}_m^3 = \bigcup_{i=1}^{\sqrt{m}} \mathcal{C}_m^3(i)$ and $\mathcal{C}_m^3(i)$ is called the *i-th column*, $\mathcal{C}_m^3(i) = \bigcup_{j=1}^{\sqrt[4]{m}} \mathcal{C}_m^3(i,j)$ where

$$\mathcal{C}_m^3(i,j) = \left\{(x - y = i + \varepsilon j + \varepsilon^3 k) \mid k = 1, 2, \ldots \sqrt[4]{m}\right\},$$

so

$$\mathcal{C}_m^3(i) = \left\{(x - y = i + \varepsilon j + \varepsilon^3 k) \mid j = 1, 2, \ldots, \sqrt[4]{m},\ k = 1, 2, \ldots \sqrt[4]{m}\right\}.$$

Consider any row $\mathcal{R}_m^3(i)$ and column $\mathcal{C}_m^3(i')$. The arrangement $\mathcal{R}_m^3(i) \cup \mathcal{C}_m^3(i')$ is isomorphic to $\mathcal{R}^2_{\sqrt{m}} \cup \mathcal{C}^2_{\sqrt{m}}$ from the arrangement $\mathcal{A}^2_{\sqrt{m}}$. Let $\mathcal{U}_m^3(i,i')$ (resp. $\mathcal{V}_m^3(i,i')$) be the copy of $\mathcal{U}^2_{\sqrt{m}}$ (resp. $\mathcal{V}^2_{\sqrt{m}}$) under the same isomorphism. Let

$$\mathcal{U}_m^3 = \bigcup_{i=1}^{\sqrt{m}} \bigcup_{i'=1}^{\sqrt{m}} \mathcal{U}_m^3(i,i'),$$

and

$$\mathcal{V}_m^3 = \bigcup_{i=1}^{\sqrt{m}} \bigcup_{i'=1}^{\sqrt{m}} \mathcal{V}_m^3(i,i').$$

In other words, for any row $\mathcal{R}_m^3(i)$ and column $\mathcal{C}_m^3(i')$, add the lines corresponding to $\mathcal{U}^2_{\sqrt{m}}$ and $\mathcal{V}^2_{\sqrt{m}}$ so that we get an arrangement isomorphic to $\mathcal{A}^2_{\sqrt{m}}$.

Because of the slopes of the lines in $\mathcal{U}_m^3(i,i')$, $\mathcal{U}_m^3(i,i') = \mathcal{U}_m^3(i+2, i'-1)$ and $|\mathcal{U}_m^3(i,i')| < \sqrt{m}$, therefore, $|\mathcal{U}_m^3| < 3\sqrt{m}\sqrt{m} = 3m$. Similarly, $\mathcal{V}_m^3(i,i') = \mathcal{V}_m^3(i+2, i'-3)$ and $|\mathcal{V}_m^3(i,i')| < \sqrt[4]{m}$, therefore, $|\mathcal{V}_m^3| < 5\sqrt{m}\sqrt[4]{m} < m$. Cearly, $|\mathcal{R}_m^3| = |\mathcal{C}_m^3| = m$, so $|\mathcal{A}_m^3| < 6m$ (see Fig. 3).

In $\mathcal{A}_m^3$, in the crossing of any row and column we have an arrangement isomorphic to $\mathcal{A}_{\sqrt{m}}^2$, so there is a monotone path of length at least $\sqrt{m}^{3/2} = m^{3/4}$. We want to link all of them with some additional lines, just like in the construction of $\mathcal{A}_m^2$. The problem is that the crossing of row $\mathcal{R}_m^3(i)$ and column $\mathcal{C}_m^3(i')$ is exactly below the crossing of $\mathcal{R}_m^3(i+1)$ and $\mathcal{C}_m^3(i'-1)$. Let T be an affine transformation, $T(x,y) = (x + \sqrt{\varepsilon}y, y)$ and let $\mathcal{B}_m^3 = T(\mathcal{A}_m^3)$. It is not hard to see that for ε small enough, all lines with positive (resp. negative) slope will still have positive (resp. negative) slope. So, in the crossing of any row and column of $\mathcal{B}_m^3$ we still have a monotone path of length $m^{3/4}$. But now, all crossings of the rows and columns can be separated from each other by vertical lines. These lines can be perturbed to lines of very large positive or negative slopes, such that they can be used to link the monotone paths in consecutive crossings. Let $\mathcal{L}$ be the set of these lines. Then $|\mathcal{L}| = m - 1$, so $|\mathcal{B}_m^3 \cup \mathcal{L}| < 7m$.

There are m row-column crossings, and in each of them we have a monotone path of length at least $m^{3/4}$ so the monotone path containing all of them has length at least $m^{3/4}m = m^{7/4}$.

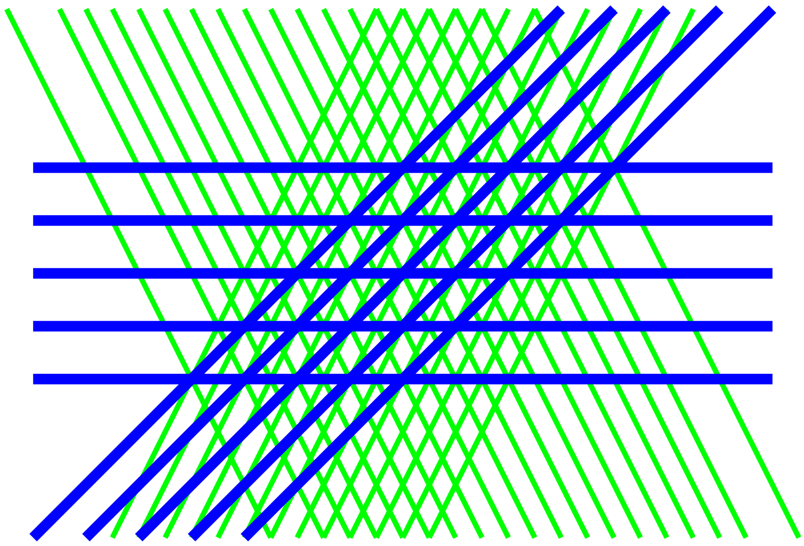

Figure 3. $\mathcal{A}_m^3$

Remarks. 1. As mentioned in the introduction, a monotone path has length at most $\binom{n}{2} + 1$ in any arrangement of n lines. This can be improved by the following observation. Take a monotone path of length $5m$ and divide it into m inervals, each of length 5. Notice that above or below each of these intervals there is a crossing of the lines which is not on the path (see Fig. 4). So, if there are n lines and a monotone path of length k, then $\binom{n}{2} \geq k - 1 + \lfloor k/5 \rfloor$ so $5n^2/12 > k$. Considering longer intervals, the constant can be further improved, but we were unable to give a $o(n^2)$ upper bound.

2. If instead the number of turns, we define the length of the path as the number of intersection points on it, it is easy to construct an arrangement of n lines with a monotone path of length $\Omega(n^2)$.

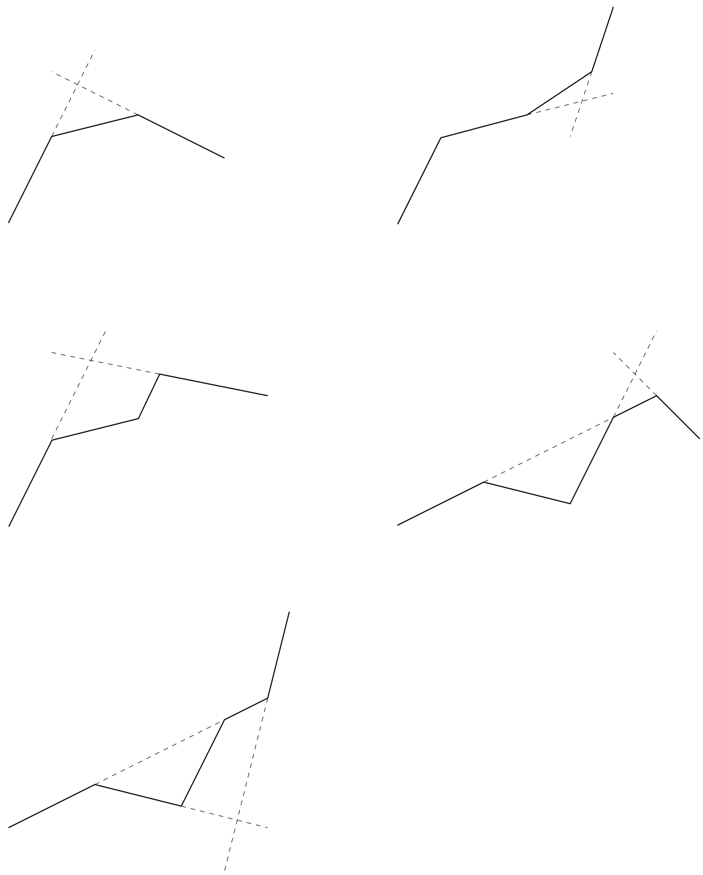

Figure 4. *A monotone path of length* 5 *has an unused crossing above or below it.*

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