A modular version of the Erdős-Szekeres theorem

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Abstract

Bialostocki, Dierker, and Voxman proved that for any $n \geq p+2$, there is an integer $B(n,p)$ with the following property. Every set of $B(n,p)$ points in general position in the plane has n points in convex position such that the number of points in the interior of their convex hull is $0 \bmod p$. They conjectured that the same is true for *all* pairs $n \geq 3$, $p \geq 2$. In this note, we show that every sufficiently large point set determining no triangle with more than one point in its interior has n elements that form the vertex set of an empty convex n-gon. As a consequence, we show that the above conjecture is true for all $n \geq 5p/6 + O(1)$.

1 Introduction

We say that a set of points in the plane is in *general position* if no three of them are collinear. Throughout this paper, $\mathcal{X}$ will denote a set of points in the plane in general position. Let $\mathrm{vert}(\mathcal{X})$ denote the vertex set of the convex hull of $\mathcal{X}$. A polygon is said to be *empty*, if it contains no elements of $\mathcal{X}$ in its interior. If every triple in $\mathrm{vert}(\mathcal{X})$ determines an empty triangle, then $\mathcal{X} = \mathrm{vert}(\mathcal{X})$ is in *convex position* or, in short, *convex*.

According to a well known theorem of Erdős and Szekeres [ES1, ES2], for any integer $n \geq 3$, there exists $E(n) = O(4^n)$ with the property that every set $\mathcal{X}$ of at least $E(n)$ points in general position in the plane has n elements in convex position. (In this case, we say that $\mathcal{X}$ *determines* a convex n-gon.) For a long time it appeared to be only a technicality that none of the existing proofs yielded the stronger result that every sufficiently large point set contains the vertex set of an *empty* convex n-gon. Harborth

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[Ha] showed that every 10-element point set determines an empty convex *pentagon*, and that this does not remain true for all 9-element sets. Finally, in 1983 Horton [Ho] surprised most experts by a simple recursive construction of arbitrarily large finite point sets determining no empty convex *heptagons*. The corresponding problem for *hexagons* is still open.

Bialostocki, Dierker, and Voxman [BDV] proposed the following elegant "modular" version of the original problem.

Conjecture. *For any $n \geq 3$ and $p \geq 2$, there exists an integer $B(n,p)$ such that every set of $B(n,p)$ points in general position in the plane determines a convex n-gon such that the number of points in its interior is $0 \bmod p$.*

Bialostocki et al. verified this conjecture for every $n \geq p+2$. The original upper bound on $B(n,p)$ was later improved by Caro [C], but his proof also relied heavily on the assumption $n \geq p+2$.

In the present note we somewhat relax this condition.

Theorem 1. *For any $n \geq 5p/6 + O(1)$, there exists an integer $B(n,p)$ such that every set of $B(n,p)$ points in general position in the plane determines a convex n-gon such that the number of points in its interior is $0 \bmod p$.*

If every triple in $\mathrm{vert}(\mathcal{X})$ determines a triangle with *at most one* point in its interior, then $\mathcal{X}$ is said to be *almost convex.*

Our proof of Theorem 1 is based on the following

Theorem 2. *For any $n \geq 3$, there exists an integer $K(n)$ such that every almost convex set of at least $K(n)$ points in general position in the plane determines an empty convex n-gon. Moreover, we have $K(n) = \Omega(2^{n/2})$.*

In Sections 2 and 3, we establish Theorems 2 and 1, respectively.

2 Almost convex sets – Proof of Theorem 2

Let $\mathcal{X}$ be a set of points in the plane in general position. For any triple $x, y, z \in \mathcal{X}$, let $\triangle xyz$ stand for the triangle determined by x, y, z. Let $\mathrm{conv}(\mathcal{X})$ denote the convex hull of $\mathcal{X}$. Given any convex polygon C, let $\mathrm{int}(C)$ denote the interior of C.

First, we rephrase the definition of almost convexity. Let $\mathcal{X}$ denote a set of n points in the plane in general position.

Lemma 2.1. *$\mathcal{X}$ is almost convex if and only if at least one of the following two conditions is satisfied.*

(i) Every triangle determined by $\mathcal{X}$ contains at most one point of $\mathcal{X}$ in its interior.

(ii) For every subset $\mathcal{Y} \subseteq \mathcal{X}$ with $|\mathcal{Y}| \geq 3$, we have $|\mathrm{vert}(\mathcal{Y})| \geq \lceil |\mathcal{Y}|/2 \rceil + 1$.

Proof: To prove part (i), let $x, y, z \in \mathcal{X}$, and assume that none of these points lie on the boundary of $\mathrm{conv}(\mathcal{X})$. (The other cases can be settled analogously.) Let u_1, u_2 be the intersection points of the line xy with the boundary of $\mathrm{conv}(\mathcal{X})$, and let $z_i z_i'$ be the edge of $\mathrm{conv}(\mathcal{X})$ such that $u_i \in z_i z_i'$

$(i = 1, 2)$. There is an edge $z_3 z_3'$ of $\operatorname{conv}(\mathcal{X})$ such that the $\triangle z_1 z_3 z_3'$ contains z. Consequently, $C = \operatorname{conv}(\{z_1, z_2, z_3, z_1', z_2', z_3'\}) \supseteq \triangle xyz$. Since $\mathcal{X}$ is almost convex, $\operatorname{int}(C)$ contains at most 4 points of $\mathcal{X}$, so there cannot be more than one point of $\mathcal{X}$ in the interior of $\triangle xyz$.

Next we prove that every almost convex set $\mathcal{X}$ satisfies condition (ii). Suppose that a subset $\mathcal{Y}$ of $\mathcal{X}$ contains at least 3 points. It follows from part (i) that $\mathcal{Y}$ is almost convex. Consequently, $|\operatorname{vert}(\mathcal{Y})| \geq \lceil |\mathcal{Y}|/2 \rceil + 1$, as required. On the other hand, if the convex hull of every 5-element subset of $\mathcal{X}$ has at least 4 vertices, then $\mathcal{X}$ is almost convex. ◻

Part (ii) of Lemma 2.1 immediately implies

Corollary 2.2. *Every subset of an almost convex set is almost convex.*

We need the following recursive construction. Let $\mathcal{R}_1$ be a set of two points in the plane. Assume that we have already defined $\mathcal{R}_1, \ldots, \mathcal{R}_j$ so that

1. $\mathcal{X}_j := \mathcal{R}_1 \cup \ldots \cup \mathcal{R}_j$ is in general position,
2. the vertex set of the polygon $\Gamma_j := \operatorname{conv}(\mathcal{X}_j)$ is $\mathcal{R}_j$, and
3. any triangle determined by $\mathcal{R}_j$ contains precisely one point of $\mathcal{X}_j$ in its interior.

Let $z_1, z_2, \ldots, z_r$ denote the vertices of Γ_j in clockwise order, and let $\varepsilon_j, \delta_j > 0$. For any $1 \leq i \leq r$, let ℓ_i denote the line through z_i orthogonal to the bisector of the angle of Γ_j at z_i. Let z_i' and z_i'' be two points on ℓ_i, at distance ε_i from z_i. Finally, move z_i' and z_i'' away from Γ_j by a distance δ_j, in the direction orthogonal to ℓ_i, and denote the resulting points by u_i' and u_i'', respectively. (See Fig. 1.)

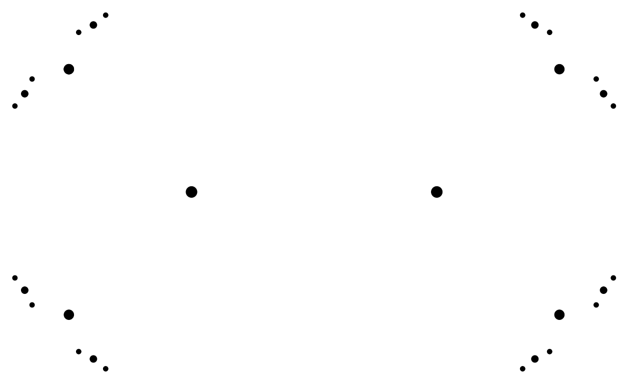

Figure 1.

It is easy to see that if ε_j and δ_j/ε_j are sufficiently small, then $\mathcal{R}_{j+1} := \{u_i', u_i'' \mid i = 1, 2, \ldots, r\}$ also satisfies the above three conditions.

We have to verify only the last condition. If $a \in \{u'_i, u''_i\}$, $b \in \{u'_j, u''_j\}$, and $c \in \{u'_k, u''_k\}$ are three points of $\mathcal{R}_{j+1}$, for three disctinct indices i, j, k, then any point of $\mathcal{X}_{j+1} = \mathcal{X}_j \cup \mathcal{R}_{j+1}$ which belongs to the interior of $\triangle abc$ must coincide with the unique point of $\mathcal{X}_j$ in the interior of $\triangle z_i z_j z_k$. If there exist $i \neq k$ such that $a = u'_i, b = u''_i$, and $c \in \{u'_k, u''_k\}$, then the only point of $\mathcal{X}_{j+1}$ inside $\triangle abc$ is z_i.

Obviously, we have $|\mathcal{X}_k| = 2^{k+1} - 2$ for every $k \geq 1$. Since no three vertices of an empty convex polygon determined by $\mathcal{X}_k$ belong to the same $\mathcal{R}_i$, it follows that any such polygon has at most $2k$ vertices. Consequently, if $K(n)$ exists, its order of magnitude is at least $2^{n/2}$.

Next we prove the existence of $K(n)$.

In the sequel, we use the following notation. For any subset $\mathcal{Y} \subseteq \mathcal{X}$, let $\mathcal{Y}'$ denote the set of all points of $\mathcal{X}$ belonging to the interior of the convex hull of $\mathcal{Y}$.

Lemma 2.3. *Suppose that $\mathcal{R}_1, \ldots, \mathcal{R}_k \subseteq \mathcal{X}$ are in general position in the plane, and they satisfy the following conditions:*

(i) $|\mathcal{R}_1| \geq 2$;

(ii) $\mathcal{R}_j$ is in convex position, for $1 \leq j \leq k$;

(iii) every triangle of $\mathcal{R}_j$, $1 \leq j \leq k$, has precisely one point of $\mathcal{X}$ in its interior;

(iv) $\mathcal{R}_{j-1} = \text{vert}(\mathcal{R}'_j) = \text{vert}\,(\text{int conv}(\mathcal{R}_j) \cap \mathcal{X})$, for every $1 < j \leq k$.

Then we have

(a) $|\mathcal{R}_{j+1}| = 2|\mathcal{R}_j|$, for every $1 \leq j \leq k-1$.

(b) If $z_1, \ldots, z_r$ denote the vertices of $\mathcal{R}_j$ in clockwise order, then the vertices of $\mathcal{R}_{j+1}$ can be labeled in clockwise order by $c(z_1), d(z_1), \ldots, c(z_r), d(z_r)$ such that every z_i $(1 \leq i \leq r)$ lies in the intersection of $\triangle d(z_{i-1})c(z_i)d(z_i)$ and $\triangle c(z_i)d(z_i)c(z_{i+1})$, where the indices are taken modulo r.

(c) $\mathcal{X}$ determines an empty convex $2k$-gon.

Proof: It follows from the properties of the sets $\mathcal{R}_j$ that $|\mathcal{R}'_j| = |\mathcal{R}_j| - 2$ and $|\mathcal{R}'_{j+1}| = |\mathcal{R}_{j+1}| - 2$, for every $1 \leq j < k$. We also have that

$$|\mathcal{R}'_{j+1}| = |\mathcal{R}_j| + |\mathcal{R}'_j|,$$

which proves part (a).

To establish part (b), denote by $u_1, u_2, \ldots, u_{2r}$ the vertices of $\mathcal{R}_{j+1}$ in clockwise order. Consider the triangles $T_i = \triangle u_i u_{i+1} u_{i+2}$, for $1 \leq i \leq 2r$. Each triangle T_i contains exactly one point of $\mathcal{X}$, and it must belong to $\mathcal{R}_j$. Since $T_1, T_3, \ldots, T_{2r-1}$ are openly disjoint, each point of the r-element set $\mathcal{R}_j$ must lie in one of them. The same is true for $T_2, T_4, \ldots, T_{2r}$. Thus, there are only two possibilities: each of the regions $T_1 \cap T_2, T_3 \cap T_4, \ldots, T_{2r-1} \cap T_{2r}$ contains precisely one point of $\mathcal{R}_j$, or each of $T_2 \cap T_3, T_4 \cap T_5, \ldots, T_{2r} \cap T_1$ contains exactly one point of $\mathcal{R}_j$. In either case we are done.

Finally, we prove part (c). Let x_1 and y_1 denote two consecutive vertices of $\mathcal{R}_1$ in the clockwise order. Using the notation in part (b), let $x_{j+1} := d(x_j)$ and $y_{j+1} := c(y_j)$, for $j = 1, \ldots, k-1$. We show that $x_1, x_2, \ldots, x_k, y_k, y_{k-1}, \ldots, y_1$, in this order, induce an empty convex polygon.

For every $1 \leq j < k$, x_j and y_j lie inside the polygon $\text{conv}(\mathcal{R}_{j+1})$, whose 4 consecutive vertices are $c(x_j), d(x_j) = x_{j+1}, c(y_j) = y_{j+1}$, and $d(y_j)$. It follows from part (b) that the line $x_j y_j$ intersects sides $c(x_j)d(x_j)$ and $c(y_j)d(y_j)$ of this polygon. Thus, $D_j = x_j x_{j+1} y_{j+1} y_j$ is a convex quadrilateral.

Furthermore, the line $x_j y_j$ separates $x_{j+1}y_{j+1}$ from $\mathcal{R}'_{j+1}$, and D_j is empty. To complete the proof, it suffices to check that these quadrilaterals fit together appropriately. That is, for $1 < j < k$, the angles $\alpha_j = \angle x_{j-1}x_j y_j + \angle y_j x_j x_{j+1}$ and $\beta_i = \angle y_{i-1}y_i x_i + \angle x_i y_i y_{i+1}$ are smaller than π. To see that $\alpha_j < \pi$, notice that it follows from part (b) that both lines $d(c(x_{j-1}))x_{j+1}$ and $c(x_j)y_{j+1}$ separate x_j from x_{j-1}. Consequently, x_j lies inside $\triangle x_{j-1}x_{j+1}c(x_j)$, so x_{j-1}, y_{j+1}, and y_j are on the same side of the line $x_j x_{j+1}$. The other inequality can be checked analogously. ⊒

Lemma 2.4. *For any positive integers $n \geq 3$ and k, there exists $L(n,k)$ such that every almost convex set $\mathcal{X}$ of at least $L(n,k)$ points contains either an empty convex n-gon, or a sequence of subsets $\mathcal{R}_1, \ldots, \mathcal{R}_k$ satisfying conditions (i)–(iv) in Lemma 2.3.*

Suppose for a moment that we have already established Lemma 2.4. Now we can prove Theorem 2 as follows.

Let $K(n) = L(n, \lceil n/2 \rceil)$, and let $\mathcal{X}$ be an almost convex set whose size is at least $K(n)$. By Lemma 2.4, $\mathcal{X}$ either contains an empty convex n-gon, and we are done, or it has a sequence of subsets, $\mathcal{R}_1, \ldots, \mathcal{R}_k$ ($k = \lceil n/2 \rceil$) satisfying conditions (i)–(iv). In the latter case, Lemma 2.3(c) guarantees the existence of an empty n-gon or $(n+1)$-gon, depending on the parity of n. This completes the proof of Theorem 2.

It remains to verify Lemma 2.4.

By Ramsey's theorem, there exists a smallest integer $r = r_3(n,m)$ with the following property. For any 2-coloring of the edges of a complete 3-uniform hypergraph of at least r vertices, there is either a set of n vertices, all of whose triples are colored with the first color, or a set of m vertices, all of whose triples are colored with the second color.

Let $m_1 = 2$, and for $j = 1, 2, \ldots, k$ define recursively the numbers $n_j := r_3(n, m_j)$ and $m_{j+1} := 2n_j - 1$. Let $L(n,k) = 2n_k - 3$, and consider an almost convex set $\mathcal{X}$ of size at least $L(n,k)$. It follows from Lemma 2.1 (ii) that $|\mathrm{vert}(\mathcal{X})| \geq n_k$. The set $\mathcal{X}_k := \mathrm{vert}(\mathcal{X})$ is almost convex. Color every triangle T determined by $\mathcal{X}_k$ with 0 or 1: with the number of points of $\mathcal{X}$ in the interior of T. According to the definition of n_k, in $\mathcal{X}_k$ we can find either an n-element subset, all of whose triples are of color 0, or an m_k-element subset, $\mathcal{Y}_k$, all of whose triples are of color 1. In the former case, there is an empty convex n-gon. In the latter case, $\mathcal{Y}_k$ is a convex set, all of whose triangles have precisely one point of $\mathcal{X}$ in their interiors.

Using the notation introduced before Lemma 2.3, let $\mathcal{X}_{k-1} := \mathrm{vert}(\mathcal{Y}'_k)$. By Corollary 2.2, $\mathcal{X}_{k-1}$ is almost convex, and for any three consecutive vertices of $\mathrm{conv}(\mathcal{Y}_k)$, the unique point of $\mathcal{X}$ in the interior of the triangle determined by them belongs to $\mathcal{X}_{k-1}$. Consequently, we have $|\mathcal{X}_{k-1}| \geq \lceil |\mathcal{Y}_k|/2 \rceil \geq n_{k-1}$.

Repeating the above procedure with $\mathcal{X}_{k-1}$ in place of $\mathcal{X}_k$, we can find either an empty convex n-gon or an m_{k-1}-element subset $\mathcal{Y}_{k-1} \subseteq \mathcal{X}_{k-1}$ in convex position, whose every triple has precisely one point in its interior. Set $\mathcal{X}_{k-2} := \mathrm{vert}(\mathcal{Y}'_{k-1})$, and continue. At some point we either find an empty convex n-gon, or, after k repetitions, we obtain a sequence of sets, $\mathcal{X}_k \supseteq \mathcal{Y}_k, \ldots, \mathcal{X}_1 \supseteq \mathcal{Y}_1$, such that for $j = 1, \ldots, k$

(i) $|\mathcal{Y}_1| \geq m_1 = 2$;

(ii) $\mathcal{X}_j$ and $\mathcal{Y}_j$ are in convex position;

(iii) every triangle determined by $\mathcal{Y}_j$ has exactly one point of $\mathcal{X}$ in its interior;

(iv) $\mathcal{X}_{j-1} = \text{vert}(\mathcal{Y}'_j)$.

Thus, the sets $\mathcal{Y}_j$ have all the properties (i)–(iv) in Lemma 2.4 (and Lemma 2.3) required from $\mathcal{R}_j$, except that instead of the last property we have the somewhat weaker relation $\mathcal{Y}_{j-1} \subseteq \text{vert}(\mathcal{Y}'_j)$.

We finish the proof of Lemma 2.4 by recursively constructing a sequence of sets $\mathcal{R}_1 \subseteq \mathcal{Y}_1, \ldots, \mathcal{R}_k \subseteq \mathcal{Y}_k$ meeting the requirements of the lemma. Let $\mathcal{R}_1 = \mathcal{Y}_1$, and assume that for some $j < k$ we have already found $\mathcal{R}_1, \ldots, \mathcal{R}_j$ such that $\mathcal{R}_{i-1} = \text{vert}(\mathcal{R}'_i)$ for $1 < i \le j$, i.e., condition (iv) is satisfied. (The other conditions are *hereditary*: they are satisfied for the sets $\mathcal{Y}_i$, so they automatically hold for $\mathcal{R}_i$.)

The following statement, applied to $A = \mathcal{R}_j$ and $B = \mathcal{Y}_{j+1}$, shows that there exists $\mathcal{R}_{j+1} \subseteq \mathcal{Y}_{j+1}$ such that $\mathcal{R}_j = \text{vert}(\mathcal{R}'_{j+1})$. This completes the recursion step and the proof of Lemma 2.4, and hence of Theorem 2.

Proposition 2.5. *Let $\mathcal{A} \subseteq \mathcal{Y}_j$ and $\mathcal{B} \subseteq \mathcal{Y}_{j+1}$ satisfy $\mathcal{A} \subseteq \text{vert}(\mathcal{B}')$. Then there exists a subset $\mathcal{C} \subseteq \mathcal{B}$ such that $\mathcal{A} = \text{vert}(\mathcal{C}')$.*

Proof: Suppose that $\mathcal{A} \ne \text{vert}(\mathcal{B}')$, and let $w \in \text{vert}(\mathcal{B}') \setminus \mathcal{A}$.

We claim that $\text{conv}(\mathcal{B})$ has three consecutive vertices, a, b, c, (in this order) such that the triangle determined by them contains w in its interior.

To verify this claim, observe that any line ℓ through w, tangent to $\text{conv}(\mathcal{B}')$, separates at most two vertices of $\mathcal{B}$ from $\mathcal{B}'$. If ℓ separates precisely one such vertex, then this vertex and the two neighboring vertices determine a triangle which contains w in its interior. If ℓ separates two such vertices, x and y, then it is easy to see that one of the triangles uxy and xyv must contain w in its interior, where u and v denote the vertices of $\text{conv}(\mathcal{B})$ immediately preceding and following $\{x, y\}$, respectively. This proves the claim.

To finish the proof of the lemma, let $\mathcal{B}_1$ denote the set obtained from $\mathcal{B}$ by deleting the point b whose existence is guaranteed by the claim. We have that $\mathcal{B}'_1 = \mathcal{B}' \setminus \{w\}$, and $\mathcal{A} \subseteq \text{vert}(\mathcal{B}'_1)$. Note that $\text{vert}(\mathcal{B}'_1)$ is not necessarily a subset of $\text{vert}(\mathcal{B}')$.

If $\text{vert}(\mathcal{B}'_1) = \mathcal{A}$, then $\mathcal{C} := \mathcal{B}_1$ will meet the requirements. Otherwise, repeat the argument with $\mathcal{B}_1$ in place of $\mathcal{B}$ to obtain a subset $\mathcal{B}_2 \subset \mathcal{B}_1$ with $\mathcal{A} \subseteq \text{vert}(\mathcal{B}'_2)$, etc. After finitely many steps, this procedure must terminate. □

3 Proof of Theorem 1

Let $n \ge 5p/6 + O(1)$, and let $\mathcal{X}$ be a set of N points in the plane. If $n > p + 1$, then the assertion was established in [BDV]. Thus, we may assume that $n \le p + 1$ and that p is sufficiently large. In fact, it follows from our argument that the theorem holds for $n \ge 5p/6 + 6$, provided that $p \ge 264$.

By the Erdős-Szekeres Theorem, there exists a subset $\mathcal{X}' \subset \mathcal{X}$ of $N' \ge \log_4 N$ points in convex position. Let $x_1, \ldots, x_{N'}$ denote the points of $\mathcal{X}'$ listed in clockwise order.

Definition 3.1. For any set C, let $\langle C \rangle$ denote the number of points of $\mathcal{X}$ in the interior of the convex hull of C, and let $\langle C \rangle_p$ denote the same number reduced modulo p.

A convex polygon C is said to be *modulo p empty* or, shortly, *p-empty*, if $\langle C\rangle_p = 0$. Given an ordered triple $x_ix_jx_k$ ($i<j<k$) and a point $x \in \mathcal{X}$ in the interior of $T = \triangle x_ix_jx_k$, we say that x is the *lowest point* of $\mathcal{X}$ in T (with respect to its "long" side, x_ix_k) if no point of $\mathcal{X}$ in T, different from x_i and x_k, is closer to the line x_ix_k than x is. (By slightly perturbing the elements of $\mathcal{X}$, if necessary, we can assume that this point is uniquely determined.)

Color the triples $\{x, x', x''\} \subset \mathcal{X}'$ with $p+1$ colors, $0, 1, \ldots p$, according to the following rule.

- $\{x, x', x''\}$ gets color p if $\langle x, x', x''\rangle = 1$.
- $\{x, x', x''\}$ gets color 1 if $\langle x, x', x''\rangle_p = 1$ and $\langle x, x', x''\rangle \neq 1$.
- For $0 \le i < p$, $i \neq 1$, $\{x, x', x''\}$ gets color i if $\langle x, x', x''\rangle_p = i$.

It follows from Ramsey's Theorem, that there is an M-element subset $\mathcal{Y} \subset \mathcal{X}'$, $M = \Omega(\log\log\log N)$, all of whose triples are of the same color, say, color q. Let $y_1, \ldots, y_M$ be an enumeration of the vertices of $\mathcal{Y}$, in clockwise order.

Claim 3.2. *If p and q are not relatively prime and N (hence, M) is sufficiently large, then $\mathcal{X}$ determines a p-empty convex n-gon.*

Proof: Suppose that $(p, q) = d > 1$. Then there exists an integer s, $p/2 \le s \le 2p/3$ such that $sq \equiv 0 \bmod p$.

If $q = p$, then $\mathcal{X} \cap \mathrm{conv}(\mathcal{Y})$ is an *almost convex* set, whose size is at least M, and the result follows from Theorem 2. Otherwise, consider any triangulation of the polygon $P = y_1y_3y_5\ldots y_{2s+3}$. Obviously, P consists of s triangles, so it is p-empty. Since $2p/3 + 2 \le n \le p+3$, we have $0 \le n-s-2 \le s+1$. Thus, for $i = 1, 2, \ldots, n-s-2$, there is a lowest point $w_i \in \mathcal{X}$ in $\triangle y_{2i-1}y_{2i}y_{2i+1}$. Using the fact that, for every i, $y_{2i-1}w_iy_{2i+1}$ is an empty triangle, we obtain that $\mathrm{conv}(y_1, y_3, \ldots, y_{2s+3}, w_1, \ldots, w_{n-s-2})$ is a p-empty convex n-gon. ◽

Thus, we can and will assume in the sequel that p and q are relatively prime.

Definition 3.3. For any triangle $T = \triangle y_iy_jy_k$ ($i<j<k$) determined by $\mathcal{Y}$, and for any point $x \in \mathcal{X}$ belonging to T, we say that $\triangle y_ixy_k$ is a *base sub-triangle*. It is called *standard* if $\langle y_i, x, y_k\rangle_p = 0$ or q.

A convex quadrilateral $y_ixx'y_k$ is called a *base sub-quadrilateral*, if $x, x' \in \mathcal{X}$ lie in the interior of T. It is *standard* if $\langle y_i, x, x', y_k\rangle \equiv 0$, q or $2q \bmod p$.

Let $\Phi(T)$ (and $\Gamma(T)$) be defined as the set of all numbers that occur as the remainder of the number points in a base sub-triangle (resp., base sub-quadrilateral) of T upon division by p. That is, let

$$\Phi(T) = \{\langle y_i, x, y_k\rangle_p \mid x \in \mathcal{X}, x \in \mathrm{int}(T) \cup \{x_j\}\},$$

$$\Gamma(T) = \{\langle y_i, x, x', y_k\rangle_p \mid x, x' \in \mathcal{X},\ x, x' \in \mathrm{int}(T),\ y_ixx'y_k \text{ convex}\}\ .$$

Clearly, $\Phi(T)$ can take at most 2^p different "values" (sets), and the same is true for $\Gamma(T)$. Therefore, by Ramsey's Theorem, we can find a subset $\mathcal{Z} \subset \mathcal{Y}$, $\mathcal{Z} = \{z_1, z_2, \ldots, z_K\}$ in clockwise order, such that

$$K = \Omega(\log\log M) = \Omega(\log\log\log\log\log N)$$

and the pair $(\Phi(T), \Gamma(T))$ is the same for every triangle $T = \triangle z_i z_j z_k$, $i < j < k$.

Claim 3.4. *If any triangle determined by $\mathcal{Z}$ has a non-standard base sub-triangle (hence, all of them do) and N (hence, K) is sufficiently large, then $\mathcal{X}$ determines a p-empty convex n-gon.*

Proof: Suppose that there exists a non-standard base sub-triangle S with $\langle S\rangle_p = s$, and let t, $0 \le t < p$, denote the unique solution of the congruence $tq \equiv s \bmod p$. Since S is non-standard, $s \ne 0$ and $t \ne 0, 1$. It follows from the choice of the set $\mathcal{Z}$ that in *every* triangle $\triangle z_i z_j z_k$, $i < j < k$, there is a point $x \in \mathcal{X}$ such that $\langle z_i, x, z_k\rangle_p = s$. Letting $l = p - t$, we clearly have $1 \le l \le p - 2$. We distinguish two cases.

Case 1: $1 \le l \le 2p/3$. Since $n \ge 2p/3 + 3$, we can write $n - 2 = a(l + 1) + b$, where $a \ge 1$ and $0 \le b < l + 1$ are suitable integers. Clearly, we have $al + 1 \ge a + b$.

The convex polygon $z_1 z_3 z_5 \dots z_{2al+3}$ has $al + 2$ vertices, so its triangulations consist of al triangles. For $i = 1, 2, \dots a$, let x_i be a point of $\mathcal{X}$ in $\triangle z_{2i-1} z_{2i} z_{2i+1}$ such that $\langle z_{2i-1}, x_i, z_{2i+1}\rangle_p = s$. For $i = a + 1, a + 2, \dots a + b$, let x_i be the lowest point of $\mathcal{X}$ in $\triangle z_{2i-1} z_{2i} z_{2i+1}$ so that $\triangle z_{2i-1} x_i z_{2i+1}$ is empty.

Then $P = \mathrm{conv}(z_1, z_3, z_5, \dots, z_{2al+3}, x_1, x_2, \dots, x_{a+b})$ is a polygon with $al + 2 + a + b = n$ vertices, and $\langle P\rangle \equiv alq + as \equiv alq + atq \equiv aq(l + t) \equiv 0 \bmod p$.

Case 2: $2p/3 < l \le p - 2$. Since $n \le p + 1$, we can write $p + 2 - n = a(t - 1) - b$, where $a \ge 1$ and $0 \le b < t - 1$ are suitable integers. Then we have $n = p + 2 - a(t - 1) + b$. Using the fact that $n \ge 2p/3 + 3$, one can easily check that $p - at + 1 \ge a + b$.

The convex polygon $z_1 z_3 z_5 \dots z_{2(p-at)+3}$ has $p - at + 2$ vertices, so its triangulations consist of $p - at$ triangles. For $i = 1, 2, \dots a$, let x_i be a point of $\mathcal{X}$ in $\triangle z_{2i-1} z_{2i} z_{2i+1}$ such that $\langle z_{2i-1}, x_i, z_{2i+1}\rangle_p = s$. For $i = a + 1, a + 2, \dots a + b$, let x_i be the lowest point of $\mathcal{X}$ in $\triangle z_{2i-1} z_{2i} z_{2i+1}$ so that $\triangle z_{2i-1} x_i z_{2i+1}$ is empty.

Then $P = \mathrm{conv}(z_1, z_3, z_5, \dots, z_{2(p-at)+3}, x_1, x_2, \dots, x_{a+b})$ is a polygon with $p - at + 2 + a + b = n$ vertices, and $\langle P\rangle \equiv q(p - at) + as \equiv 0 \bmod p$. ◻

Claim 3.5. *If any triangle determined by $\mathcal{Z}$ has a non-standard base sub-quadrilateral (hence, all of them do) and N (hence, K) is sufficiently large, then $\mathcal{X}$ determines a p-empty convex n-gon.*

Proof: Suppose that there exists a non-standard base sub-quadrilateral S with $\langle S\rangle_p = s$, and, as before, let t denote the unique solution of the congruence $tq \equiv s \bmod p$ in the interval $[0, p)$. Since S is non-standard, we have $s \ne 0$ and $t \ne 0, 1, 2$. It follows that *every* triangle $\triangle z_i z_j z_k$, $i < j < k$ contains two points $x, x' \in \mathcal{X}$ such that $z_i x x' z_k$ is a convex quadrilateral and $\langle z_i, x, x', z_k\rangle_p = s$. Letting $l = p - t$, we clearly have $1 \le l \le p - 3$. We distinguish two cases.

Case 1: $1 \le l \le 2p/3$. Since $n \ge 2p/3 + 4$, we can write $n - 2 = a(l + 2) + b$, where $a \ge 1$ and $0 \le b < l + 2$ are suitable integers. Clearly, we have $al + 2 \ge a + b$.

The convex polygon $z_1 z_3 z_5 \dots z_{2al+3}$ has $al + 2$ vertices, so its triangulations consist of al triangles. For $i = 1, 2, \dots a$, let x_i and x_i' be two points of $\mathcal{X}$ in $\triangle z_{2i-1} z_{2i} z_{2i+1}$ such that $z_{2i-1} x_i x_i' z_{2i+1}$ is convex and $\langle z_{2i-1}, x_i, x_i', z_{2i+1}\rangle_p = s$. For $i = a + 1, a + 2, \dots a + b$, let x_i be the lowest point of $\mathcal{X}$ in $\triangle z_{2i-1} z_{2i} z_{2i+1}$. More precisely, in the exceptional case of $a + b = al + 2$, let x_{a+b} be a point in $\triangle z_{2al+3} z_K z_1$ such that $\triangle z_{2al+3} x_{a+b} z_1$ is empty.

Then $P = \text{conv}(z_1, z_3, z_5, \dots, z_{2al+3}, x_1, x_1', x_2, x_2', \dots, x_a, x_a', x_{a+1}, \dots, x_{a+b})$ is a convex polygon with $al + 2 + 2a + b = n$ vertices, and $\langle P \rangle \equiv alq + as \equiv alq + atq \equiv aq(l+t) \equiv 0 \bmod p$.

Case 2: $2p/3 < l \leq p - 3$. Since $n \leq p + 1$, we can write $p + 2 - n = a(t-2) - b$, where $a \geq 1$ and $0 \leq b < t - 2$ are suitable inetegers. Then $n = p + 2 - a(t-2) + b$. Using the fact that $n \geq 3p/4 + 4$, one can easily check that $p - at + 1 \geq a + b$.

The convex polygon $z_1 z_3 z_5 \dots z_{2(p-at)+3}$ has $p - at + 2$ vertices, so its triangulations consist of $p - at$ triangles. For $i = 1, 2, \dots a$, let x_i and x_i' be two points of $\mathcal{X}$ in $\triangle z_{2i-1} z_{2i} z_{2i+1}$ such that $z_{2i-1} x_i x_i' z_{2i+1}$ is a convex quadrilateral with $\langle z_{2i-1}, x_i, x_i', z_{2i+1} \rangle_p = s$. For $i = a+1, a+2, \dots a+b$, let x_i be the lowest point of $\mathcal{X}$ in $\triangle z_{2i-1} z_{2i} z_{2i+1}$.

Then $P = \text{conv}(z_1, z_3, z_5, \dots, z_{2(p-at)+3}, x_1, x_1', x_2, x_2', \dots, x_a, x_a', x_{a+1}, \dots, x_{a+b})$ is a convex polygon with $p - at + 2 + 2a + b = n$ vertices, and $\langle P \rangle \equiv q(p - at) + as \equiv 0 \bmod p$. ◻

From now on we assume that all base sub-triangles and base sub-quadrilaterals of the triangles determined by $\mathcal{Z}$ are standard.

Definition 3.6. For any triangle $T = \triangle z_i z_j z_k$, $i < j < k$, define a partial order on the points in the interior of T as follows. For $x, y \in T$, $x \prec_T y$ if and only if $\triangle z_i y z_k$ contains x. The *rank* of y is the largest number a for which there exist $x_1, x_2, \dots x_a$ in T such that $x_1 \prec_T x_2, \dots \prec_T x_a \prec_T y$.

Claim 3.7. *Let $T = \triangle z_i z_j z_k$, $i < j < k$.*

If $q \neq 1, \frac{p+1}{2}$, then there exist $x_0, x_1, \dots, x_{q-1}$ in T such that $z_i x_0 x_1 \dots x_{q-1} z_k$ is an empty convex $(q+2)$-gon.

If $q = \frac{p+1}{2}$, then there exist x_0, x_1 in T such that $z_i x_0 x_1 z_k$ is an empty convex quadrilateral.

Proof: Suppose that $q \neq 1$. Let $x_0, x_1, \dots, x_r$ be the points of rank 0 in the interior of T, listed in counter-clockwise order of visibility from z_j. It follows from the fact that every base sub-triangle is standard that $r \geq q - 1$. For every $0 \leq l \leq r - 1$, the quadrilateral $z_i x_l x_{l+1} z_k$ is convex and empty.

If, in addition, $q \neq \frac{p+1}{2}$, then there is no base sub-quadrilateral containing precisely one element of $\mathcal{X}$, for such a quadrilateral would be non-standard. Consequently, $z_i x_l x_{l+1} x_{l+2} z_k$ is an empty convex pentagon for $0 \leq l \leq q - 3$, and the claim is true. ◻

Claim 3.8. *Suppose that $K \geq 4p - 1$ and $q = 1$. Then $\mathcal{X}$ determines a p-empty convex n-gon.*

Proof: Consider the triangle $T = \triangle z_i z_j z_k$, where $i = 1, j = 2p$ and $k = 4p - 1$. Clearly, T (as any other triangle determined by $\mathcal{Z}$) satisfies $\langle T \rangle_p = 1$ and $\langle T \rangle \neq 1$.

Let x denote any point of rank r in T. Since every base sub-triangle is standard, it follows by an easy induction that $\langle z_i, x, z_k \rangle \geq \frac{r}{2} p$ if r is even, and $\langle z_i, x, z_k \rangle \geq \frac{r-1}{2} p + 1$ if r is odd.

Suppose first that T does not contain a point of rank 4. Then T contains at least $p + 1$ points, all of rank $0, 1, 2$ or 3. Let $P_0 := T$. We show how to construct a sequence of convex polygons $P_1, P_2, \dots, P_s$ satisfying the conditions

(i) z_j and z_k are vertices of P_t $(1 \leq t \leq s)$;
(ii) P_t has at most 6 vertices $(1 \leq t \leq s)$;
(iii) every point of $\mathcal{X}$ in P_t belongs to the closure of P_{t+1} $(0 \leq t \leq s - 1)$;

(iv) P_s is empty.

Suppose that we have already defined P_t for some $t \geq 0$. If $\langle P_t \rangle = 0$, then set $s := t$. Otherwise, construct $P_{t+1} = z_j y_1 \dots y_r z_k$, where $1 \leq r \leq 4$, as follows. Let y_1 be the first point of $\mathcal{X}$ lying in P_t, in counter-clockwise order of visibility from z_j. Let $\mathcal{T}_1$ denote the set of points of $\mathcal{X}$ lying in P_t but not contained in $\triangle z_j y_1 z_k$.

If $\mathcal{T}_1 = \emptyset$, then letting $r = 1$, $P_{t+1} = z_j y_1 z_k$ meets all the requirements. Otherwise, let y_2 be the first point of $\mathcal{T}_1$ in counter-clockwise order of visibility from y_1. Clearly, $z_j y_1 y_2 z_k$ is a convex quadrilateral, and the rank of y_2 is smaller than that of y_1. Let $\mathcal{T}_2$ denote the set of points of $\mathcal{T}_1$ not contained in the quadrilateral $z_j y_1 y_2 z_k$. If $\mathcal{T}_2 = \emptyset$, then letting $r = 2$, $P_{t+1} = z_j y_1 y_2 z_k$ meets all the requirements. Otherwise, let y_3 be the first point of $\mathcal{T}_2$ in counter-clockwise order of visibility from y_2. Clearly, $z_j y_1 y_2 y_3 z_k$ is a convex pentagon, and the rank of y_3 is smaller than that of y_2. Finally, let $\mathcal{T}_3$ denote the set of points of $\mathcal{T}_2$ not contained in the pentagon $z_j y_1 y_2 y_3 z_k$. If $\mathcal{T}_3 = \emptyset$, then letting $r = 3$, $P_{t+1} = z_j y_1 y_2 y_3 z_k$ meets all the requirements. Otherwise, let y_4 be the first point of $\mathcal{T}_3$ in counter-clockwise order of visibility from y_3. Clearly, $z_j y_1 y_2 y_3 y_4 z_k$ is a convex hexagon, and the rank of y_4 is smaller than that of y_3. Therefore, the rank of y_4 is 0, and every point of $\mathcal{T}_3$ is contained in the hexagon $z_j y_1 y_2 y_3 y_4 z_k$, which satisfies all the conditions (i)–(iv).

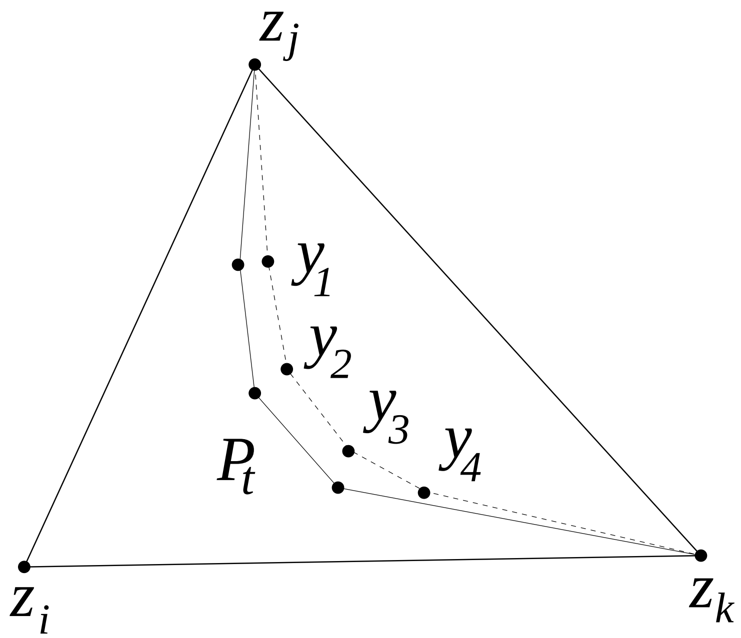

Figure 2.

Suppose next that T contains a point x of rank 4. Let x' denote the intersection point of the line $z_j x$ and the segment $z_i z_k$. Now $\triangle z_i x z_k$ contains at least $2p$ points of $\mathcal{X}$. Thus, we may assume without loss of generality that $\triangle x x' z_k$ contains at least p points of $\mathcal{X}$, all of rank $0, 1, 2$ or 3. In this case, let

$P_0 := z_j x' z_k$.

In the same way as above, one can construct a sequence of convex polygons $P_1, P_2, \ldots, P_s$ satisfying the conditions

(i) z_j, z_k, and x are vertices of P_t $(1 \le t \le s)$;
(ii) P_t has at least 4 and at most 7 vertices $(1 \le t \le s)$;
(iii) every point of $\mathcal{X}$ in P_t belongs to the closure of P_{t+1} $(0 \le t \le s-1)$;
(iv) $P_s \cap \triangle z_i x z_k$ is empty.

In both cases, it follows from the properties of the polygons P_t that $\langle P_t \rangle > \langle P_{t+1} \rangle \ge \langle P_t \rangle - 4$, for $0 \le t \le s-1$. Furthermore, we have $\langle P_0 \rangle - \langle P_s \rangle \ge p - 4$. Therefore, there exists an integer $1 \le t' \le s$ such that $\langle P_{t'} \rangle \equiv 7 + r - n$ for some $0 \le r \le 7$. Then $P = \text{conv}(P_{t'}, z_{j+2}, z_{j+4}, \ldots, z_{j+2(n-7-r)})$ is a p-empty polygon. Suppose $P_{t'}$ has $7 - r'$ vertices for some $0 \le r' \le 4$. For $m = 1, 2, \ldots, r + r'$, let w_m be the lowest point of $\mathcal{X}$ in $\triangle z_{j+2m-2} z_{j+2m-1} z_{j+2m}$. Then $\text{conv}(P, w_1, w_2, \ldots, w_{r+r'})$ is a p-empty n-gon. ◻

Claim 3.9. *Suppose that $K \ge 4p - 1$ and $2 \le q \le 6$. Then $\mathcal{X}$ determines a p-empty convex n-gon.*

Proof: Consider the triangle $T = \triangle z_i z_j z_k$, where $i = 1, j = 2p$ and $k = 4p - 1$. Let x be the point of $\mathcal{X}$ in $\text{int}(T)$, closest to the line $z_i z_j$.

Then $\triangle z_i x z_k$ is a standard base sub-triangle, so that $\langle z_i, x, z_k \rangle_p = 0$ or q. Since $\langle T \rangle_p = q$, we have $\langle z_j, x, z_k \rangle_p = p - 1$ or $q - 1$. In the first case, choose an integer $1 \le a \le p - 1$ such that $aq \equiv 1 \bmod p$. In the second case, choose an integer $1 \le a \le p - 1$ such that $aq \equiv p - q + 1 \bmod p$. In either case, $(p - q + 1)/q \le a \le (p(q-1) + 1)/q$.

The polygon $\text{conv}(z_j, z_{j+2}, z_{j+4}, \ldots, z_{j+2a}, z_k, x)$ has $a+3$ vertices and is p-empty. Since $n \ge 5p/6 + 4$, $q \le 6$ and $p \ge 23$, we have $a + 3 \le n \le p + 1 \le a(q+1) + 3$. Thus, there is a non-negative integer $f \le aq$ such that $f + a + 3 = n$. Note that $q < \frac{p+1}{2}$, so we can apply Claim 3.7 to conclude that, for every $1 \le m \le a$, there is a q-element subset $\mathcal{U}_m \subseteq \mathcal{X}$ in the interior of $\triangle z_{j+2m-2} z_{j+2m-1} z_{j+2m}$, which, together with z_{j+2m-2} and z_{j+2m}, forms the vertex set of an empty convex $(q+2)$-gon. Let $\mathcal{U}$ be an f-element subset of $\mathcal{U}_1 \cup \ldots \cup \mathcal{U}_a$. Then $\text{conv}(z_j, z_{j+2}, z_{j+4}, \ldots, z_{j+2a}, z_k, x, \mathcal{U})$ is a p-empty n-gon. ◻

For the rest of the proof, we assume that $K \ge 4p - 1$ and $q \ge 7$. Fix $T = \triangle z_i z_j z_k$, where $i = 1, j = 2p$ and $k = 4p - 1$.

Claim 3.10. *Suppose that all points of $\mathcal{X}$ in the interior of T have rank 0. Then $\mathcal{X}$ determines a p-empty convex n-gon.*

Proof: Let $x_0, x_1, x_2, \ldots x_s$ denote the points of $\mathcal{X}$ in the interior of T, listed in counter-clockwise order of visibility from z_j. Clearly, we have $s \ge q - 1$. There is an integer $1 \le a \le q - 1$ such that $\left\lfloor \frac{ap}{q} \right\rfloor + 3 \le n \le \left\lfloor \frac{(a+1)p}{q} \right\rfloor + 2$. Then $n = \left\lfloor \frac{ap}{q} \right\rfloor + 3 + b$, where $0 \le b \le \left\lfloor \frac{p}{q} \right\rfloor \le \left\lfloor \frac{ap}{q} \right\rfloor$. Write $ap \equiv c \bmod q$ with $1 \le c \le q - 1$.

The convex polygon $P = z_i z_{i+2} \ldots z_{i+2\lfloor ap/q \rfloor} z_j x_c$ has $\left\lfloor \frac{ap}{q} \right\rfloor + 3$ vertices and contains $\langle P \rangle \equiv q \left\lfloor \frac{ap}{q} \right\rfloor + c \equiv 0 \bmod p$ points in its interior. For $l = 1, 2, \ldots b$, let y_l be the lowest point of $\mathcal{X}$ in $\triangle z_{i+2l-2} z_{i+2l-1} z_{i+2l}$

with respect to the side $z_{i+2l-2}z_{i+2l}$. Then $\text{conv}(z_j, z_{i+2}, \ldots, z_{i+2\lfloor ap/q \rfloor}, z_j, x_c, y_1, \ldots y_b)$ is a p-empty convex polygon with $\left\lfloor \frac{ap}{q} \right\rfloor + 3 + b = n$ vertices. ◻

It remains to prove

Claim 3.11. *Suppose that there is a point of $\mathcal{X}$ in the interior of T, whose rank is 1. Then $\mathcal{X}$ determines a p-empty convex n-gon.*

Proof: Let $x \in \mathcal{X}$ be a point of rank 1 in the interior of T. Then $\triangle z_i x z_k$ is a standard, non-empty base sub-triangle with at least q points in its interior, all of which have rank 0. Let $x_0, x_1, \ldots, x_r$ denote the points of $\mathcal{X}$ in the interior of $\triangle z_i x z_k$, listed in counter-clockwise order of visibility from z_j. Suppose that the line $z_j x$ separates $x_0, \ldots, x_t$ from $x_{t+1}, \ldots, x_r$. Since $r \geq q-1$, we may assume without loss of generality that $t \geq t_0 = \lceil q/2 \rceil - 1$.

Letting $s_0 := \langle z_i, x_0, x, z_j \rangle$, we have $\langle z_i, x_m, x, z_j \rangle = s_0 + m$, for $0 \leq m \leq t$. Choose an integer $1 \leq s_0' \leq p$ satisfying $s_0' \equiv s_0 \bmod p$. Let $I \subset \{1, 2, \ldots, q\}$ be an interval of consecutive integers, defined as follows:

$$I = \begin{cases} \{2, 3, \ldots, \lfloor q/2 \rfloor + 2\}, & \text{if } 7 \leq q \leq 11; \\ \{\lceil q/3 \rceil - 1, \ldots, \lfloor 5q/6 \rfloor\}, & \text{if } 12 \leq q \neq (p+1)/2; \\ \{\lceil q/3 \rceil + 1, \ldots, \lfloor 5q/6 \rfloor + 2\}, & \text{if } q = (p+1)/2. \end{cases}$$

In view of the fact that $(p, q) = 1$, we have that $|\{bp \bmod q \mid b \in I\}| = |I| \geq \lfloor q/2 \rfloor + 1$. Furthermore, $|\{a \bmod q \mid s_0' \leq a \leq s_0' + t_0\}| = t_0 + 1 = \lceil q/2 \rceil$. Thus, by the pigeonhole principle, there are integers a, b satifying $s_0' \leq a \leq s_0' + t_0$, $b \in I$ such that $bp \equiv a \bmod q$. Let $a = cq + r$, $0 \leq r < q$. Then $0 \leq c < p/q + 1$ and $bp = Cq + r$, where $C = \lfloor bp/q \rfloor$. Let $a' = a - s_0'$. Clearly, we have $C - c \geq 0$ and $0 \leq a' \leq t$.

The polygon $P = z_i z_{i+2} z_{i+4} \ldots z_{i+2(C-c)} z_j x x_{a'}$ has $C - c + 4$ vertices and $\langle P \rangle \equiv (C-c)q + s_0 + a' = Cq + r - a + s_0 + a' = bp \equiv 0 \bmod p$.

By modifying P, we will increase the number of vertices to n without changing the number of interior points. For $m = 1, 2, \ldots, C - c$, let $\mathcal{U}_m \subset \triangle z_{i+2m-2} z_{i+2m-1} z_{i+2m}$ denote a set of q points if $q \neq \frac{p+1}{2}$ and a set of 2 points if $q = \frac{p+1}{2}$, whose existence is guaranteed in Claim 3.7. Let $\mathcal{U} = \mathcal{U}_1 \cup \ldots \cup \mathcal{U}_{C-c}$. Then we have

$$|\mathcal{U}| = \begin{cases} q(C-c), & \text{if } q \neq \frac{p+1}{2}; \\ 2(C-c), & \text{if } q = \frac{p+1}{2}. \end{cases}$$

One can readily check that $C - c + 4 \leq C + 4 \leq 5p/6 + 6 \leq n$. It is sufficient to prove that $|\mathcal{U}| \geq n - (C - c + 4)$. Then there exists a $\mathcal{U}' \subseteq \mathcal{U}$, $|\mathcal{U}'| = n - (C - c + 4)$ such that $\text{conv}(P \cup \mathcal{U}')$ is a p-empty n-gon. We distinguish three cases.

Case 1: $7 \leq q \leq 11$. In this case, $p \geq 264 \geq 2q(q+1)$, so that $C - c > p/q - 2 \geq p/(q+1)$. Note that $q \neq \frac{p+1}{2}$. Thus, $|\mathcal{U}| + C - c = (q+1)(C-c) \geq p \geq n - 4$, and the statement follows.

Case 2: $12 \leq q \neq \frac{p+1}{2}$. We also have $p \geq 24$, so that $1/3 - 2/q - 1/(q+1) \geq 2/p$ and $C - c > p/3 - 2p/q - 2 \geq p/(q+1)$, as in the previous case.

Case 3: $q = \frac{p+1}{2}$. In this case, $c \leq 2$ and $C \geq p/3 + 1$. This implies $|\mathcal{U}| + C - c = 3(C-c) \geq p - 3 \geq n - 4$, and we are done. ◻

Note that the condition $n \geq 5p/6 + O(1)$ is heavily used in the proofs of Claims 3.9 and 3.11, and our arguments do not allow to replace it by a weaker bound.

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