Note on geometric graphs

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Abstract

A *geometric graph* is a graph drawn in the plane so that the vertices are represented by points in general position, the edges are represented by straight line segments connecting the corresponding points. We show that a geometric graph of n vertices with no $k+1$ pairwise disjoint edges has at most 2^9k^2n edges.

1 Introduction

A *geometric graph* G is a graph drawn in the plane by (possibly crossing) straight line segments, i.e., it is defined as a pair $G = (V, E)$, where V is a set of points in general position in the plane and E is a set of closed segments whose endpoints belong to V.

The following question was raised by Avital and Hanani [AH], Kupitz [K], Erdős and Perles. Determine the smallest number $e_k(n)$ such that any geometric graph with n vertices and $m > e_k(n)$ edges contains $k+1$ pairwise disjoint edges.

It follows from a result of Kupitz [K] that $e_k(n) \geq kn$ for any $k \leq n/2$. Pach and Törőcsik [PT] proved that $e_k(n) \leq k^4n$ for any fixed k, which was the first upper bound linear in n. Both the upper and lower bounds were improved by Tóth and Valtr [TV] to $\frac{3}{2}(k-1)n - 2k^2 \leq e_k(n) \leq k^3(n+1)$ $(k \leq n/2)$. In this note we further improve the upper bound.

Theorem 1. *For any* $k < n/2$,

$$e_k(n) \leq 2^9k^2n.$$

Let G be a geometric graph. For any vertex v, let $x(v)$ and $y(v)$ denote its x- and y-coordinate, respectively. An edge e is said to *lie below* an edge e', if every vertical line intersecting both e and e' intersects e strictly below e'.

Define four binary relations $\prec_i$ $(i = 1, \ldots, 4)$ on the edge set E as follows (see also [PT, PA, TV]). Let $e = v_1v_2, e' = v_1'v_2'$ be two disjoint edges of G, where $x(v_1) < x(v_2)$ and $x(v_1') < x(v_2')$. Then (see Fig. 1.)

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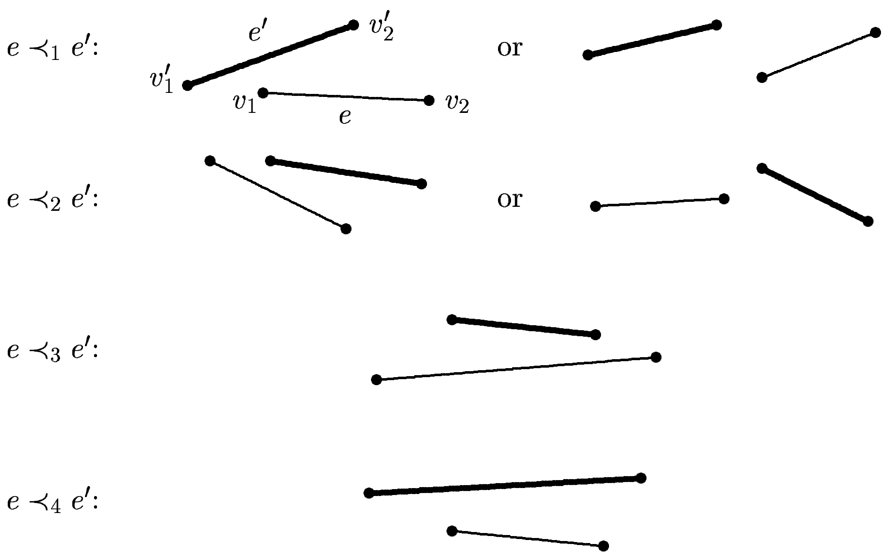


Figure 1.

$$\begin{array}{lll}
e \prec_1 e', & \text{if} & x(v_1) \geq x(v_1'),\ \ x(v_2) \geq x(v_2'),\ \ \text{and } e \text{ lies below } e', \\
e \prec_2 e', & \text{if} & x(v_1) \leq x(v_1'),\ \ x(v_2) \leq x(v_2'),\ \ \text{and } e \text{ lies below } e', \\
e \prec_3 e', & \text{if} & x(v_1) \leq x(v_1'),\ \ x(v_2) \geq x(v_2'),\ \ \text{and } e \text{ lies below } e', \\
e \prec_4 e', & \text{if} & x(v_1) \geq x(v_1'),\ \ x(v_2) \leq x(v_2'),\ \ \text{and } e \text{ lies below } e'.
\end{array}$$

Each of the relations $\prec_i$ is a partial ordering, and any pair of disjoint edges of G is comparable by at least one of them. Theorem 1 is a direct consequence of the following stronger statement.

Theorem 2. *Let $k \leq n/2$ and let G be a geometric graph with no $k+1$ edges forming a chain with respect to any of the partial orders $\prec_1, \prec_2, \prec_3, \prec_4$. Then*

$$e(G) \leq 2^9 k^2 n.$$

For $k = O(\sqrt{n})$ this result can not be improved apart from the value of the constant.

The relations $\prec_1, \prec_2, \prec_3, \prec_4$ were introduced by Pach and Törőcsik [PT]. In fact, their result was analogous to Theorem 2, with the weaker bound $e \leq k^4 n$.

2 Proof of Theorem 2.

For any graph G, let $e(G)$ denote the number of edges of G. Let G be a geometric graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and with no $k+1$ edges forming a chain in any of the partial orderings $\prec_1, \ldots, \prec_4$. If there are two vertices with the same x-coordinates, we can perturb them to have different x-coordinates. It is easy to see that this way we did not create any additional chain. Therefore, we can suppose without loss of generality that all vertices have different x-coordinates and the vertices are numbered from left to right.

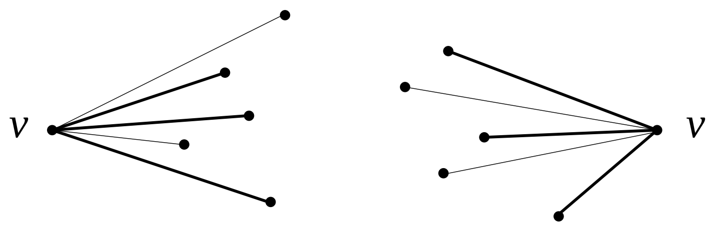
For any vertex v_i, the *left edges* (resp. *right edges*) of v_i are those edges v_iv_j of G, where $i > j$ (resp. $i < j$). The *left degree* l_i (resp. the *right degree* r_i) of v_i is the number of left edges (resp. right edges) of v_i.

Lemma. *Let $X = \{x_1, x_2, \ldots, x_m\}$ be a sequence of different real numbers. Then there are pairwise disjoint monotone subsequences $X_1, X_2, \ldots, X_l \subset X$ such that for $i = 1, 2, \ldots, l$, $|X_i| = \lceil\sqrt{m/2}\rceil$, and $|X_1| + |X_2| + \cdots + |X_l| \geq m/2$.*

Proof. Take a monotone subsequence of size $\lceil\sqrt{m/2}\rceil$ of X and delete it from X. Continue as long as there are at least $m/2$ elements of X left. It can be done by the Erdős-Szekeres Theorem [ES35]. □

Return to the proof of Theorem 2. Do the following procedure on G, for $i = 1, 2, \ldots, n$.

> Right Decomposition Procedure[i]. Let $v = v_i$, $r = r_i$ and let $e_1, e_2, \ldots, e_r$ be the right edges of v in clockwise order (such that the clockwise angle enclosed by e_1 and e_r is less than $180°$). Let $x(e_j)$ denote the x-coordinate of the endpoint of e_j different from v. By the Lemma, the sequence $x(e_1), x(e_2), \ldots, x(e_r)$ contains monotone subsequences, each of size $\lceil\sqrt{r/2}\rceil$ such that their total size is at least $r/2$. It defines a partition of the corresponding edges into subsequences. Call each subset of those edges which belong to the same subsequence, *right-block* of edges at v_i. Delete those edges which do not belong to any of the subsequences. For any remaining edge e_j, we say that the *type* of e_j is *right-increasing* (resp. *right-decreasing*) if $x(e_j)$ belongs to an *increasing* (resp. *decreasing*) subsequence. (See Fig. 2.)



right increasing subsequence left increasing subsequence

Figure 2.

Call the resulting graph G_1. Clearly, $e(G_1) \geq e(G)/2$. Since every edge of G_1 is either of type right-increasing or right-decreasing, at least half of the edges are of the same type, say, right-increasing. (The other case can be treated analogously, as explained in the remark at the end of the paper.) Delete all right-decreasing edges from G_1, and call the resulting graph G_2. It follows that $e(G_2) \geq e(G)/4$.

Let $l'_1, l'_2, \ldots, l'_n$ be the left degrees of $v_1, v_2, \ldots, v_n$ respectively, in G_2. Since $G_2 \subset G$, $l'_i \leq l_i$. Apply the LEFT DECOMPOSITION PROCEDURE on G_2, analogous to the RIGHT DECOMPOSITION PROCEDURE. Let the resulting graph be G_3, we have that $e(G_3) \geq e(G)/8$. Suppose that at least half of the edges of G_3 are left-increasing. (The other case can be treated analogously, as explained in the remark at the end of the paper.) Delete all left-decreasing edges from G_3, and call the resulting graph G_4. It follows that $e(G_4) \geq e(G)/16$.

For two edges of G_4 with a common endpoint, $e_1 = v_iv_j$, $e_2 = v_iv_k$ we say that e_2 is a *right-zag* of e_1, if both e_1 and e_2 are right edges of v_i, and e_2 follows immediately after e_1 in the same right-block at v_i. Analogously, for $e_1 = v_iv_j$ and $e_2 = v_iv_k$ we say that e_2 is a *left-zag* of e_1, if both e_1 and e_2 are left edges of v_i, and e_2 follows immediately after e_1 in the same left-block at v_i.

A path $e_1e_2 \ldots e_m$ of G_4 is said to be a *zig-zag path* if one of the following three conditions holds.

(i) $m = 1$

(ii) For any $1 \leq i \leq m-1$, e_{i+1} is a right-zag of e_i if i is odd and a left-zag if i is even.

(iii) For any $1 \leq i \leq m-1$, e_{i+1} is a right-zag of e_i if i is even and a left-zag if i is odd.

Observe that each edge of G_4 has at most one right-zag and one left-zag. Also, each edge is a right-zag and a left-zag of at most one edge. Therefore, each edge of G_4 is contained in at most *two* maximal zig-zag paths.

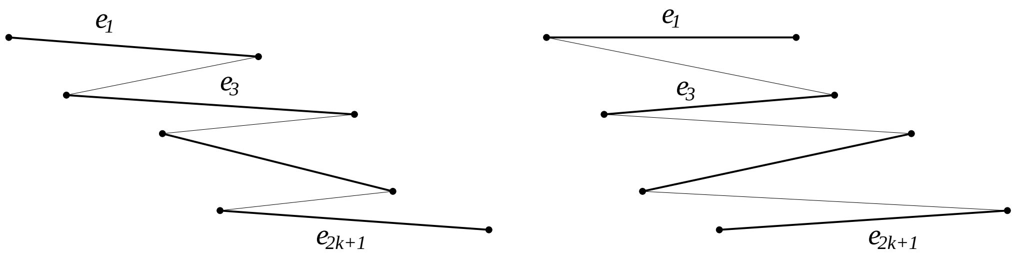
Claim 1. Every zigzag path in G_4 has at most $2k$ edges.

Proof. Suppose that $e_1e_2 \ldots e_{2k+1}$ is a zig-zag path and let $1 \leq i \leq 2k-2$. First we show that $e_i \prec_1 e_{i+2}$. Suppose that $e_i = v_av_b$, $e_{i+1} = v_bv_c$, and $e_{i+2} = v_cv_d$. We distinguish two cases.

Case 1. e_{i+1} is a right-zag of e_i and e_{i+2} is a left-zag of e_{i+1}. Then e_{i+1} follows e_i in a right block of v_b, so $x(v_a) < x(v_c)$. Also, e_{i+2} follows e_{i+1} in a right block of v_c, so $x(v_b) < x(v_d)$. Clearly, e_{i+2} is below e_i, so $e_i \prec_1 e_{i+2}$.

Case 2. e_{i+1} is a left-zag of e_i and e_{i+2} is a right-zag of e_{i+1}. Then e_{i+1} follows e_i in a left block of v_b, so $x(v_a) < x(v_c)$. Also, e_{i+2} follows e_{i+1} in a left block of v_c, so $x(v_b) < x(v_d)$. Clearly, e_{i+2} is below e_i, so $e_i \prec_1 e_{i+2}$.

Consequently, $e_1 \prec_1 e_3 \prec_1 e_5 \prec_1 \ldots \prec_1 e_{2k+1}$ so there is a chain of length $k+1$, a contradiction (see Fig. 3.). This concludes the proof of Claim 1. □



$$e_1 \prec_1 e_3 \prec_1 \ldots \prec_1 e_{2k+1}$$

Figure 3.

Claim 2. There are at most $\sqrt{2e(G)/n}$ maximal zig-zag paths.

Proof. For each vertex v_i, the number of maximal zig-zag paths starting at v_i is at most the number of blocks of edges at v_i. Since each right block in G has size $\lceil\sqrt{r_i/2}\rceil$, the number of right blocks at v_i in G is at most $\sqrt{r_i/2}$. Therefore, the number of right blocks at v_i in G_4 is also at most $\sqrt{r_i/2}$. Similarly, the number of left blocks at v_i in G_2 is at most $\sqrt{l'_i/2}$, so the number of left blocks at v_i in G_4 is at most $\sqrt{l'_i/2} \leq \sqrt{l_i/2}$. Therefore, for the total number Z of maximal zig-zag paths in G_4 we have that

$$Z \leq \sum_{i=1}^{n}\left(\sqrt{r_i/2}+\sqrt{l_i/2}\right) \leq \sqrt{n}\sqrt{\sum_{i=1}^{n} r_i + l_i} = \sqrt{2e(G)n}.$$

□

Each edge of G_4 is covered by at most two maximal zig-zag paths, hence using Claims 1 and 2 we get that

$$e(G_4) \leq \frac{1}{2}2k\sqrt{2e(G)n}.$$

Therefore,

$$\frac{e(G)}{16} \leq \frac{1}{2}2k\sqrt{2e(G)n},$$

which implies that

$$e(G) \leq 2^9k^2n.$$

This concludes the proof of the upper bound. For the lower bound assume that $k \leq \sqrt{n/2}$ and consider the following geometric graph $G(k,n)$. Take a slightly perturbed $k \times k$ piece of a unit square grid and rotate it slightly anticlockwise direction. Place the remaining $n-k^2$ points very far to the right and connect each vertex in the lattice with each of the remaining vertices. $G(k,n)$ has n vertices, $k^2n-k^4 \geq k^2n/2$ edges, and it is easy to see that there are no $k+1$ edges that form a chain with respect to any of the relations $\prec_i$. If $\sqrt{n/2} \le k \le c\sqrt{n}$ then consider $G(k', n)$ with $k' = \sqrt{n/2}$ (suppose for simplicity that it is an integer). $G(k', n)$ has n vertices, $n^2/4 \ge k^2 n/4c^2$ edges, and there are no $k+1$ edges that form a chain with respect to any of our relations $\prec_i$. $\square$

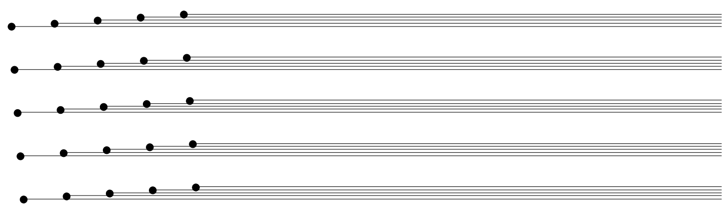


Figure 4.

Remarks. 1. It the proof of the upper bound, we assumed that the edges of G_4 belong to *right increasing* and *left increasing* blocks. It the other three cases the proof is analogous. The only difference is that in the proof of Claim 1 we have to use $\prec_2, \prec_3$, or $\prec_4$ in place of $\prec_1$. See Fig. 5.

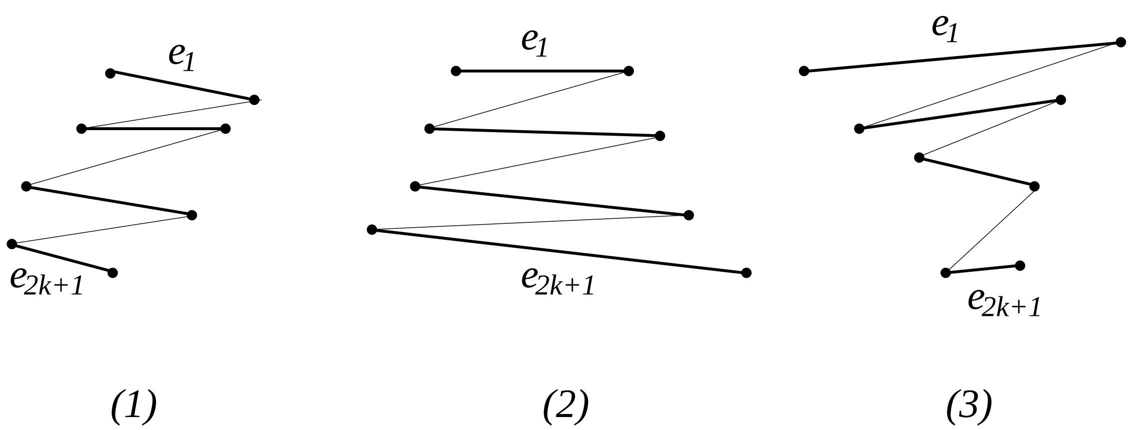


Figure 5.

(1) The edges of G_4 belong to *right decreasing* and *left decreasing* blocks. Then $e_1 \prec_2 e_3 \prec_2 \ldots \prec_2 e_{2k+1}$.
(2) The edges of G_4 belong to *right increasing* and *left decreasing* blocks. Then $e_1 \prec_3 e_3 \prec_3 \ldots \prec_3 e_{2k+1}$.
(3) The edges of G_4 belong to *right decreasing* and *left increasing* blocks. Then $e_1 \prec_4 e_3 \prec_4 \ldots \prec_4 e_{2k+1}$.

2. Theorem 2 guarantees that any geometric graph with n vertices and $e > 2^9k^2n$ edges contains $k+1$ edges that form a chain. Following the proof of Theorem 2, it is easy to design a polynomial algorithm that finds such a set of $k+1$ edges.

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