ON THE HOLLOW ENCLOSED BY CONVEX SETS

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Abstract

For $n \leq d$, a family $\mathscr{F} = \{C_0, C_1, \ldots, C_n\}$ of compact convex sets in $\mathbb{R}^d$ is called an n-critical family provided any n members of $\mathscr{F}$ have a non-empty intersection, but $\bigcap_{i=0}^{n} C_i = \varnothing$. If $n = d$ then a lemma on the intersection of convex sets due to Klee implies that the $d+1$ members of the d-critical family enclose a 'hollow' in $\mathbb{R}^d$, a bounded connected component of $\mathbb{R}^d \setminus \bigcup_{i=0}^{n} C_i$. Here we prove that the closure of the convex hull of a hollow in $\mathbb{R}^d$ is a d-simplex.[1]

Besides the Helly-theorem on intervals in $\mathbb{R}^1$ a less notable property is that two disjoint intervals can be separated by a point, in other words, there is a 'hollow' (an interval) between them, a gap, which cannot be bridged with two intervals having empty intersection. This separation or gap property, trivial as it is, helps characterize the intersection patterns of

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convex sets in $\mathbb{R}^1$ in terms of 'interval graphs'. Actually, the gap property implies the foremost necessary condition that an interval graph must be chordal, namely, each cycle of length more than three has a chord (see [5]). Just as Helly's theorem is established in $\mathbb{R}^d$, for every $d \geq 1$, the separation or gap property has extensions to higher dimension.

A family of compact convex sets $C_0, C_1, \ldots, C_n \subset \mathbb{R}^d$ is called here an n-*critical* family if $\bigcap_{i \neq j} C_i \neq \varnothing$, for every $j = 0, 1, \ldots, n$, but $\bigcap_{i=0}^{n} C_i = \varnothing$. The denotation 'critical'[2] becomes clear when in some finite family of sets with empty intersection we consider a 'smallest' subset with the same property, a 'critical subfamily'.

Convexity and compactness in the definition of a critical family was chosen here with combinatorial geometry applications in mind (see [13]). However, in intersection or covering theorems of topology, when a finite or infinite family of sets appears, the compactness requirement of the members might be relaxed (see [17]), and the condition $\bigcap_{i=0}^{n} C_i = \varnothing$ is usually replaced with its contrapositive that $\bigcup_{i=0}^{n} C_i$ is a convex set, which denies the hollow (see [14]). Meanwhile, the primary condition that $\bigcap_{i \neq j} C_i \neq \varnothing$, for every $j = 0, 1, \ldots, n$, is unchanged and displays a topology variation of n-criticality in the different contexts.

The role of n-critical families (or its variations) in Euclidean spaces was recognized by Klee [14, 15], Berge [3], and Ghouila-Houri [9] in the study of intersection properties of convex sets. These properties are closely related to fixed point theorems and minimax theorems as explored by Fan [6]. As a result, the intersection theorems and their applications were extended further in functional analysis and in topology by Balaj [1], Ben-El-Mechaiekh [2], Fan [6, 7], Horvath [12] and others, by replacing the Euclidean space with general topological vector spaces. All these investigations are originated in classical topology results such as the Sperner's lemma [20], and its generalizations starting with the Knaster, Mazurkievicz, Kuratowski-theorem [16, 17].

Observe that by Helly's theorem [11], there is no n-critical family in $\mathbb{R}^d$ provided $n > d$. A fundamental lemma due to Klee [14] implies that for $n = d$ there is a bounded domain $D \subseteq \mathbb{R}^d \setminus \bigcup_{i=0}^{d} C_i$ called here the *hollow* enclosed by the d-critical family in $\mathbb{R}^d$ (Corollary 1.3). Section 1 contains different proofs of Klee's fundamental covering lemma displaying its many faceted connections to combinatorial topology. In Section 2 it is proved that the closure of the convex hull of a hollow in $\mathbb{R}^d$ is a d-simplex (Theorem 2.1). An immediate corollary of the hollow theorem, related to an early result of Ghouila-Houri [9], is formulated in Section 3 (Theorem 3.2). The note concludes with a separation property of n-critical families in $\mathbb{R}^d$, actually a corollary of a more general separation result by Klee [14, Theorem 1], for the case $n < d$, when there is no hollow enclosed by the family (Theorem 3.3).

[2] The concept of criticality was introduced in graph theory by T. Gallai [8]

Given a set $X \subset \mathbb{R}^d$, the convex hull, the closure, and the boundary of X is denoted by $\mathrm{Conv}(X)$, $\mathrm{cl}(X)$, and ∂X, respectively.

1 Klee's lemma

A basic lemma discovered by Klee [14] and independently by Berge [3] captures a fundamental intersection property of n-critical families. We include here three proofs using different techniques and displaying a many faceted connections of the lemma to topology. The first purely geometry proof is using the standard separation theorem of disjoint compact convex sets (c.f. [15]). The second proof was outlined by Berge [3] and applies a combinatorial topology result deduced from Sperner's lemma [20]. The last proof uses the KKM lemma from fixed-point theory due to Knaster, Kuratowski, and Mazurkievicz [16].

Lemma 1.1. [Klee [14], Berge [3]]. *Let $C_0, C_1, \ldots, C_n \subset \mathbb{R}^d$ be compact convex sets such that $\bigcap_{\substack{i=1\\ i\neq j}}^{n} C_i \neq \varnothing$, for every $j = 0, 1, \ldots, n$. If $\bigcup_{i=0}^{n} C_i$ is convex, then $\bigcap_{i=0}^{n} C_i \neq \varnothing$.*

Proof. The proof is induction on n. The case $n = 0$ is trivial; assume that $n \geq 1$ and the claim is true for n convex sets. If $\bigcap_{i=0}^{n} C_i = \varnothing$, then C_n and $A = \bigcap_{i=0}^{n-1} C_i$ are disjoint compact convex sets, thus they can be strictly separated with a hyperplane H such that $H \cap A = H \cap C_n = \varnothing$. Let $C_i' = H \cap C_i$, $0 \leq i \leq n-1$.

For every $j = 0, \ldots, n-1$, the condition $\bigcap_{\substack{i=1\\ i\neq j}}^{n} C_i = C_n \cap \left(\bigcap_{\substack{i=1\\ i\neq j}}^{n-1} C_i \right) \neq \varnothing$ combined with $H \cap C_n = \varnothing$ imply that $H \cap \left(\bigcap_{\substack{i=1\\ i\neq j}}^{n-1} C_i \right) = \bigcap_{\substack{i=1\\ i\neq j}}^{n-1} C_i' \neq \varnothing$. Because $\bigcup_{i=0}^{n-1} C_i' = (H \cap C_n) \cup \left(\bigcup_{i=0}^{n-1} H \cap C_i \right) = H \cap \left(\bigcup_{i=0}^{n} C_i \right)$ is convex, we obtain by induction that $\bigcap_{i=0}^{n-1} C_i' = H \cap \left(\bigcap_{i=0}^{n-1} C_i \right) = H \cap A \neq \varnothing$, a contradiction. $\square$

Second proof of Lemma 1.1. Let $a_j \in \bigcap_{i \neq j} C_i$, for $j = 0, 1, \ldots, n$, and set $S = \mathrm{Conv}(\{a_0, \ldots, a_n\})$ for the convex hull of these $n+1$ points. If S is not a simplex, then they span an affine subspace of dimension $n-1$ or less, then by Helly's theorem the claim $\bigcap_{i=0}^{n} C_i \neq \varnothing$ follows. We assume now that S is an n-simplex. Since the facet $S^{(j)} \subset S$ opposite a_j is included in C_j and $\bigcup_{i=0}^{n} C_i$ is convex, we have $S \subseteq \bigcup_{i=0}^{n} C_i$.

We take a simplicial subdivision of S with arbitrary small mesh[3]. A

[3] mesh = the maximum diameter of the simplices of the subdivision

Sperner coloring[4] of the vertices of the subdivision is defined next. For a vertex v of the subdivision let the color of v be any index $j \in \{0, 1, \ldots, n\}$ such that $v \in C_{j-1} \setminus C_j$ (where $C_{-1} = C_n$). A color j exists for every $v \in S$, since otherwise, $v \in \bigcap_{i=0}^{n} C_i$, and the claim follows. Observe, if j is the color of $v \in \mathrm{Conv}(\{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\})$, then $j \in \{i_0, i_1, \ldots, i_k\}$ follows by the convexity of C_j, and because $v \notin C_j$. Then by Sperner's lemma, there is an n-simplex whose vertices are multicolored with $n+1$ different colors.

By repeating the procedure with simplicial sudivisions of S with mesh $\varepsilon \searrow 0$, there is a convergent subsequence of the multicolored subdividing simplices approaching a point $p \in S$. This limit point satisfies $p \in C_{j-1}$, for every $j = 0, 1, \ldots, n$, thus $\bigcap_{i=0}^{n} C_i \neq \varnothing$ follows. □

The KKM lemma due to Knaster, Kuratowski, and Mazurkievicz [16] is known as a remarkable intersection theorem for closed covers of a Euclidean simplex. Extending the Sperner lemma [20] the KKM lemma was the starting point of further generalizations to topological vector spaces [2, 12, 17]; these variations have been applied in mathematical fixed-point theory [7].

A set-valued map Γ of the points of an arbitrary set $X \subset \mathbb{R}^d$ into sets of $\mathbb{R}^d$ is called a *KKM map on X* if for every finite subset $N \subseteq X$, $\mathrm{Conv}(N) \subseteq \bigcup_{x \in N} \Gamma(x)$. Ben-El-Mechaiekh [2] proves a particular version of the KKM theorem stated as follows.

Theorem 1.2. *If Γ is a KKM map on $X \subset \mathbb{R}^d$ such that, for every $x \in X$, $\Gamma(x)$ is a non-empty closed convex subset of $\mathbb{R}^d$, then the family $\mathscr{F} = \{\Gamma(x)\}_{x \in X}$ has the finite intersection property, that is the intersection of the members of any finite subfamily of $\mathscr{F}$ is nonempty.* □

For finite sets X the claim in Theorem 1.2 simply becomes $\bigcap_{x \in X} \Gamma(x) \neq \varnothing$. As observed by Ben-El-Mechaiekh [2], Klee's fundamental intersection theorem (Lemma 1.1) follows from the finite version of Theorem 1.2.

Third proof of Lemma 1.1. Let $a_j \in \bigcap_{i \neq j} C_i$, for $j = 0, 1, \ldots, n$. Define the map $\Gamma(a_i) \mapsto C_{i-1}$, for $i = 0, 1, \ldots, n$, (where $C_{-1} = C_n$). We verify that Γ is a KKM map on $A = \{a_0, a_1, \ldots, a_n\}$; let $N \subseteq A$.

For $N = A$, because $A \subset \bigcup_{i=0}^{n} C_i$ and $C = \bigcup_{i=0}^{n} C_i$ is convex, we obtain $\mathrm{Conv}(N) = \mathrm{Conv}(A) \subset C = \bigcup_{a_i \in N} \Gamma(a_i)$. For $N \neq A$, let j be an index such that $a_j \in N$, and $a_{j-1} \notin N$. Observe that $N \subset C_{j-1}$, and since C_{j-1} is convex, we obtain $\mathrm{Conv}(N) \subset C_{j-1} = \Gamma(a_j) \subset \bigcup_{a_i \in N} \Gamma(a_i)$. By Theorem 1.2, $\bigcap_{a_i \in A} \Gamma(a_i) = \bigcap_{i=0}^{n} C_i \neq \varnothing$ follows. □

Corollary 1.3. *If $\{C_0, C_1, \ldots, C_d\}$ is a d-critical family in $\mathbb{R}^d$, then $\mathbb{R}^d \setminus$*

[4] a vertex v_i of the n-simplex $(v_0, \ldots, v_n)$ is colored with i, $i = 0, 1, \ldots, n$, furthermore; if $v \in \mathrm{Conv}(\{v_{i_0}, v_{i_1}, \ldots, v_{i_k}\})$ then the color of v is any index from $\{i_0, i_1, \ldots, i_k\}$

$\bigcup_{i=0}^{d} C_i$ *has a bounded connected component D, that is every ray emanating from any point of D intersects some* C_i*,* $0 \leq i \leq d$*.*

Proof. Let $a_j \in \bigcap_{i \neq j} C_i$, for $j = 0, 1, \dots, d$. If $E \subset \mathbb{R}^d$ is the affine space of dimension less than d, then the contradiction $\bigcap_{i=0}^{n} C_i \neq \varnothing$ is obtained by Helly's theorem. Let $S = \mathrm{Conv}(\{a_0, \dots, a_n\})$ be the d-simplex; notice that each face of S is contained in $\bigcup_{i=0}^{n} C_i$. The compact convex sets $C_i' = C_i \cap S$, $i = 0, 1, \dots, n$, form a d-critical family, thus by Lemma 1.1 $\bigcup_{i=0}^{d} C_i' \subset S$ is not convex, which means that S does not cover $\bigcup_{i=0}^{d} C_i$. Let $p \in S \setminus \bigcup_{i=0}^{d} C_i$. Because $\partial S \subseteq \bigcup_{i=0}^{d} C_i$, every ray emanating from p intersects C_j, for some $0 \leq j \leq d$. □

2 The Hollow theorem

Theorem 2.1. *If* $\mathscr{F} = \{C_0, \dots, C_d\}$ *is a* $d-$*critical family in* $\mathbb{R}^d$*, then one of the connected components of* $\mathbb{R}^d \setminus \bigcup_{i=0}^{d} C_i$ *is a non-empty bounded region D, and the closure of* $\mathrm{Conv}(D)$ *is a d-simplex.*

Proof. The claim is true for $d = 1$; let $d \geq 2$ and assume that the claim is true for $d - 1$. By Corollary 1.3, the hollow D enclosed by $\mathscr{F}$ exists. Furthermore, D is an open set, $\partial D \subseteq \partial C_0 \cup \ldots \cup \partial C_d$, and D is contained in any d-simplex S with vertices in $\bigcap_{h \neq j} C_h$, $j = 0, \dots, d$. Since S is closed, cl(Conv(D) $\subset S$.

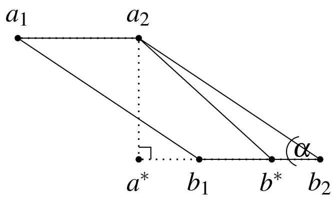
For $j = 0, \dots, d$, let $p_j \in \bigcap_{h \neq j} C_h$ be a closest point of $\bigcap_{h \neq j} C_h$ to C_j. We claim that $p_0, \dots, p_d$ are unique points of ∂D. Assume that this claim is true, and let S be the d-simplex with vertices $p_0, \dots, p_d$. Because cl(Conv(D)) is convex and the vertices of S belong to ∂D, we have $S \subset$ cl(Conv(D)). On the other hand, we know cl(Conv(D))$\subset S$, thus cl(Conv(D)) $= S$ follows.

1. We show that the simplex S is unique. Suppose that the points $a_1, a_2 \in \bigcap_{h \neq d} C_h$ and $b_1, b_2 \in C_d$ are such that the minimum distance between $\bigcap_{h \neq d} C_h$ and C_d is $m = |\overline{a_1 b_1}| = |\overline{a_2 b_2}|$.[5] Let the position vectors of a_i and b_i be $\mathbf{a_i}$ and $\mathbf{b_i}$, respectively. By convexity, $\mathbf{a} = \frac{1}{2}(\mathbf{a_1} + \mathbf{a_2}) \in \bigcap_{h \neq d} C_h$ and $\mathbf{b} = \frac{1}{2}(\mathbf{b_1} + \mathbf{b_2}) \in C_d$, hence $(\mathbf{a} - \mathbf{b})^2 \geq m^2$. Using $(\mathbf{a_1} - \mathbf{b_1})^2 = (\mathbf{a_2} - \mathbf{b_2})^2 = m^2$ and setting γ for the angle between $\mathbf{a_1} - \mathbf{b_1}$ and $\mathbf{a_2} - \mathbf{b_2}$ we obtain

$$
\begin{aligned}
2m^2 \leq 2(\mathbf{a} - \mathbf{b})^2 &= \tfrac{1}{2}(\mathbf{a_1} - \mathbf{b_1} + \mathbf{a_2} - \mathbf{b_2})^2 \\
&= \tfrac{1}{2}[(\mathbf{a_1} - \mathbf{b_1})^2 + (\mathbf{a_2} - \mathbf{b_2})^2] + (\mathbf{a_1} - \mathbf{b_1})(\mathbf{a_2} - \mathbf{b_2}) \\
&= m^2 + m^2 \cos\gamma \leq 2m^2.
\end{aligned}
$$

[5] $\overline{ab}$ is the line segment between points a and b

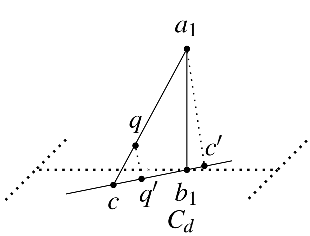
This implies $\cos\gamma = 1$, that is $\overline{a_1 b_1} \parallel \overline{a_2 b_2}$, hence either $\overline{a_1 b_1} = \overline{a_2 b_2}$ or (a_1, a_2, b_2, b_1) is a parallelogram.

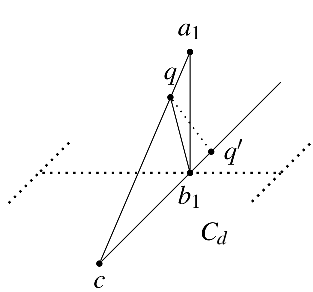


Assume that $\overline{a_1 b_1}$ and $\overline{a_2 b_2}$ are distinct segments. If (a_1, a_2, b_2, b_1) is not a rectangle, then set $\alpha = \angle a_2 b_2 b_1 < \pi/2$. Let a^* be the orthogonal projection of a_2 on the line through b_1, b_2, and let $b^* \in \overline{b_1 b_2} \cap \overline{a^* b_2}$. Then $b^* \in C_d$, and in the right triangle (a_2, a^*, b_2) we have $|\overline{a_2 b^*}| < |\overline{a_2 b_2}| = m$, a contradiction. Thus we obtain that (a_1, a_2, b_2, b_1) is a rectangle.

The open ball of radius m centered at a_1 is disjoint from C_d, hence the hyperplane through b_1, b_2 and perpendicular to $\overline{a_1 b_1}$ is a supporting hyperplane to C_d. For every $j = 0, \ldots, d-1$, select a point $c_j \in \bigcap_{h \neq j} C_h$. Apply Radon's theorem [19] on the $(d+2)-$element set $R = \{a_1, a_2, c_0, \ldots, c_{d-1}\}$. Let $J_1 \cup J_2 = R$ be the Radon-partition, and let $q \in \mathrm{Conv}(J_1) \cap \mathrm{Conv}(J_2)$. If $c_j \notin J_1$, then $\mathrm{Conv}(J_1) \subset C_j$, and if $c_j \notin J_2$, then $\mathrm{Conv}(J_2) \subset C_j$; therefore, $q \in \mathrm{Conv}(J_1) \cap \mathrm{Conv}(J_2) \subset C_j$, for $j = 0, \ldots, d-1$. Thus we obtain that $q \in \bigcap_{j=0}^{d-1} C_j$, which implies $q \notin C_d$. Because $\mathrm{Conv}(J_i \setminus \{a_1, a_2\}) \subset C_d$ and $q \notin C_d$, points a_1, a_2 are in distinct partition classes, say $a_i \in J_i$. Since $a_1 \neq a_2$, we may assume that $q \neq a_1$; denote m_0 the distance of q from C_d. Clearly, $m \leq m_0$.

Because $\overline{a_1 q} \subset \mathrm{Conv}(J_1)$ and $\mathrm{Conv}(J_1 \setminus \{a_1\}) \subseteq C_d$, the line through a_1 and q intersects C_d at some point $c \in C_d$. Our argument proceeds on the plane containing the triangle (a_1, b_1, c). Let q' and c' be the points on the line through c and b_1 such that $\overline{qq'} \perp \overline{cb_1}$ and $\overline{a_1 c'} \perp \overline{cb_1}$ (see the figures).

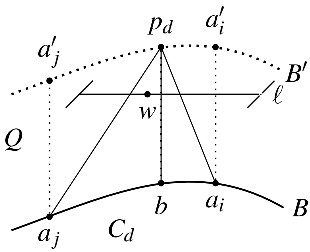




If $q' \in \overline{cb_1}$ then by convexity, $q' \in C_d$. This implies that $m_0 \leq |\overline{qq'}| < |\overline{a_1 c'}| \leq |\overline{a_1 b_1}| = m \leq m_0$, a contradiction (see the figure on the left). If $b_1 \in \overline{cq'}$ then we have $\angle b_1 q a_1 = \pi - \angle cqb_1 \geq \pi - \angle cqq' > \pi/2$ (see on the right). Therefore, $m_0 \leq |\overline{qb_1}| < |\overline{a_1 b_1}| = m \leq m_0$, a contradiction.

We conclude that $a_1 = a_2$, thus p_d is uniquely determined as the closest point in $\bigcap_{h\neq d} C_h$ to C_d. Similarly, each point $p_i \in \bigcup_{h\neq i} C_h$ closest to C_i, $i = 0,\dots,d-1$, is uniquely determined. Furthermore, because $\bigcap_{i=0}^{d} C_i = \varnothing$, $S = (p_0,\dots,p_d)$ is a $d-$simplex.

2. Next we show that $p_d \in \partial H$. Let $b \in \partial C_d$ be the closest point in C_d to $p_d \in \bigcap_{h\neq d} C_h$. For $i = 0,1,\dots,d-1$, let $a_i \in \partial C_d \cap \left(\bigcap_{h\neq i} C_h\right)$. We translate the point b to p_d, and assume that the same translation takes the points $a_0,\dots,a_{d-1}$ into $a'_0,\dots a'_{d-1}$, respectively. Define $B = \partial C_d \cap \mathrm{Conv}(\{b,a_0,\dots,a_{d-1}\} \cup \{p_d,a'_0,\dots a'_{d-1}\})$, and let B' be the translation of B sending b into p_d. Observe that $\bigcap_{h\neq d} C_h$ has no point in the interior of $Q = \mathrm{Conv}(B \cup B')$.



Now we take a hyperplane ℓ strictly separating p_d from B and sufficiently close to p_d. The intersection of $C = \mathrm{Conv}(\{p_d,a_0,\dots,a_{d-1}\})$ with ℓ is inside the interior of Q; let $L = \ell \cap C$. The convex sets $C'_i = C_i \cap L$, $i = 0,1,\dots,d-1$, form a $(d-1)$-critical family $\mathscr{F}'$ in the hyperplane ℓ. By induction, the hollow enclosed by $\mathscr{F}'$ in ℓ contains a point $w \in L \setminus \left(\bigcup_{i=0}^{d-1} C_i\right)$. The simplex $\mathrm{Conv}(\{p_d,a_0,\dots,a_{d-1}\})$ contains the hollow H enclosed by $\mathscr{F}$ in $\mathbb{R}^d$, which implies that $w \in H$.

Because ℓ can be taken arbitrarily close to p_d, the point $w \in H$ becomes arbitrarily close to p_d. Thus we obtain $p_d \in \partial H$, and similarly, $p_i \in \partial H$, $0 \le i \le d-1$. Therefore, cl(Conv(H))=Conv($\{p_0,p_1,\dots,p_d\}$). □

3 Conclusion

Given a $d-$critical family $\mathscr{F} = \{C_0,\dots,C_d\}$ in $\mathbb{R}^d$, a *cage* is defined as a closed set containing $d+1$ *base points*, $a_i \in \bigcap_{h\neq i} C_h$, $0 \le i \le d$. A convex cage M carried by $\mathscr{F}$ contains the hollow $D \subset \mathbb{R}^d \setminus \bigcup_{i=0}^{d} C_i$ enclosed by the family, because D is included in the convex hull of the base points of M. The generalization of Berge's theorem [3] due to Ghouila-Houri [9] implies the following property of a convex cage (as a special case).

Proposition 3.1. *Let $\mathscr{F} = \{C_0,\dots,C_d\}$ be a $d-$critical family in $\mathbb{R}^d$, and let F be a closed set containing the hollow D enclosed by $\mathscr{F}$. If M is a convex cage carried by $\mathscr{F}$, then $F \cap M$ is also a cage.* □

When applying Proposition 3.1 with $F = \mathrm{cl}(\mathrm{Conv}(D))$, then the $d+1$ base points of the cage $F \cap M$ may depend on the choice of M. Theorem 2.1 implies that this is not the case, Proposition 3.1 is true in a stronger form, namely, there is a unique convex cage minimal by inclusion, the d-simplex $\mathrm{cl}(\mathrm{Conv}(D))$.

Theorem 3.2. *Let $\mathscr{F} = \{C_0, \ldots, C_d\}$ be a $d-$critical family in $\mathbb{R}^d$. Then there exist $d+1$ base points, which belong to every convex cage M carried by $\mathscr{F}$.* □

If $n < d$ then there is no hollow enclosed by the members of an n-critical family in $\mathbb{R}^d$. In particular, the two compact convex members of a 1-critical family do not enclose a hollow in $\mathbb{R}^2$; nevertheless, since they are disjoint, they can be strictly separated by a line. A result due to Klee [14, Theorem 1] extends this separation property in $\mathbb{R}^d$ for any n-sets.[6] Klee's separation theorem has an immediate corollary for n-critical families below; a simple proof (extending easily the induction proof of Lemma 1.1 given above) is due to Breen [4].

Theorem 3.3. (Breen [4]). *For $1 \le n \le d$, let $\{C_0, C_1, \ldots, C_n\}$ be an n-critical family in $\mathbb{R}^d$, and let $a_i \in \bigcap_{h \neq i} C_h$, $0 \le i \le n$. Then in $\mathbb{R}^d$ there are two affine subspaces, W of dimension n and V of dimension $d-n$ (called a stabbing affine subspace), meeting in a single point p and such that*

(a) *$V \cap C_i = \varnothing$ and $a_i \in W$, for every $0 \le i \le n$, and*

(b) *the set $W \bigcap \left(\bigcup_{i=0}^{n} C_i\right)$ surrounds[7] $\{p\}$ in W.* □

The special version of Theorems 2.1 and 3.2 for $d = 2$ was originally developed and applied by Jobson et al. [13, Lemma 1] in the study of an extremal problem involving forbidden planar convex hypergraphs. It is worth noting that the characterization of d-dimensional convex hypergraphs[8] is not known for $d \ge 2$. For $d = 1$ the convex hypergraphs are called interval graphs; and as it is well known, their characterization was done by Lekkerkerker and Boland [18] in terms of forbidden obstructions, and by Gilmore and Hoffman [10] using the ordering and the separation property of the real line.

Having Theorem 3.3, one could try to generalize the Hollow Theorem (Theorem 2.1), that is, for an n-critical family $\{C_0, C_1, \ldots, C_n\}$ in $\mathbb{R}^d$, one might ask for some kind of 'geombinatorial' description of the set of all stabbing $(d-n)$-dimensional affine spaces V. At this point we do not even have a reasonable conjecture.

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[6] the concept of an *n-set* is a variation of n-criticality used by Klee [14]

[7] Q surrounds P in A if $A \setminus Q$ has a connected component which is bounded and contains P

[8] vertices are convex sets in $\mathbb{R}^d$, and $d+1$ vertices form a hyperedge if and only if they have nonempty intersection

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