An Erdős-Szekeres type problem in the plane

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Abstract

Let $f(k,n)$, $n \geq k \geq 3$, denote the smallest positive integer such that any set of $f(k,n)$ points, in general position in the plane, contains n points whose convex hull has at least k vertices. We give lower and upper estimates on $f(k,n)$, both in the form $\Theta(kn) + 2^{\Theta(k)}$.

1 Introduction

A classical result of Erdős and Szekeres [ES1] states that, for every integer $n \geq 3$ there is a smallest positive integer $g(n)$ such that among any $g(n)$ points, in general position in the plane, there exist n points in convex position. The best known bounds for $g(n)$ are the following.

Theorem 1.1. [ES2, TV]

$$2^{n-2} + 1 \leq g(n) \leq \binom{2n-5}{n-2} + 2 \ .$$

The following generalization was motivated in [K]. For integers $n \geq k \geq 3$, let $f(k,n)$ be the smallest number with the property that among any $f(k,n)$ points in general position

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in the plane, there exist n points whose convex hull has at least k vertices. Clearly $f(k,n)$ exists and satisfies $g(k) \le f(k,n) \le g(n)$.

It follows from a canonical version of the Erdős-Szekeres theorem (see [BV, PS]) that, for any fixed k, $f(k,n)$ is a linear function of n. The coefficient of n however is of order $2^{\Omega(k^2)}$.

In this note we obtain the following improvements.

Theorem 1.2. *For arbitrary integers $n \ge k \ge 3$,*

$$\frac{(k-1)(n-1)}{2} + 2^{k/2-4} \le f(k,n) \le 2kn + 2^{8k} \ .$$

Better results are available for small values of k.

Theorem 1.3. $f(4,n) = \lceil 3n/2 \rceil - 1$.

Theorem 1.4. $2n - 1 \le f(5,n) \le 7n - 23$.

We prove these results in Sections 2 and 3, respectively. Section 4 contains the proof of the upper bound in Theorem 1.2, while the lower bound is proved in Section 5.

2 The case $k = 4$

Proof of Theorem 1.3. The lower bound follows from Theorem 1.2. To prove the upper bound, let P denote any set of at least $\lceil 3n/2 \rceil - 1$ points, in general position in the plane. If $\mathrm{conv}(P)$ has at least 4 vertices, then we are done. Therefore we may assume that $\mathrm{conv}(P)$ has only 3 vertices which we denote, in counter-clockwise order, by p_1, p_2, p_3. Let $q_1 = p_1$. We define the points $q_2, \dots, q_{\lceil n/2 \rceil}$ recursively as follows. Suppose that, for some $i \le \lceil n/2 \rceil - 1$ the convex hull of $P \setminus \{q_1, \dots, q_i\}$ has at least 4 vertices. In this case we have found at least $\lceil 3n/2 \rceil - 1 - i \ge n$ points whose convex hull has at least 4 vertices, and we are done. Thus, we may assume that the convex hull of $P \setminus \{q_1, \dots, q_i\}$ is a triangle $p_2 p_3 q_{i+1}$.

This way we have obtained points $q_1 = p_1, q_2, \dots, q_{\lceil n/2 \rceil}$ such that

$$P' = P \setminus \{q_1, \dots, q_{\lceil n/2 \rceil}, p_2, p_3\} \subset \triangle p_2 p_3 q_{\lceil n/2 \rceil} \subset \triangle p_2 p_3 q_{\lceil n/2 \rceil - 1} \subset \dots \subset \triangle p_2 p_3 q_1 \ .$$

Consider the points $q_2, \dots, q_{\lceil n/2 \rceil}$, in counter-clockwise order of visibility from p_1, and denote by r_1 and r_2 the first and the last points, respectively. Let s_i $(i = 1, 2)$ denote the intersection point of line $p_1 r_i$ with segment $p_2 p_3$. Note that $|P'| \ge n - 3$. Thus, we may assume, without any loss of generality, that the convex quadrilateral $p_2 r_1 r_2 s_2$ contains at least $\lceil (n-3)/2 \rceil$ points of P'. Denote the set of these points by P''. In this case p_1, p_2 and

r_2 are extremal points of the set $P^* = P'' \cup \{q_1, q_2, \dots, q_{\lceil n/2 \rceil}, p_2\}$, which has at least $\lceil (n-3)/2 \rceil + \lceil n/2 \rceil + 1 = n$ elements. Moreover, every point of P' lies inside triangle $p_2 r_2 p_3$, consequently, every point of P'' lies inside triangle $p_2 r_2 s_2$. Thus, P^* has at least one more extremal point. This completes the proof of the theorem.

3 The method of convex and concave chains

Theorem 3.1. *For arbitrary integers $n \geq k \geq 3$,*

$$f(k,n) \leq \binom{2k-5}{k-2} n \,.$$

Proof. Fix k an n. Let P denote a set of points, in general position in the plane, whose cardinality N is large enough. Let p denote one of its extremal points, and number the other points of P as $p_1, p_2, \dots, p_{N-1}$, in clockwise order of visibility from p. A *convex chain of length* ℓ with left (resp. right) endpoint p_{i_1} (resp. $p_{i_{\ell'}}$) is any sequence of $\ell' \geq \ell$ points $p_{i_1}, p_{i_2}, \dots, p_{i_{\ell'}}$ $(i_1 < i_2 < \dots < i_{\ell'})$, such that $pp_{i_1}p_{i_2}\dots p_{i_{\ell'}}$ is a convex $(\ell'+1)$-gon which contains at least $n-k-\ell'+\ell$ points of P in its interior. Similarly, a *concave chain of length* ℓ with left (resp. right) endpoint p_{i_1} (resp. $p_{i_{\ell'}}$) is any sequence of $\ell' \geq \ell$ points $p_{i_1}, p_{i_2}, \dots, p_{i_{\ell'}}$ $(i_1 < i_2 < \dots < i_{\ell'})$, such that the region bounded by the segments $p_{i_j}p_{i_{j+1}}$ $(1 \leq j \leq \ell-1)$ and the rays starting at point p and incident to points p_1 and p_ℓ, respectively, is an unbounded convex region which contains at least $n-k-\ell'+\ell$ points of P in its interior.

For $i,j \geq 2$, let $g_{k,n}(i,j)$ denote the smallest integer such that, for an arbitrary set P with N large enough, and for an arbitrary choice of its extremal point p, any $g_{k,n}(i,j)$-element subset of $\{p_1, p_2, \dots, p_{N-1}\}$ contains either a concave chain of length i or a convex chain of length j. When it does not cause any ambiguity, we simply write $g(i,j)$ for $g_{k,n}(i,j)$. It is immediate, that $g_{k,n}(2,j) = g_{k,n}(i,2) = n-k+2$ for any $i,j \geq 2$.

Lemma 3.2. *For $i,j \geq 3$, we have $g_{k,n}(i,j) \leq g_{k,n}(i-1,j) + g_{k,n}(i,j-1) - 1$.*

Proof. The proof is analogous to one of the original proofs of the Erdős-Szekeres theorem [ES1]. Suppose that N is large enough, and let $S \subset \{p_1, p_2, \dots, p_{N-1}\}$, $|S| = g(i-1,j) + g(i,j-1) - 1$. If S contains a concave chain of length i, we are done. Otherwise, since $|S| \geq g(i,j-1)$, it contains a convex chain of length $j-1$. Delete its left endpoint from S. Since we still have at least $g(i,j-1)$ points, there is another convex chain of length $j-1$. Delete its left endpoint from S again and continue as long as the remaining set has at least $g(i,j-1)$ points. We deleted $g(i-1,j)$ points of S, all of them are left endpoints of a convex chain of length $j-1$. By definition of $g(i-1,j)$, the set of deleted points contains either a convex chain of size j or a concave chain of size $i-1$. In the first case we are done. In the second case, let q be the right endpoint of that concave chain and let r be its second point from the right. q is also the left endpoint of some convex chain of length $j-1$, let s be its second point from the left. Now it is easy to see that depending on the angle $\angle rqs$, either the concave chain can be extended by s or the convex chain can be extended by r, concluding the proof of the lemma.

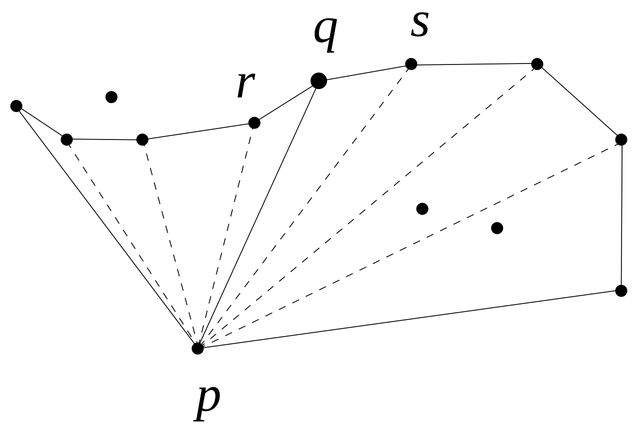


Figure 1.

Since $g_{k,n}(i,2) = g_{k,n}(2,j) < n$, it follows by induction that $g_{k,n}(i,j) < \binom{i+j-4}{i-2}n$, in particular, $g_{k,n}(k,k-1) \leq \binom{2k-5}{k-2}n - 1$. Consequently, if $N \geq \binom{2k-5}{k-2}n$, then either P contains a concave chain of length k, or it contains a convex chain of length $k-1$, and the result follows.

Proof of Theorem 1.4. The lower bound follows from Theorem 1.2. To prove the upper bound, notice first that $g_{k,n}(3,3) = n-k+3$. By repeated application of Lemma 3.2 we obtain

$$\begin{aligned} g_{k,n}(5,4) = g(5,4) &\leq g(4,4) + g(5,3) - 1 \\ &\leq g(3,4) + 2g(4,3) + g(5,2) - 3 \\ &\leq g(2,4) + 3g(3,3) + 2g(4,2) + g(5,2) - 6 \\ &= 3(n-k+3) + 4(n-k+2) - 6 \\ &= 7n - 7k + 11\ . \end{aligned}$$

Consequently, $f(5,n) \leq g_{5,n}(5,4) + 1 \leq 7n - 23$.

4 The upper bound

Proof of Theorem 1.2 (upper bound). Obviously, $f(3,n) = n$. Thus, in the sequel we assume $k \geq 4$. We prove the following estimate:

$$f(k,n) \leq \max\{(k-1)(2n-8k+19), 0\} + \max\{n-k+1, g(4k-10)\}\ .$$

Combining this with Theorem 1.1 the upper bound in Theorem 1.2 follows.

Let P be any set of $N \geq \max\{(k-1)(2n-8k+19), 0\} + \max\{n-k+1, g(4k-10)\}$ points, in general position in the plane. Peel off convex layers from P as follows. Let $P_0 = P$ and Q_0 be the vertices of the convex hull of P. If we already have P_i and Q_i, let $P_{i+1} = P_i \setminus Q_i$ and let Q_{i+1} be the set of vertices of the convex hull of P_{i+1}. If there is a smallest integer $i \leq 2n - 8k + 19$ such that $|Q_i| \geq k$, then it is easy to check that $|P_i| \geq n$. That is, we found at least n points whose convex hull has at least k vertices, and we are done.

We can therefore assume that $|Q_i| \leq k-1$ for $1 \leq i \leq t = \max\{2n-8k+19, 0\}$, implying that $P' = P_{t+1}$ has at least $g(4k-10)$ points. Consequently, P' contains the vertex set of a convex polygon $K = p_1, p_2, \ldots, p_{4k-10}$, in counter-clockwise order. The segments $p_{k-2}p_{k-1}$ and $p_{3k-7}p_{3k-6}$ are opposite sides of the polygon K, and we may assume, without any loss of generality, that rays r_1, starting at p_{k-2} and passing through p_{k-1}, and r_2, starting at p_{3k-6} and passing through p_{3k-7}, do not intersect each other.

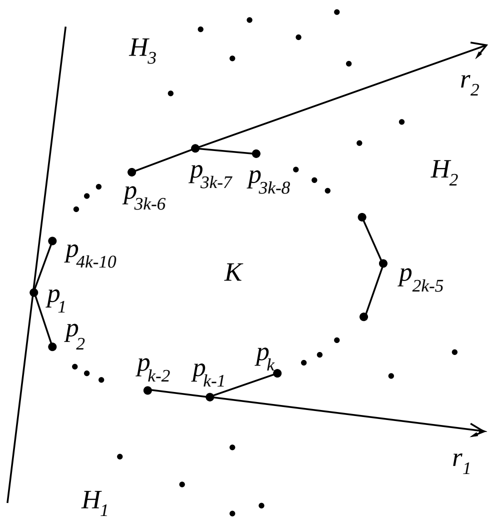


Figure 2.

Consider any open half plane H whose supporting line is incident to p_1 such that H contains points $p_2, p_3, \ldots p_{4k-10}$. The polygonal chains $(\cup_{i=1}^{k-3} p_i p_{i+1}) \cup r_1$ and $p_1 p_{4k-10} \cup (\cup_{i=3k-6}^{4k-9} p_i p_{i+1}) \cup r_2$ divide H into 3 open regions H_1, H_2, H_3, of which the middle one, H_2, contains vertices $p_k, p_{k+1}, \ldots, p_{3k-8}$ of K (see the Figure). Thus, $|P \cap H_2| \geq 2k - 7$.

It follows from the construction of the convex layers Q_i that $H \cap Q_i \neq \emptyset$ for $i = 1, 2, \ldots, t$. Consequently, $|P \cap H| \geq t + 4k - 11$. Define

$$R_1 = (P \cap (H_1 \cup H_2)) \cup \{p_2, p_3, \ldots, p_{k-1}\}$$

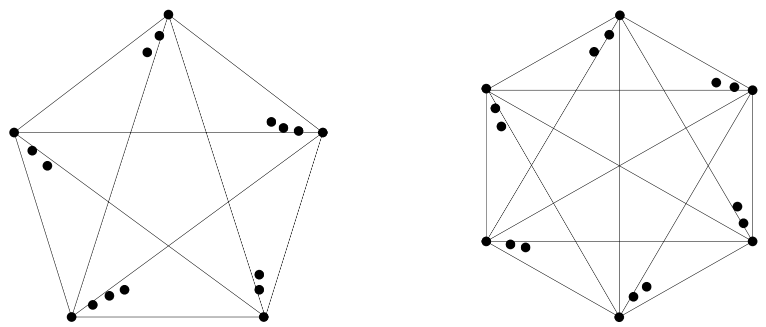
and

$$R_2 = (P \cap (H_3 \cup H_2)) \cup \{p_{3k-7}, p_{3k-6}, \dots, p_{4k-10}\} ,$$

then we have $|R_1|+|R_2| = |P \cap H|+|P \cap H_2| \geq t+6k-18 \geq 2n-2k+1$. If $|R_1| \geq n-k+1$, then $R_1 \cup \{p_1, p_2, \dots, p_{k-1}\}$ contains at least n points, and has at least k extremal points, including $p_1, p_2, \dots, p_{k-1}$. We argue similarly if $|R_2| \geq n-k+1$.

5 The construction

Proof of Theorem 1.2 (lower bound). In fact, we prove that $\left\lfloor \frac{(k-1)(n-1)}{2} \right\rfloor + a_k \leq f(k,n)$, where $a_k = 2^{\lfloor k/2 \rfloor - 3} + 1$ if $k \geq 6$ and $a_k = 1$ otherwise. First, for any $n \geq k \geq 4$ we obtain a set $P_{k,n}$ of $\left\lfloor \frac{(k-1)(n-1)}{2} \right\rfloor$ points, in general position in the plane, which does not contain n points whose convex hull has at least k vertices. Let $v_1, v_2, \dots, v_{k-1}$ denote, in this order, the vertices of a regular $(k-1)$-gon. Write $v_0 = v_{k-1}$, $v_k = v_1$ and $v_{k+1} = v_2$. For every $1 \leq i \leq k-1$, construct points $v_{i1} = v_i, v_{i2}, \dots, v_{it_i}$, where $t_i = \lfloor (n-1)/2 \rfloor$ if i is odd, and $t_i = \lceil (n-1)/2 \rceil$ if i is even, such that $v_i v_{i2} v_{i3} \dots v_{it_i} v_{i+1}$ is a convex polygon lying in the intersection of triangles $v_{i-1} v_i v_{i+1}$ and $v_i v_{i+1} v_{i+2}$ and, with the notation $K_i = \{v_i, v_{i2}, \dots, v_{it_i}\}$, every line $v_{ij} v_{ik}$ separates K_{i+1} from v_{i-1}.



$P_{6,8}$ $\qquad$ $P_{7,7}$

Figure 3.

Claim 5.1. *Suppose $u_1, u_2, \dots, u_r \in P_{k,n}$ are vertices of a convex polygon K.*

- *(i) If three of the u_α are vertices of some K_i, then K lies in the triangle $v_i v_{i+1} v_{i+2}$, with vertex v_{i+2} omitted.*
- *(ii) If two of the u_α are vertices of some K_i and K does not lie in triangle $v_i v_{i+1} v_{i+2}$, with vertex v_{i+2} omitted, then none of the u_α is of the form $v_{(i+1)j}$.*

It follows that if a subset of $P_{k,n}$ does not lie in triangle $v_iv_{i+1}v_{i+2}$, with vertex v_{i+2} omitted, then its convex hull may have at most $k-1$ vertices. On the other hand if a subset of $P_{k,n}$ does lie in triangle $v_iv_{i+1}v_{i+2}$, with vertex v_{i+2} omitted, then it has at most $n-1$ points. Thus, $P_{k,n}$ does not contain n points whose convex hull has at least k vertices, as we claimed above. This proves the lower bound in the case $k < 6$.

If $k \geq 6$ we can extend $P_{k,n}$ with $2^{\lfloor k/2 \rfloor - 3}$ points as follows. The segments v_iv_j divide the convex polygon $v_1v_2 \ldots v_{k-1}$ into finitely many regions. Denote by S the region which contains the centre of the polygon if $k-1$ is odd. If $k-1$ is even, then there are several regions which have the centre of the polygon on their boundary, let in this case S be one of these regions.

Claim 5.2. *Any line through an inner point of S which is not incident to any v_i separates $k/2$ of the v_i from the others if k is even; and separates either $\lfloor k/2 \rfloor$ or $\lceil k/2 \rceil$ of the v_i from the others if k is odd.*

In view of Theorem 1.1, there is a set E_k of $2^{\lfloor k/2 \rfloor - 3}$ points, in general position in the plane, which does not contain $\lfloor k/2 \rfloor - 1$ points in convex position. Let S_k denote the image of E_k under a suitable similarity such that S_k lies inside the region S and $P'_{k,n} = P_{k,n} \cup S_k$ is in general position. We claim that $P'_{k,n}$ does not contain n points whose convex hull has at least k vertices.

Notice first that S, and so S_k, too, is disjoint from any triangle $v_iv_{i+1}v_{i+2}$. Thus, if the vertex set of the convex hull of some subset of $P'_{k,n}$ is disjoint from S_k, then either the convex hull has at most $k-1$ vertices or the subset itself has less than n points. On the other hand, if the vertex set of the convex hull of some subset of $P'_{k,n}$ is not disjoint from S_k, then the convex hull has at most $\lfloor k/2 \rfloor - 2$ vertices in S_k. Moreover, it follows from Claims 5.2 and 5.1(ii) that the convex hull cannot have more than $\lceil k/2 \rceil + 1$ vertices in $P_{k,n}$. Altogether, it cannot have more than $k-1$ vertices in $P'_{k,n}$.

This completes the proof of the theorem.

Remark. Most likely S_k can be replaced by a larger set, maybe even of the size $c2^{2k}$, but any essential improvement would certainly require a lot of technical details.

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