Erdős-Szekeres-type theorems for segments and non-crossing convex sets

János Pach[*]
Courant Institute, NYU and Hungarian Academy of Sciences

Géza Tóth[†]
DIMACS Center, Rutgers University and Hungarian Academy of Sciences

Abstract

A family $\mathcal{F}$ of convex sets is said to be in *convex position*, if none of its members is contained in the convex hull of the others. It is proved that there is a function $N(n)$ with the following property. If $\mathcal{F}$ is a family of at least $N(n)$ plane convex sets with non-empty interiors, such that any *two* members of $\mathcal{F}$ have at most two boundary points in common and any *three* are in convex position, then $\mathcal{F}$ has n members in convex position. This result generalizes a theorem of T. Bisztriczky and G. Fejes Tóth [BF1]. The statement does not remain true, if two members of $\mathcal{F}$ may share four boundary points. This follows from the fact that there exist infinitely many straight-line segments such that any three are in convex position, but no four are. However, there is a function $M(n)$ such that every family of at least $M(n)$ segments, any *four* of which are in convex position, has n members in convex position.

1 Introduction

Erdős and Szekeres [ES1], [ES2] proved that any set of more than $\binom{2n-4}{n-2}$ points in general position in the plane contains n points which are in convex position, i.e., they form the vertex set of a convex n-gon. T. Bisztriczky and G. Fejes Tóth [BF1], [BF2], [F] extended this result to families of convex sets.

Throughout this paper, by a *family* $\mathcal{F} = \{B_1, \ldots, B_t\}$ we always mean a family of compact convex sets in the plane in *general position,* i.e., no three of them have a common supporting line, and no two are tangent to each other. $B_i \in \mathcal{F}$ is said to be a *vertex* of $\mathcal{F}$ if B_i is not

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contained in the convex hull of the union of the others, i.e., if bd conv$(\cup\mathcal{F})$, the boundary of the convex hull of the union of all members of $\mathcal{F}$, contains a piece of the boundary of B_i. $\mathcal{F}$ is said to be in *convex position* if every member B_i, $(i = 1, \ldots, t)$ of $\mathcal{F}$ is a vertex of $\mathcal{F}$. Evidently, any two members of $\mathcal{F}$ are in convex position.

T. Bisztriczky and G. Fejes Tóth proved that there exists a function $N(n)$ such that if $\mathcal{F}$ is a family of *pairwise disjoint* convex sets, $|\mathcal{F}| > N(n)$ and any *three* members of $\mathcal{F}$ are in convex position, then $\mathcal{F}$ has n members in convex position. In [PT], we have shown that this is true with $N(n) < 16^n$.

The aim of this paper is to extend the above result to families of not necessarily disjoint sets. A compact convex set in the plane with non-empty interior is called a *convex body.* Two convex bodies are said to be *non-crossing*, if they have at most two boundary points in common.

Theorem 1. *For every n, there exists an integer $N = N(n) > 0$ with the following property.*

Every family of at least N pairwise non-crossing convex bodies in the plane such that any three of them are in convex position, has n members in convex position.

Theorem 1 cannot be generalized to families of convex bodies whose boundaries may have four intersection points per pair. Indeed, this can be shown by replacing in the following theorem each segment by a very narrow ellipse.

Theorem 2. *There is an infinite family of straight-line segments in the plane such that any three of them are in convex position but no four are.*

However, the next result shows that an analogue of Theorem 1 is true for families of segments, assuming that any *four* of them are in convex position.

Theorem 3. *For every n, there exists an integer $M = M(n) > 0$ with the following property.*

Every family of at least M straight-line segments in the plane such that any four of them are in convex position, has n members in convex position.

In fact, we conjecture that for any $k > 2$, there exist a constant m_k and a function $M_k(n)$ with the following property. Every family of at least $M_k(n)$ convex bodies in the plane such that any two share at most k boundary points and any m_k are in convex position, has n members in convex position.

2 Proof of Theorem 1

Since $\mathcal{F}$ is in general position, small perturbations of the bodies do not effect whether or not a subset of $\mathcal{F}$ is in convex position. Therefore, we can assume that the boundary of every member of $\mathcal{F}$ is smooth and no three members of $\mathcal{F}$ share a common boundary point.

Let $\mathcal{F} = \{B_1, B_2, \ldots, B_N\}$ be a family of non-crossing convex bodies in the plane. By Ramsey's theorem, $\mathcal{F}$ has $\log_4 N$ members which are either pairwise disjoint or pairwise intersecting. In the first case, it follows from the (improved version [PT]) of the Bisztriczky-Fejes Tóth theorem [BF1] that there are many (at least $\log_{16} \log_4 N$) members in convex position, which exceeds n if N is large enough. So we can assume that $\{B_1, B_2, \ldots, B_{N'}\}$ is a subfamily of pairwise intersecting bodies, $N' \geq \log_4 N$.

We classify the ordered triples (B_i, B_j, B_k), $i < j < k$, as follows. Let $I = \mathrm{bd}(B_i) \cap B_j, I' = \mathrm{bd}(B_i) \cap B_k$. Go along $\mathrm{bd}(B_i)$ in clockwise direction. Denote the starting point and the endpoint of I (resp. of I') by s and e (resp. s' and e'). The type of the ordered triple (B_i, B_j, B_k), $i < j < k$, is determined by the clockwise order of s, s', e, e' along $\mathrm{bd}(B_i)$, and some other conditions, in the following way.

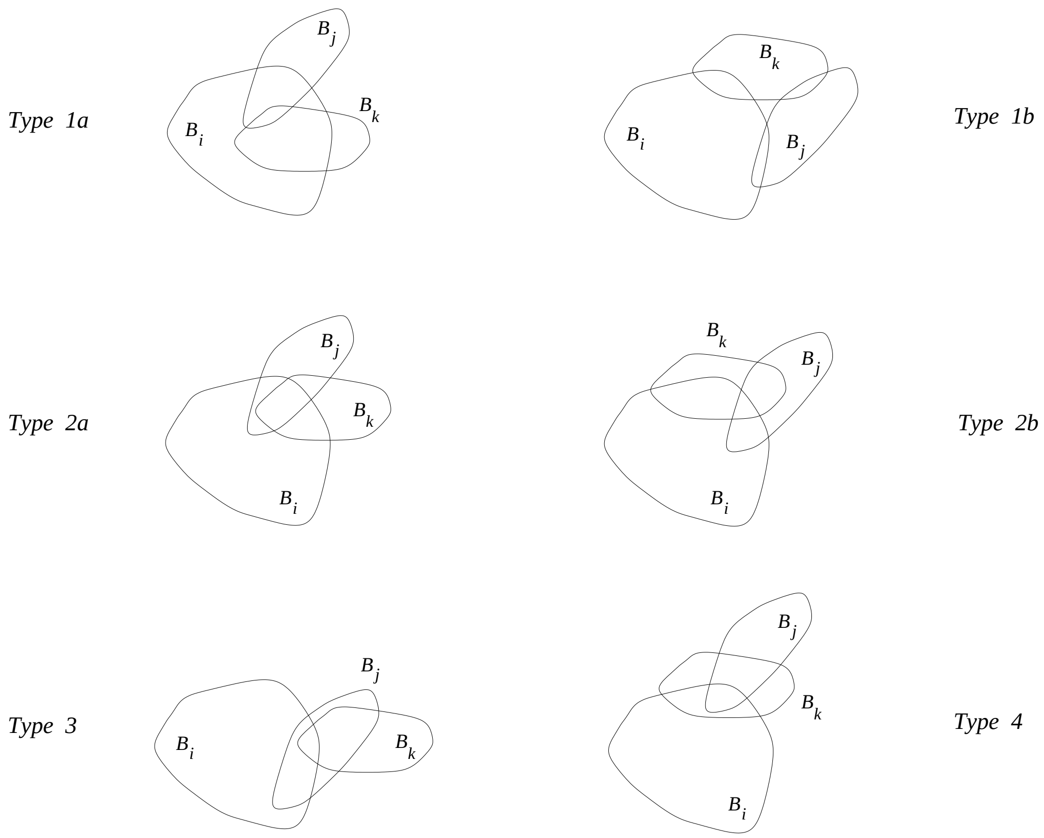


Fig. 1.

Type	*clockwise order of s, s', e, e' on bd(B_i), and additional conditions*
1a	$ses'e'$ and $B_i \cap B_j \cap B_k \neq \emptyset$
1b	$ses'e'$ and $B_j \cap B_k \neq \emptyset$, $B_i \cap B_j \cap B_k = \emptyset$
2a	$ss'ee'$
2b	$s'se'e$
3	$ss'e'e$
4	$s'see'$

By Ramsey's theorem, if N is large enough, then there is an arbitrarily large subfamily $\{B_1, B_2, \ldots, B_f\}$, all of whose ordered triples are of the same type. An easy case analysis shows that there are no five bodies such that all of their ordered triples are of type 1b. By symmetry, we do not have to treat types 2a and 2b separately: it is enough to consider, say, type 2a. So, we are left with four cases according to the type of the ordered triples of $\{B_1, \ldots, B_f\}$.

Case 1. *All triples are of type 1a (see Fig. 2).*

Considering the triples (B_1, B_i, B_{i+1}), we can conclude that the intervals $I_i = \mathrm{bd}(B_1) \cap B_i$ are pairwise disjoint and $I_2, I_3, \ldots, I_f$ follow each other on bd(B_1) in, say, clockwise order. Since B_i and B_j $(1 < i, j \leq f)$ intersect each other inside B_1, they do not intersect each other outside B_1. Let x be the starting point of I_2. We can assume that x is the leftmost point of bd(B_1).

Let y be the rightmost point of bd(B_1). There is an f' such that $I_2, I_3, \ldots, I_{f'}$ are on the arc xy and $I_{f'+2}, I_{f'+3}, \ldots, I_f$ are on the arc yx. By symmetry, we can assume that $f' \geq f/2 - 2$. Let l_x (resp. l_y) be the vertical supporting line of B_1 at x (resp. y). Since there is at most one B_i $(2 \leq i \leq f')$ such that B_i meets both l_x and l_y, there is a g such that $B_2, B_3, \ldots, B_g$ do not intersect l_y and $B_{g+2}, B_{g+3}, \ldots, B_f$ do not intersect l_x. Again, by symmetry, we can assume that $g \geq f'/2 - 2 \geq f/4 - 3$.

Claim 1. (i) *B_1 is a vertex of $\mathcal{G} = \{B_1, B_2, \ldots, B_g\}$.*
(ii) *B_g is a vertex of $\mathcal{G}$.*
(iii) *$\mathcal{G}$ is in convex position.*

Proof. (i) Since l_y is a supporting line of B_1 and each B_i, $(2 \leq i \leq g)$ is to the left of it, we have that $y \in \mathrm{bd}(\mathrm{conv}\ \cup_{i=1}^{g} B_i)$ and B_1 is a vertex of $\mathcal{G}$.

Since the bodies $B_2, B_3, \ldots, B_g$ do not intersect each other outside B_1, for any $1 < i < j < k \leq g$, B_i, B_j and B_k appear on $\mathrm{bd}(\cup_{i=1}^{l} B_i)$ in this clockwise order. So, if B_i, B_j, B_k each appears on the convex hull of $B_1, B_2, \ldots, B_g$, they appear in the same order.

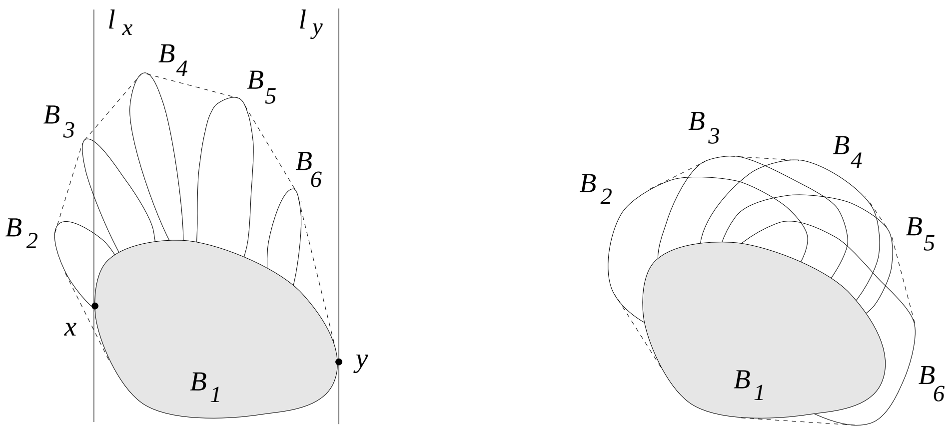
(ii) Suppose that B_g is not a vertex of $\mathcal{G}$. Let $c = \max\{i \mid B_i \text{ is a vertex of } \mathcal{G}\}$. Then $B_g \subset \mathrm{conv}(B_1 \cup B_c)$, a contradiction.

(iii) We show that for any fixed $1 < i < g$, B_i is a vertex of $\mathcal{G}$. Suppose on the contrary that B_i is not a vertex of $\mathcal{G}$. Let

$$c = \max\{j < i \mid B_j \text{ is a vertex of } \mathcal{G}\},\ \ d = \min\{j > i \mid B_j \text{ is a vertex of } \mathcal{G}\}.$$

Since B_1 and B_g are both vertices of $\mathcal{G}$, c and d are well defined. The triple (B_c, B_i, B_d) is of type 1a, therefore $B_i \subset \operatorname{conv}(B_c \cup B_d)$, a contradiction.

If we choose N large enough, g can be arbitrarily large, therefore in Case 1 we are done. □



Case 1 *Case 2*

Fig. 2.

Case 2. *All triples are of type 2a (see Fig. 2).*

Let s_i and e_i denote the starting point resp. endpoint of $I_i = \operatorname{bd}(B_1) \cap B_i$, in clockwise order. Considering the triples (B_1, B_i, B_{i+1}) and (B_1, B_2, B_i), it is easy to deduce that the (clockwise) order of the points s_i and e_i $(i = 2, \dots, f)$ along $\operatorname{bd}(B_1)$ is $s_2, s_3, \dots, s_f, e_2, e_3, \dots, e_f$, and $\operatorname{bd}(\cup_{i=1}^{f} B_i)$ is composed of arcs belonging to $\operatorname{bd}(B_1), \operatorname{bd}(B_2), \dots, \operatorname{bd}(B_f)$ in this order. Therefore, those members of $\{B_1, B_2, \dots, B_f\}$ which contribute to the boundary of $\operatorname{conv} \cup_{i=1}^{f} B_i$, appear along this boundary in their original order.

Claim 2. (i) *B_1 is a vertex of $\mathcal{G} = \{B_1, B_2, \dots, B_f\}$.*
(ii) *B_f is a vertex of $\mathcal{G}$.*
(iii) *$\mathcal{G}$ is in convex position.*

Proof. (i) Suppose that B_1 is not a vertex of $\mathcal{G}$. Let

$$c = \min\{i \mid B_i \text{ is a vertex of } \mathcal{G}\},\ \ d = \max\{i \mid B_i \text{ is a vertex of } \mathcal{G}\}.$$

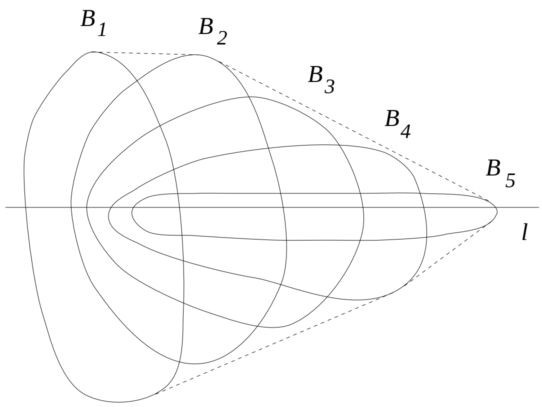
Then $B_1 \subset \operatorname{conv}(B_c \cup B_d)$, a contradiction.

(ii) Suppose that B_f is not a vertex of $\mathcal{G}$. Let $c = \max\{i \mid B_i \text{ is a vertex of } \mathcal{G}\}$.

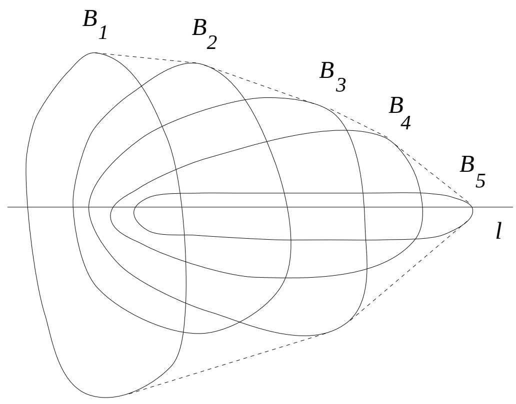
Then $B_f \subset \text{conv}(B_1 \cup B_c)$, a contradiction.

(iii) The proof is exactly the same as the proof of (ii).

If we choose N large enough, f can be arbitrarily large, therefore in Case 2 we are done. $\square$



Case 3



Subcase 3a

Fig. 3.

Case 3. *All triples are of type 3.*

Let $1 \leq i < j \leq f-1$. Since $B_{j+1} \not\subset \text{conv}(B_i \cup B_j)$ and the ordered triple (B_i, B_j, B_{j+1}) is of type 3, $\text{bd}(B_j)$ and $\text{bd}(B_{j+1})$ intersect each other in two points outside B_i. Thus, they do not intersect each other inside B_i, so $B_i \cap B_j \supset B_i \cap B_{j+1}$ (see Fig. 3).

Therefore, we can take an oriented line l passing through $B_1 \cap B_f$ and $B_f \setminus B_{f-1}$ with the following property. Let $I_i = l \cap B_i$, with starting point s_i and endpoint e_i. Then the order of the points s_i and e_i $(i = 1, \ldots, f)$ along l is $s_1, s_2, \ldots, s_f, e_1, e_2, \ldots, e_f$.

We distinguish two further subtypes of triples (B_i, B_j, B_k), $1 \leq i < j < k \leq f$, in the following way. There are four uniquely determined points, p_1, q_1, p_2, q_2, on $\text{bd}(B_i \cup B_j \cup B_k)$, in this clockwise order, such that the piece of $\text{bd}(B_i \cup B_j \cup B_k)$ which belongs to B_k (resp. B_i) is the arc p_1q_1 (resp. p_2q_2).

We say that the triple (B_i, B_j, B_k) is of type 3a (resp. 3b) if the arc q_1p_2 (resp. q_2p_1) of $\text{bd}(B_i \cup B_j \cup B_k)$ has a part which also belongs to $\text{bd}(B_j)$. Since (B_i, B_j, B_k) is in convex position, it is of type 3a or 3b (or both).

By Ramsey's theorem, there is a subfamily of size $g = \log_4 f$, all of whose ordered triples are of the same subtype.

For simplicity, denote this subfamily by $\{B_1, B_2, \ldots, B_g\}$). By symmetry, we can assume that all triples are of subtype 3a.

Claim 3. (i) B_1 *is a vertex of* $\mathcal{G} = \{B_1, B_2, \ldots, B_g\}$.
(ii) B_g *is a vertex of* $\mathcal{G}$.
(iii) $\mathcal{G}$ *is in convex position.*

Proof. (i) For any $1 < i \leq g$, let

$$S_i = \{x \in \text{bd}(B_1) \mid B_1 \text{ has a supporting line at } x \text{ which avoids } B_i\}.$$

Every S_i is an interval (connected arc) of $\text{bd}(B_1)$, and none of them contains e_1. Any triple (B_1, B_i, B_j) is in convex position, therefore $S_i \cap S_j \neq \emptyset$ (i. e., the sets S_i are pairwise intersecting). This implies that there exists a $y \in \cap_{i=2}^{g} S_i$ and a supporting line of B_1 at y, which avoids all other bodies.

(ii) The proof is analogous to the proof of (i).

(iii) Let p be a point of $\text{bd}(B_1) \cap \text{bd}(\text{conv } \cup_{i=1}^{l} B_i)$ and let q be a point of $\text{bd}(B_g) \cap \text{bd}(\text{conv } \cup_{i=1}^{g} B_i)$. We show that for any fixed $1 < i < g$, B_i appears on the (clockwise) arc pq of $\text{bd}(\text{conv } \cup_{i=1}^{g} B_i)$. Suppose, for a contradicition, that B_i does not appear on the arc pq.

Let

$$c = \max\{j < i \mid B_j \text{ appears on the arc } pq \text{ of } \text{bd}(\text{conv } \cup_{i=1}^{g} B_i)\},$$
$$d = \min\{j > i \mid B_j \text{ appears on the arc } pq \text{ of } \text{bd}(\text{conv } \cup_{i=1}^{g} B_i)\}.$$

Since B_1 and B_g both appear on the arc pq, c and d are well defined. Then the triple (B_c, B_i, B_d) is not of subtype 3a, a contradiction.

If we choose N large enough, g can be arbitrarily large, therefore in Case 3 we are done. □

Case 4. *All triples are of type 4.*

Let $1 \leq i < j < k \leq m$. Since $B_j \not\subset \text{conv}(B_i \cup B_k)$ and the ordered triple (B_i, B_j, B_k) is of type 4, $\text{bd}(B_j)$ and $\text{bd}(B_k)$ intersect each other in two points outside B_i. Therefore, the oriented triple (B_k, B_j, B_i) is of type 1a, and, considering $(B_f, B_{f-1}, \ldots, B_1)$, we can proceed as in Case 1.

Thus, in all cases we can find n bodies in convex position provided that N is large enough. This completes the proof of Theorem 1. □

3 Proof of Theorem 2

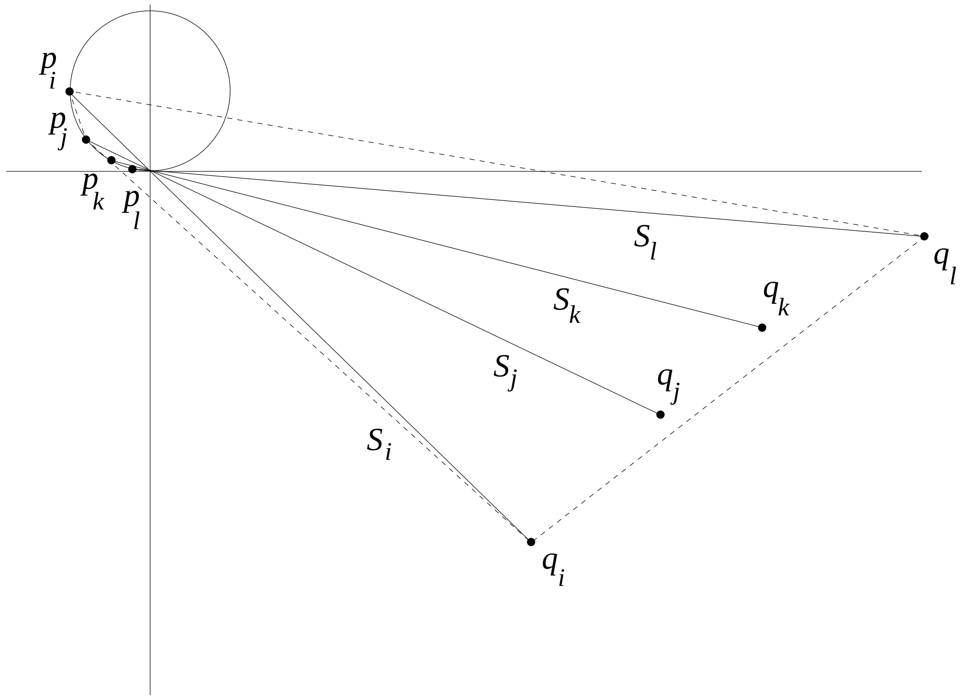


Fig. 4.

For $i = 1, 2, \ldots$, let

$$p_i = \left(-\sin\left(\frac{\pi}{2^i}\right) \ , \ \ 1 - \cos\left(\frac{\pi}{2^i}\right) \right),$$

$$a_i = \frac{1 - \cos(\frac{\pi}{2^i})}{\sin(\frac{\pi}{2^i})}$$

and, for some sufficiently large K, let

$$q_i = \left(\frac{K}{\sqrt{a_i}} \ , \ \ -K \cdot \sqrt{a_i} \right).$$

Finally, let $\mathcal{S}$ consist of all segments $S_i = p_i q_i$ for $i = 1, 2, \ldots$ (see Fig. 4).

Each segment S_i passes through the origin. One of its endpoints, p_i, is on the circle $x^2 + (y - 1)^2 = 1$ so that p_{i+1} is the midpoint of the arc $\overline{Op_i}$. The other endpoint lies on the hyperbola $xy = -K^2$. Since K is large enough, the segments are very long.

Let $i < j < l$. Notice that S_i intersects the line $p_j p_l$. Therefore, p_l is not a vertex of $\mathrm{conv}(S_i, S_j, S_l)$. Since the hyperbola $xy = -K^2$ is concave, q_j cannot be a vertex of $\mathrm{conv}(S_i, S_j, S_l)$ either. Thus, the vertices of $\mathrm{conv}(S_i, S_j, S_l)$ are p_i, p_j, q_i, q_l, in counter-clockwise order, which shows that any three segments are in convex position.

Now let $i < j < k < l$. By the above observations, the vertices of $\mathrm{conv}(S_i, S_j, S_k, S_l)$ are p_i, p_j, q_i, q_l. Thus, $S_k \subset \mathrm{conv}(S_i, S_j, S_l)$. $\square$

4 Proof of Theorem 3

Let pq denote the closed straight-line segment connecting two points, p and q. Let $\mathcal{S}$ be a family of M segments in the plane in general position, i.e., no two of them are parallel, no three endpoints are collinear, and we may assume that no two endpoints have the same x-coordinate. By Ramsey's theorem, $\mathcal{S}$ has at least $\log_4 M$ members which are either pairwise disjoint or pairwise crossing. In the first case, we can apply the (improved version of the) Bisztriczky-Fejes Tóth theorem to conclude that $\mathcal{S}$ has many (i.e., at least $\log_{16} \log_4 M$) members in convex position, which exceeds n, provided that M is large enough.

So, we can assume that $\mathcal{S}$ has $\log_4 M$ pairwise crossing members $S_1, S_2, \ldots$, and we can also suppose without loss of generality that they are listed in increasing order of slopes.

We classify the triples in $\mathcal{S}$, as follows. Let p_i and q_i denote the left endpoint and the right endpoint of S_i, respectively. We say that two triples in $\mathcal{S}'$, (S_i, S_j, S_k) and $(S_{i'}, S_{j'}, S_{k'})$ $i < j < k, i' < j' < k'$, are of the same *type* if the following conditions are satisfied:

(i) the orientation of $p_i p_j p_k$ is the same as the orientation of $p_{i'} p_{j'} p_{k'}$;

(ii) the orientation of $q_i q_j q_k$ is the same as the orientation of $q_{i'} q_{j'} q_{k'}$;

(iii) for any $\alpha, \beta, \gamma, \delta \in \{i, j, k\}$, the half-lines $\overrightarrow{p_\alpha p_\beta}$ and $\overrightarrow{q_\gamma q_\delta}$ cross each other if and only if $\overrightarrow{p_{\alpha'} p_{\beta'}}$ and $\overrightarrow{q_{\gamma'} q_{\delta'}}$ do.

Note that it follows immediately from (iii) that

(iv) for any $\alpha, \beta, \gamma, \delta \in \{i, j, k\}$, $\overrightarrow{p_\alpha p_\beta}$ intersects the segment $q_\gamma q_\delta$ if and only if $\overrightarrow{p_{\alpha'} p_{\beta'}}$ intersects $q_{\gamma'} q_{\delta'}$. Similarly, $\overrightarrow{q_\alpha q_\beta}$ intersects $p_\gamma p_\delta$ if and only if $\overrightarrow{q_{\alpha'} q_{\beta'}}$ intersects $p_{\gamma'} p_{\delta'}$.

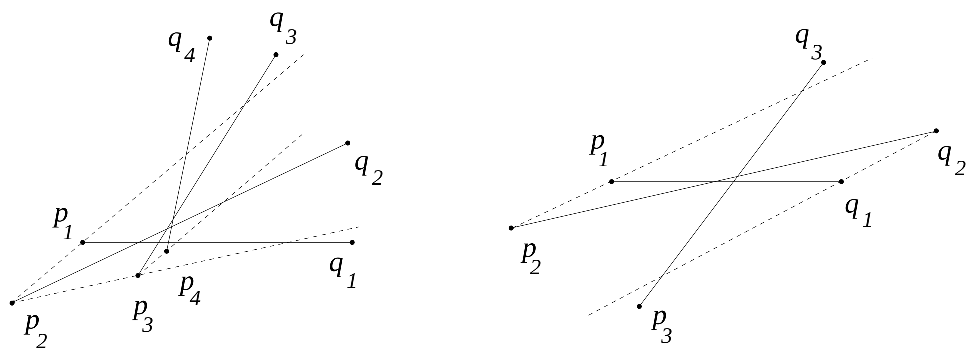
Applying Ramsey's theorem to the triples of $\mathcal{S}$, we obtain that there exists a subfamily $\mathcal{S}' \subseteq \mathcal{S}$ consisting of at least $f = f(M)$ segments (denoted by $S_1, S_2, \ldots S_f$, for simplicity), all of whose triples (S_i, S_j, S_k) $i < j < k$, are of the same type. Here $f(M)$ is a suitable function which tends to infinity, as $M \to \infty$. In what follows, we will show that $\mathcal{S}'$ is in convex position. This will complete the proof of the theorem, because if M is sufficiently large, then $|\mathcal{S}'| \geq f(M) \geq n$ holds.

Let $S_1, S_2, S_3 \in \mathcal{S}'$. It cannot occur that the orientations of $p_1 p_2 p_3$ and $q_1 q_2 q_3$ are both clockwise; otherwise p_2 (resp. q_2) would lie inside the triangle $p_1 p_3 c$ (resp. $q_1 q_3 c$), where c is is the intersection of S_1 and S_3. Thus, S_2 would be contained in the convex hull of $S_1 \cup S_3$, contradicting our assumption that any four segments are in convex position. Therefore, we have to distinguish two essentially different cases (up to symmetry about the y-axis).

Case 1: *$p_1p_2p_3$ and $q_1q_2q_3$ are both counter-clockwise oriented.*

Subcase 1.1: The half-line $\overrightarrow{p_2p_1}$ intersects q_2q_3, and $\overrightarrow{p_2p_3}$ intersects q_1q_2 (see Fig. 5).

Then $\overrightarrow{p_3p_2}$ cannot intersect q_3q_4. This implies that (S_2, S_3, S_4) cannot have the same type as (S_1, S_2, S_3), contradicting the definition of $\mathcal{S}'$. Thus, this subcase cannot occur.



Subcase 1.1 *Subcase 1.3*

Fig. 5.

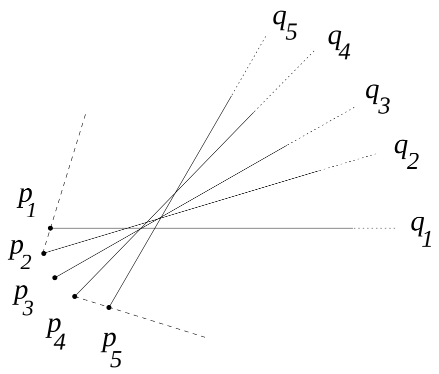
Subcase 1.2: $\overrightarrow{p_2p_1}$ does not intersect q_2q_3, and $\overrightarrow{p_2p_3}$ does not intersect q_1q_2 (see Fig. 6).

In this case, every p_i $1 \leq i \leq f$, is a vertex of the convex hull of $\cup\mathcal{S}'$, so $\mathcal{S}'$ is in convex position. Indeed, assume for contradiction that the line p_ip_{i+1} is not a supporting line of conv$(\cup\mathcal{S}')$. Then there is a q_j on the right-hand side of $\overrightarrow{p_ip_{i+1}}$. If $j < i$, we find that $\overrightarrow{p_ip_{i+1}}$ intersects q_jq_i, contradicting the fact that (S_j, S_i, S_{i+1}) has the same type as (S_1, S_2, S_3). If $j > i+1$, then $\overrightarrow{p_{i+1}p_i}$ intersects $q_{i+1}q_j$, contradicting the fact that (S_i, S_{i+1}, S_j) has the same type as (S_1, S_2, S_3).

We are left with one of the following cases: either (i) $\overrightarrow{p_2p_1}$ *intersects* q_2q_3*, and* $\overrightarrow{p_2p_3}$ *does not intersect* q_2q_1 or (ii) $\overrightarrow{p_2p_1}$ *does not intersect* q_2q_3*, and* $\overrightarrow{p_2p_3}$ *intersects* q_2q_1. Since we can reverse the numbering of the segments, we can assume that (i) holds. We can interchange the roles of the p's and q's, so we can also assume that either (iii) $\overrightarrow{q_2q_1}$ *intersects* p_2p_3*, and* $\overrightarrow{q_2q_3}$ *does not intersect* p_2p_1 or (iv) $\overrightarrow{q_2q_1}$ *does not intersect* p_2p_3*, and* $\overrightarrow{q_2q_3}$ *intersects* p_2p_1. Moreover, since (i) holds, (iv) cannot hold. Therefore, we have to consider only the following case.

Subcase 1.3: $\overrightarrow{p_2p_1}$ intersects q_2q_3, and $\overrightarrow{q_2q_1}$ intersects p_2p_3 (see Fig. 5).

Now we have $S_1 \subset \text{conv}(S_2 \cup S_3)$, a contradiction.



Subcases 1.2 and 2.2

Fig. 6.

Case 2: *$p_1p_2p_3$ is counter-clockwise and $q_1q_2q_3$ is clockwise oriented.*

Subcase 2.1: $\overrightarrow{p_2p_3}$ intersects q_1q_2 (see Fig. 4).

In this case, $S_3 \subset \text{conv}(S_1 \cup S_2 \cup S_4)$, a contradiction.

By symmetry, we also have a contradiction if $\overrightarrow{p_2p_1}$ intersects q_2q_3. Therefore, the only remaining case is the following.

Subcase 2.2: $\overrightarrow{p_2p_3}$ does not intersect q_1q_2, and $\overrightarrow{p_2p_1}$ does not intersect q_2q_3 (see Fig. 6).

It follows in exactly the same way as in Subcase 1.2 that every p_i $(1 \le i \le f)$ is a vertex of the convex hull of $\cup\mathcal{S}'$, hence $\mathcal{S}'$ is in convex position.

The above case analysis shows that we can always find $f(M)$ segments in $\mathcal{S}$, which are in convex position. This completes the proof of Theorem 3, because $f(M) \ge n$ if M is sufficiently large. $\square$

5 Concluding Remarks

The proof of Theorem 3 can be modified to yield the following slightly more general result.

Theorem 4. *For every n, there exists an integer $M' = M'(n) > 0$ with the following property. Let $\mathcal{F}$ be any family of at least M' convex sets in the plane such that*
(i) *the boundaries of any two intersect in at most four points,*
(ii) *no three have a point in common,*
(iii) *any four are in convex position.*

Then $\mathcal{F}$ has n members in convex position.

We conjecture that condition (ii) in Theorem 4 can be dropped.

In [KP], it was shown that every family of $n \geq 3 \cdot \binom{l}{3} \cdot k$ convex sets in the plane has either k disjoint members or l members, no 3 of which have a point in common.

Combining this result with Theorems 1 and 4, we obtain

Theorem 5. *For every $k \geq 3$ and for every n, there exists $M_k = M_k(n) > 0$ with the following property. Let $\mathcal{F}$ be any family of at least M convex sets in the plane such that*
(i) *the boundaries of any two intersect in at most four points,*
(ii) *no k have a point in common,*
(iii) *any four are in convex position.*

Then $\mathcal{F}$ has n members in convex position.

References

[BF1] T. Bisztriczky and G. Fejes Tóth, A generalization of the Erdős-Szekeres convex n-gon theorem, *Journal für die reine und angewandte Mathematik* **395** (1989), 167–170.

[BF2] T. Bisztriczky and G. Fejes Tóth, Convexly independent sets, *Combinatorica* **10** (1990), 195–202.

[ES1] P. Erdős and G. Szekeres, A combinatorial problem in geometry, *Compositio Mathematica* **2** (1935), 463–470.

[ES2] P. Erdős and G. Szekeres, On some extremum problems in elementary geometry, *Ann. Universitatis Scientiarum Budapestinensis, Eötvös, Sectio Mathematica* **III–IV** (1960–61), 53–62.

[F] G. Fejes Tóth, Recent progress on packing and covering, in: *Advances in Discrete and Computational Geometry* (South Hadley, MA, 1996; B. Chazelle et al., eds.), Contemporary Mathematics **223**, AMS, Providence, 1999, 145–162.

[KP] M. Katchalski and J. Pach, Touching convex sets in the plane, *Bulletin of the Canadian Mathematical Society* **37** (1994), 495–504.

[PT] J. Pach and G. Tóth, A generalization of the Erdős-Szekeres theorem to disjoint convex sets, *Discrete and Computational Geometry* **19** (1998), 437–445.