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ORIGINAL PAPER

Improvement on the Decay of Crossing Numbers

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Abstract We prove that the crossing number of a graph decays in a "continuous fashion" in the following sense. For any $\varepsilon > 0$ there is a $\delta > 0$ such that for a sufficiently large n, every graph G with n vertices and $m \geq n^{1+\varepsilon}$ edges, has a subgraph G' of at most $(1-\delta)m$ edges and crossing number at least $(1-\varepsilon)\text{CR}(G)$. This generalizes the result of J. Fox and Cs. Tóth.

Keywords Crossing number · Embedding method

1 Introduction

For any graph G, let $n(G)$ (resp. $m(G)$) denote the number of its vertices (resp. edges). If it is clear from the context, we simply write n and m instead of $n(G)$ and $m(G)$. The crossing number $\text{CR}(G)$ of a graph G is the minimum number of edge crossings over all drawings of G in the plane. In the optimal drawing of G, crossings are not necessarily

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distributed uniformly among the edges. Some edges could be more "responsible" for the crossing number than some other edges. For any fixed k, it is not hard to construct a graph G whose crossing number is k, but G has an edge e such that $G \setminus e$ is planar. Richter and Thomassen [7] started to investigate the following general problem. We have a graph G, and we want to remove a given number of edges. By *at least* how much does the crossing number decrease? They conjectured that there is a constant c such that every graph G with $\mathrm{CR}(G) = k$ has an edge e with $\mathrm{CR}(G - e) \geq k - c\sqrt{k}$. They only proved that G has an edge with $\mathrm{CR}(G - e) \geq \frac{2}{5}k - O(1)$.

Pach et al. [5] proved that for every graph G with $m(G) \geq \frac{103}{16}n(G)$, we have $\mathrm{CR}(G) \geq 0.032\frac{m^3}{n^2}$. It is not hard to see [6] that for *any* edge e, we have $\mathrm{CR}(G - e) \geq \mathrm{CR}(G) - m + 1$. These two results imply an improvement of the Richter–Thomassen bound if $m \geq 8.1n$, and also imply the Richter–Thomassen conjecture for graphs of $\Omega(n^2)$ edges.

Fox and Tóth [3] investigated the case where we want to delete a *positive fraction* of the edges.

Theorem A [3] *For every $\varepsilon > 0$, there is an n_ε such that every graph G with $n(G) \geq n_\varepsilon$ vertices and $m(G) \geq n(G)^{1+\varepsilon}$ edges has a subgraph G' with*

$$m(G') \leq \left(1 - \frac{\varepsilon}{24}\right) m(G)$$

and

$$\mathrm{CR}(G') \geq \left(\frac{1}{28} - o(1)\right) \mathrm{CR}(G).$$

In this note we generalize Theorem A.

Theorem *For every $\varepsilon, \gamma > 0$, there is an $n_{\varepsilon,\gamma}$ such that every graph G with $n(G) \geq n_{\varepsilon,\gamma}$ vertices and $m(G) \geq n(G)^{1+\varepsilon}$ edges has a subgraph G' with*

$$m(G') \leq \left(1 - \frac{\varepsilon\gamma}{1224}\right) m(G)$$

and

$$\mathrm{CR}(G') \geq (1 - \gamma)\mathrm{CR}(G).$$

2 Proof of the Theorem

Our proof is based on the argument of Fox and Tóth [3], the only new ingredient is Lemma 1.

Definition Let $r \geq 2$, $p \geq 1$ be integers. A $2r$*-earring of size* p is a graph which is a union of an edge uv and p edge-disjoint paths between u and v, each of length at most $2r - 1$. Edge uv is called the *main edge* of the $2r$-earring.

Lemma 1 *Let $r \geq 2$, $p \geq 1$ be integers. There exists n_0 such that every graph G with $n \geq n_0$ vertices and $m \geq 6prn^{1+1/r}$ edges contains at least $m/3pr$ edge-disjoint $2r$-earrings, each of size p.*

Proof By the result of Alon et al. [1], for some n_0, every graph with $n \geq n_0$ vertices and at least $n^{1+1/r}$ edges contains a cycle of length at most $2r$.

Suppose that G has $n \geq n_0$ vertices and $m \geq 6prn^{1+1/r}$ edges. Take a *maximal* edge-disjoint set $\{E_1, E_2, \ldots, E_x\}$ of $2r$-earrings, each of size p. Let $E = E_1 \cup E_2 \cup \cdots \cup E_x$, the set of all edges of the earrings and let $G' = G - E$. Now let E'_1 be a $2r$-earring of G' of maximum size. Note that this size is less than p. Let $G'_1 = G' - E'_1$. Similarly, let E'_2 be a $2r$-earring of G'_1 of maximum size and let $G'_2 = G'_1 - E'_2$. Continue analogously, as long as there is a $2r$-earring in the remaining graph. We obtain the $2r$-earrings $E'_1, E'_2, \ldots, E'_y$, and the remaining graph $G'' = G'_y$ does not contain any $2r$-earring. Let $E' = E'_1 \cup E'_2 \cup \cdots \cup E'_y$.

We claim that $y < n^{1+1/r}$. Suppose on the contrary that $y \geq n^{1+1/r}$. Take the main edges of $E'_1, E'_2, \ldots, E'_y$. We have at least $n^{1+1/r}$ edges so by the result of Alon et al. [1] some of them form a cycle C of length at most $2r$. Let i be the smallest index with the property that C contains the main edge of E'_i. Then C, together with E'_i would be a $2r$-earring of G'_{i-1} of greater size than E'_i, contradicting the maximality of E'_i.

Each of the earrings $E'_1, E'_2, \ldots, E'_y$ has at most $(p-1)(2r-1)+1$ edges so we have $|E'| \leq y(p-1)(2r-1)+y < (2pr-1)n^{1+1/r}$. The remaining graph, G'' does not contain any $2r$-earring, in particular, it does not contain any cycle of length at most $2r$, since it is a $2r$-earring of size one. Therefore, by [1], for the number of its edges we have $e(G'') < n^{1+1/r}$.

It follows that the set $E = \{E_1, E_2, \ldots, E_x\}$ contains at least $m - 2prn^{1+1/r} \geq \frac{2}{3}m$ edges. Each of $E_1, E_2, \ldots, E_x$ has at most $p(2r-1)+1 \leq 2pr$ edges, therefore, $x \geq m/3pr$. □

Lemma 2 [3] *Let G be a graph with n vertices, m edges, and degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. Let ℓ be the integer such that $\sum_{i=1}^{\ell-1} d_i < 4m/3$ but $\sum_{i=1}^{\ell} d_i \geq 4m/3$. If n is large enough and $m = \Omega(n \log^2 n)$ then*

$$\mathrm{CR}(G) \geq \frac{1}{65}\sum_{i=1}^{\ell} d_i^2.$$

Proof of the Theorem Let $\varepsilon, \gamma \in (0,1)$ be fixed. Choose integers r, p such that $\frac{1}{\varepsilon} < r \leq \frac{2}{\varepsilon}$, and $\frac{67}{\gamma} < p \leq \frac{68}{\gamma}$. It follows that we have $\frac{1}{r} < \varepsilon \leq \frac{2}{r}$, and $\frac{67}{p} < \gamma \leq \frac{68}{p}$. Then there is an $n_{\varepsilon,\gamma}$ with the following properties: (a) $n_{\varepsilon,\gamma} \geq n_0$ from Lemma 1, (b) $(n_{\varepsilon,\gamma})^{1+\varepsilon} > 18pr \cdot (n_{\varepsilon,\gamma})^{1+1/r}$.

Let G be a graph with $n \geq n_{\varepsilon,\gamma}$ vertices and $m \geq n^{1+\varepsilon}$ edges. Let $v_1, \ldots, v_n$ be the vertices of G, of degrees $d_1 \leq d_2 \leq \cdots \leq d_n$ and define ℓ as in Lemma 2, that is, $\sum_{i=1}^{\ell-1} d_i < 4m/3$ but $\sum_{i=1}^{\ell} d_i \geq 4m/3$. Let G_0 be the subgraph of G induced by $v_1, \ldots, v_\ell$. Observe that G_0 has $m' \geq m/3$ edges. Therefore, by Lemma 1 G_0 contains at least $m'/3pr \geq m/9pr$ edge-disjoint $2r$-earrings, each of size p.

Let M be the set of the main edges of these $2r$-earrings. We have $|M| \geq m/9pr \geq \frac{\varepsilon\gamma}{1224}m$. Let $G' = G - M$ and $G'_0 = G_0 - M$.

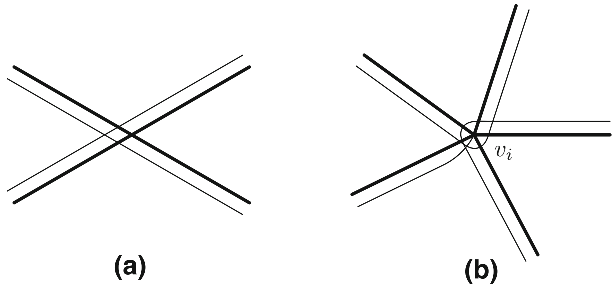

Fig. 1 The *thick edges* are edges of G', the *thin edges* are the potential edges. **a** A neighborhood of a crossing in $D(G')$ and **b** a neighborhood of a vertex v_i in G'

Take an optimal drawing $D(G')$ of the subgraph $G' \subset G$. We have to draw the missing edges to obtain a drawing of G. Our method is a randomized variation of the embedding method, which has been applied by Leighton [4], Richter and Thomassen [7], Shahrokhi et al. [8], Székely [9], and most recently by Fox and Tóth [3]. For every missing edge $e_i = u_i v_i \in M \subset G_0$, e_i is the deleted main edge of a $2r$-earring $E_i \subset G_0$. So there are p edge-disjoint paths in G_0 from u_i to v_i. For each of these paths, draw a curve from u_i to v_i infinitesimally close to that path, on either side. Call these p curves *potential* $u_i v_i$*-edges* and call the resulting drawing D. Note that a potential $u_i v_i$-edge crosses itself if the corresponding path does. In such cases, we redraw the potential $u_i v_i$-edge in the neighborhood of each self-crossing to get a noncrossing curve.

To get a drawing of G, for each $e_i = u_i v_i \in M$, choose one of the p potential $u_i v_i$-edges at random, independently and uniformly, with probability $1/p$, and draw the edge $u_i v_i$ as that curve.

There are two types of new crossings in the obtained drawing of G. First category crossings are infinitesimally close to a crossing in $D(G')$, second category crossings are infinitesimally close to a vertex of G_0 in $D(G')$.

The expected number of first category crossings is at most

$$\left(1 + \frac{2}{p} + \frac{1}{p^2}\right) \mathrm{CR}(G') = \left(1 + \frac{1}{p}\right)^2 \mathrm{CR}(G').$$

Indeed, for each edge of G', there can be at most one new edge drawn next to it, and that is drawn with probability at most $1/p$. Therefore, in the close neighborhood of a crossing in $D(G')$, the expected number of crossings is at most $(1 + \frac{2}{p} + \frac{1}{p^2})$ (see Fig. 1a).

In order to estimate the expected number of second category crossings, consider the drawing D near a vertex v_i of G_0. In the neighborhood of vertex v_i we have at most d_i original edges. Since we draw at most one potential edge along each original edge, there can be at most d_i potential edges in the neighborhood. Each potential edge can cross each original edge at most once, and any two potential edges can cross at most twice (see Fig. 1b). Therefore, the total number of first category crossings in D in the neighborhood of v_i is at most $2d_i^2$. (This bound can be substantially improved with a more careful argument, see e. g. [3], but we do not need anything better here.) To obtain the drawing of G, we keep each of the potential edges with probability $1/p$, so the expected number of crossings in the neighborhood of v_i is at most $(\frac{1}{p}+\frac{1}{p^2})d_i^2$, using the fact that the self-crossings of the potential uv-edges have been eliminated.

Therefore, the total expected number of crossings in the random drawing of G is at most $(1+\frac{2}{p}+\frac{1}{p^2})\text{CR}(G')+(\frac{1}{p}+\frac{1}{p^2})\sum_{i=1}^{\ell}d_i^2$.

There exists an embedding with at most this many crossings, therefore, by Lemma 2 we have

$$\begin{aligned}\text{CR}(G) &\leq \left(1+\frac{1}{p}\right)^2\text{CR}(G')+\left(\frac{1}{p}+\frac{1}{p^2}\right)\sum_{i=1}^{\ell}d_i^2\\ &\leq \left(1+\frac{1}{p}\right)^2\text{CR}(G')+\left(\frac{65}{p}+\frac{65}{p^2}\right)\text{CR}(G).\end{aligned}$$

It follows that

$$\left(1-\frac{65}{p}-\frac{65}{p^2}\right)\text{CR}(G)\leq\left(1+\frac{1}{p}\right)^2\text{CR}(G'),$$

so

$$\begin{aligned}\left(1-\frac{65}{p}-\frac{65}{p^2}\right)\left(1-\frac{1}{p}\right)^2\text{CR}(G) &\leq \left(1-\frac{1}{p^2}\right)^2\text{CR}(G'),\\ \left(1-\frac{65}{p}-\frac{65}{p^2}\right)\left(1-\frac{2}{p}\right)\text{CR}(G) &\leq \text{CR}(G'),\\ \left(1-\frac{67}{p}\right)\text{CR}(G) &\leq \text{CR}(G'),\end{aligned}$$

consequently,

$$(1-\gamma)\,\text{CR}(G)\leq\text{CR}(G').$$

□

3 Concluding Remarks

In the statement of our Theorem we cannot require that *every* subgraph G' with $(1-\delta)$ $m(G)$ edges has crossing number $\text{CR}(G')\geq(1-\gamma)\text{CR}(G)$, instead of just *one* such subgraph G'. In fact, the following statement holds.

Proposition 1 *For every $\varepsilon\in(0,1)$ there exist graphs G_n with $n(G_n)=\Theta(n)$ vertices and $m(G_n)=\Theta(n^{1+\varepsilon})$ edges with subgraphs $G_n'\subset G_n$ such that*

$$m(G_n')=(1-o(1))\,m(G_n)$$

and

$$\mathrm{CR}(G'_n) = o(\mathrm{CR}(G_n)).$$

Proof Roughly speaking, G_n will be the disjoint union of a large graph G'_n with low crossing number and a small graph H_n with large crossing number. More precisely, let $G = G_n$ be a disjoint union of graphs $G' = G'_n$ and $H = H_n$, where G' is a disjoint union of $\Theta(n^{1-\varepsilon})$ complete graphs, each with $\lfloor n^{\varepsilon} \rfloor$ vertices and H is a complete graph with $\lfloor n^{(3+5\varepsilon)/8} \rfloor$ vertices. We have $m(G) = \Theta(n^{1+\varepsilon})$ and $m(H) = \Theta(n^{(3+5\varepsilon)/4}) = o(m(G))$, since $\frac{3+5\varepsilon}{4} < 1 + \varepsilon$. By the crossing lemma (see e. g. [5]), $\mathrm{CR}(G) \geq \mathrm{CR}(H) = \Omega(n^{(3+5\varepsilon)/2})$, but $\mathrm{CR}(G') = O(n^{1-\varepsilon} \cdot n^{4\varepsilon}) = O(n^{1+3\varepsilon}) = o(\mathrm{CR}(G))$, because $\frac{3+5\varepsilon}{2} > 1 + 3\varepsilon$. □

In the preliminary version of this paper [2] we conjectured that we can require that a positive fraction of all subgraphs G' of G with $(1-\delta)m(G)$ edges has crossing number $\mathrm{CR}(G') \geq (1-\gamma)\mathrm{CR}(G)$. The following construction shows that the conjecture does not hold in general for graphs with less than $n^{4/3-\Omega(1)}$ edges.

Proposition 2 *For every $\varepsilon \in (0, 1/3)$ and $\delta > 0$ there exist graphs G_n with $n(G_n) = \Theta(n)$ vertices and $m(G_n) = \Theta(n^{1+\varepsilon})$ edges with the following property. Let G'_n be a random subgraph of G_n such that we choose each edge of G_n independently with probability $p = 1 - \delta$. Then*

$$\Pr\left[\mathrm{CR}(G'_n) \leq o(\mathrm{CR}(G_n))\right] > 1 - e^{-\delta n^{\Theta(1/3-\varepsilon)}}.$$

Proof As in Proposition 1, the idea is to build the graph $G = G_n$ from two disjoint graphs K and H, where K is a large graph with low crossing number and H is a small graph with high crossing number. In addition, deleting a random constant fraction of edges from H will break all the crossings in H with high probability.

Now we describe the constructions more precisely. Let $\gamma > 0$ be a constant such that $3\varepsilon + 4\gamma < 1$. Let K be a disjoint union of $\Theta(n^{1-\varepsilon})$ complete graphs, each with n^{ε} vertices (we omit the explicit rounding to keep the notation simple). We have $m(K) = \Theta(n^{1+\varepsilon})$ and $\mathrm{CR}(K) = \Theta(n^{1+3\varepsilon})$.

The graph H consists of five *main* vertices $v_1, v_2, \ldots, v_5$ and $n^{1-2\gamma}$ internally vertex disjoint paths of length n^{γ} connecting each pair v_i, v_j. The graph H has $n(H) = \Theta(n^{1-\gamma})$ vertices and $m(H) = \Theta(n^{1-\gamma})$ edges. We claim that $\mathrm{CR}(H) = n^{2-4\gamma}$. The upper bound follows from the fact that the crossing number of K_5 is 1. We take a drawing of K_5 with one crossing and replace each edge e by $n^{1-2\gamma}$ paths drawn close to e. For the lower bound take a drawing of H with minimum number of crossings. Let $p_{i,j}$ be a path with the minimum number of crossings among the paths connecting v_i and v_j. By redrawing all the other paths connecting v_i and v_j along $p_{i,j}$ the crossing number of the drawing does not change. The paths $p_{i,j}$ together form a subdivision of K_5, therefore at least one pair $p_{i,j}, p_{k,l}$ of the paths crosses. Due to the redrawing, every path connecting v_i and v_j crosses every path connecting v_k and v_l, which makes $n^{2-4\gamma}$ crossings. By the choice of γ, $n^{1+3\varepsilon} = o(n^{2-4\gamma})$, therefore $\mathrm{CR}(G) = \Theta(\mathrm{CR}(H))$ and $\mathrm{CR}(K) = o(\mathrm{CR}(G))$.

Let G' be a random subgraph of G where each edge of G is taken independently with probability $p = 1 - \delta$. Let $H' = G' \cap H$. We show that with high probability, H' is a forest, in particular $\mathrm{CR}(H') = 0$. This happens if at least one edge is missing from every path connecting two main vertices of H. The probability of such an event is at least

$$1 - n \cdot (1-\delta)^{n^\gamma} \geq 1 - e^{-\delta n^\gamma + \log n}.$$

It follows that with this probability, $\mathrm{CR}(G') \leq \mathrm{CR}(K) \leq o(\mathrm{CR}(G))$. □

Note that in the above construction the number δ does not have to be constant: it is enough to delete a random $\delta = c \log n / n^\gamma$ fraction of the edges to get the same conclusion with probability almost 1.

The question whether deleting a small random constant fraction of the edges of a graph G decreases the crossing number only by a small constant fraction remains open for graphs with more than $n^{4/3}$ edges. We do not know the answer even to the following weaker version of the question.

Problem 1 Let $\varepsilon \in (0, 2/3)$ and $p \in (0, 1)$ be constants. Does there exist $c(p) > 0$ and n_0 such that for every graph G with $n(G) > n_0$ and $m(G) > n(G)^{4/3+\varepsilon}$, a random subgraph G' of G with each edge taken with probability p has crossing number at least $c(p) \cdot \mathrm{CR}(G)$, with probability at least $1/2$?

The graphs in Proposition 2 have small number of edges responsible for almost all the crossings. Is this the only way how to force a random subgraph of G to have crossing number $o(\mathrm{CR}(G))$?

Problem 2 Let $\varepsilon > 0$. Does there exist n_0 and δ such that every graph G with $n(G) \geq n_0$ and $m(G) \geq n(G)^{1+\varepsilon}$ has a subset F of $o(m(G))$ edges such that every subgraph G' of G with $m(G') \geq (1-\delta)m(G)$ and $F \subset E(G')$ has $\mathrm{CR}(G') \geq (1-\varepsilon)\mathrm{CR}(G)$?

References

1. Alon, N., Hoory, S., Linial, N.: The Moore bound for irregular graphs. Graphs Combin. **18**, 53–57 (2002)
2. Černý, J., Kynčl, J., Tóth, G.: Improvement on the decay of crossing numbers. In: Proceedings of the 15th International Symposium on Graph Drawing (Graph Drawing 2007). Lecture Notes in Computer Science, vol. 4875, pp. 25–30. Springer, Berlin (2008)
3. Fox, J., Cs, Tóth: On the decay of crossing numbers. J. Combin. Theory Ser. B **98**(1), 33–42 (2008)
4. Leighton, T.: Complexity Issues in VLSI. MIT Press, Cambridge (1983)
5. Pach, J., Radoičić, R., Tardos, G., Tóth, G.: Improving the Crossing Lemma by finding more crossings in sparse graphs. In: Proceedings of the 19th ACM Symposium on Computational Geometry, pp. 68–75 (2004) [Also in Discrete Comput. Geom. **36**, 527–552 (2006)]
6. Pach, J., Tóth, G.: Thirteen problems on crossing numbers. Geombinatorics **9**, 194–207 (2000)
7. Richter, B., Thomassen, C.: Minimal graphs with crossing number at least k. J. Combin. Theory Ser. B **58**, 217–224 (1993)
8. Shahrokhi, F., Sýkora, O., Székely, L., Vrt'o, I.: Crossing numbers: bounds and applications. In: Intuitive Geometry (Budapest, 1995), vol. 6, pp. 179–206. Bolyai Society Mathematical Studies, Budapest (1997)
9. Székely, L.: Short proof for a theorem of Pach, Spencer, and Tóth. In: Towards a Theory of Geometric Graphs. Contemporary Mathematics, vol. 342, pp. 281–283. AMS, Providence (2004)