Note on the Erdős-Szekeres theorem

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Abstract

Let \( g(n) \) denote the least integer such that among any \( g(n) \) points in general position in the plane there are always \( n \) in convex position. In 1935, P. Erdős and G. Szekeres showed that \( g(n) \) exists and \( 2^{n-2} + 1 \leq g(n) \leq \binom{2n-4}{n-2} + 1 \). Recently, the upper bound has been slightly improved by Chung and Graham and by Kleitman and Pachter. In this note we further improve the upper bound to

\[
g(n) \leq \binom{2n-5}{n-2} + 2.
\]

In 1933, Esther Klein raised the following question. Is it true that for every \( n \) there is a least number \( g(n) \) such that among any \( g(n) \) points in general position in the plane there are always \( n \) in convex position?

This question was answered in the affirmative in a classical paper of Erdős and Szekeres [ES35]. In fact, they showed [ES35, ES60] that

\[
2^{n-2} + 1 \leq g(n) \leq \binom{2n-4}{n-2} + 1.
\]

The lower bound, \( 2^{n-2} + 1 \), is sharp for \( n = 2, 3, 4, 5 \) and has been conjectured to be sharp for all \( n \). However, the upper bound, \( \binom{2n-4}{n-2} + 1 \approx e\frac{16}{\sqrt{n}} \), was not improved for 60 years. Recently, Chung and Graham [CG97] managed to improve it by 1. Shortly after, Kleitman and Pachter [KP97] showed that \( g(n) \leq \binom{2n-4}{n-2} + 7 - 2n \).

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Inspired by these papers, in this note we get a further improvement, roughly by a factor of 2.

**Theorem.** Any set of \( \binom{2n-5}{n-2} + 2 \) points in general position in the plane contains \( n \) points in convex position.

In other words, \( g(n) \leq \binom{2n-5}{n-2} + 2 \). Since \( 2 \binom{2n-5}{n-2} = \binom{2n-1}{n-2} \), our upper bound is about half of the original bound of Erdős and Szekeres.

In the original proof, Erdős and Szekeres were looking for special convex \( n \)-gons, namely for \( n \)-caps and \( n \)-cups.

![Figure 1: A 6-cap and a 6-cup.](image)

**Definition.** The points \( (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) \), \( x_1 < x_2 < \ldots < x_n \), form an \( n \)-cap if

\[
\frac{y_2 - y_1}{x_2 - x_1} > \frac{y_3 - y_2}{x_3 - x_2} > \ldots > \frac{y_n - y_{n-1}}{x_n - x_{n-1}}.
\]

Similarly, they form an \( n \)-cup if

\[
\frac{y_2 - y_1}{x_2 - x_1} < \frac{y_3 - y_2}{x_3 - x_2} < \ldots < \frac{y_n - y_{n-1}}{x_n - x_{n-1}}.
\]

(See Fig. 1.)

**Lemma (Erdős and Szekeres [ES35]).** Let \( f(n, m) \) be the least integer such that any set of \( f(n, m) \) points in general position in the plane contains either an \( n \)-cap or an \( m \)-cup. Then

\[
f(n, m) = \binom{n + m - 4}{n - 2} + 1.
\]
Proof of Theorem.

Let $P$ be a set of points in general position in the plane and suppose that $P$ does not contain $n$ points in convex position. Let $a$ be a vertex of the convex hull of $P$. Let $b$ be a point outside the convex hull of $P$ such that none of the lines determined by the points of $P \setminus \{a\}$ intersects the segment $ab$. Finally, let $\ell$ be a line through $b$ which avoids the convex hull of $P$ (see Fig. 2).

Consider a projective transformation $T$ which maps the line $\ell$ to the line at infinity, and maps the segment $ab$ to the vertical half-line $v^{-}(a')$, emanating downwards from $a' = T(a)$. We get a point set $P' = T(P)$ from $P$. Since $\ell$ avoided the convex hull of $P$, the transformation $T$ does not change convexity on the points of $P$, that is, any subset of $P$ is in convex position if and only if the corresponding points of $P'$ are in convex position. So the assumption holds also for $P'$, no $n$ points of $P'$ are in convex position. By the choice of the point $b$, none of the lines determined by the points of $P' \setminus \{a'\}$ intersects $v^{-}(a')$. Therefore, any $m$-cap in the set $Q' = P' \setminus \{a'\}$ can be extended by $a'$ to a convex $(m + 1)$-gon.

Since no $n$ points of $P'$ are in convex position, $Q'$ cannot contain any $n$-cup or $(n-1)$-cap.
Therefore, by the Lemma,

\[ |Q'| \leq f(n, n - 1) - 1 = \binom{2n - 5}{n - 2}, \quad |P| \leq \binom{2n - 5}{n - 2} + 1, \]

and the theorem follows. \( \square \)

References


