Note on the chromatic number of the space

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Abstract

The chromatic number of the space is the minimum number of colors needed to color the points of the space so that every two points unit distance apart have different colors. We show that this number is at most 15, improving the best known previous bound of 18.

1 Introduction

The *unit distance graph* G_d of the space $\Re^d$ is the graph whose vertices correspond to the points of $\Re^d$ and two vertices are connected if and only if the corresponding points have distance 1. The classical Hadwiger-Nelson problem asks for the chromatic number of the plane, or more precisely the chromatic number of the unit distance graph of the plane. The best known bounds are four and seven, due to Nelson and Isbell, respectively.

In three dimensions, Raiskii [R70] proved that $\chi(G_3) \geq 5$ which was recently improved by Nechushtan [N00] to 6. For the upper bound, Székely and Wormald [SW89] (see also [BT96]) proved $\chi(G_3) \leq 21$ which was improved to $\chi(G_3) \leq 18$ by Coulson [C97].

Theorem. *The chromatic number of the space $\Re^3$ is at most 15.*

In higher dimensions, the best known bounds are

$$(1 + o(1)) \cdot 1.2^d \leq \chi(G_n) \leq (3 + o(1))^d$$

due to Frankl and Wilson [FW81] and Larman and Rogers [LR72] respectively. For a survey on this problem see [S91] and [C93].

2 Proof of the Theorem

Let Π_3 be the convex hull of all vectors that are obtained by permuting the coordinates of the vector $(1, 2, 3, 4)$. This is a three dimensional polytope, called the permutahedron (see [Z98]). It has 24 vertices and 14 facets, 8 regular hexagons and 6 squares (see Figure). Our 15-coloring of the space is based on a tiling of the space with copies of Π_3.

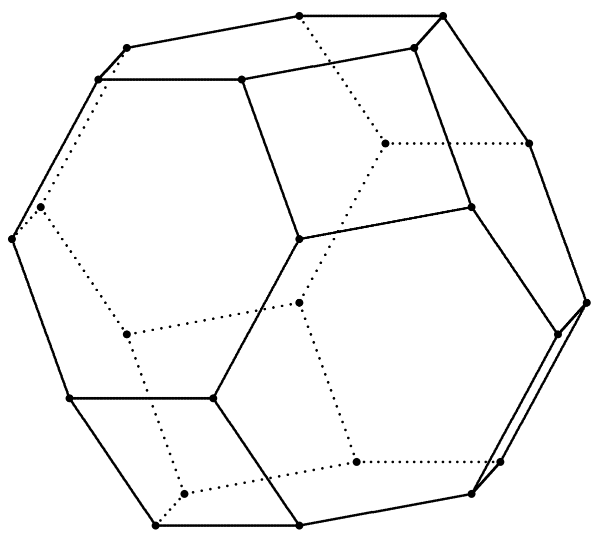

Figure. *The permutahedron* Π_3

Let $\mathcal{T}$ be isometric to Π_3, with center at the origin, and edge length $\frac{\sqrt{10}}{10}$. The vertices of $\mathcal{T}$ are all vectors that are obtained by permuting the coordinates of the vectors

$$\left(\pm\frac{\sqrt{20}}{20}, \pm\frac{\sqrt{20}}{10}, 0\right).$$

It is easy to check that $\mathcal{T}$ has diameter 1, the distance between any pair of opposite vertices. Remove those vertices of $\mathcal{T}$ which have $-\frac{\sqrt{20}}{20}$ as a coordinate. For simplicity, we still call the resulting body $\mathcal{T}$. Now the distance 1 is not realized within $\mathcal{T}$.

Let Λ be the lattice generated by

$$\vec{a} = \left(\frac{\sqrt{20}}{5}, 0, 0\right), \quad \vec{b} = \left(0, \frac{\sqrt{20}}{5}, 0\right), \quad \vec{c} = \left(\frac{\sqrt{20}}{10}, \frac{\sqrt{20}}{10}, \frac{\sqrt{20}}{10}\right).$$

Then $\{\mathcal{T} + \vec{u} \mid \vec{u} \in \Lambda\}$ is a tiling of $\Re^3$. Let $\mathcal{T}_{i,j,k} = \mathcal{T} + i\vec{a} + j\vec{b} + k\vec{c}$ for any i, j, k integers. Note that the boundary points of any $\mathcal{T}_{i,j,k}$ are covered twice.

Now we define a coloring of the space, using colors $0, 1, \ldots 14$. For any point $p \in \mathcal{T}_{i,j,k}$ let the color of p be $5i + 3j + k$ (mod 15). For multiply covered points choose any of the resulting colors. We show that it is a proper coloring, that is, the unit distance is not realized between points of the same color.

A triple (i,j,k) is called *dangerous* if there are two points, $p \in \mathcal{T}_{0,0,0}$ and $q \in \mathcal{T}_{i,j,k}$ at unit distance.

Claim. If (i,j,k) is dangerous, then $5i+3j+k \not\equiv 0 \pmod{15}$.

Proof of Claim. Clearly, (i,j,k) is dangerous if and only if $(-i,-j,-k)$ is. And $5i+3j+k \equiv 0 \pmod{15}$ if and only if $5(-i)+3(-j)-k \equiv 0 \pmod{15}$. Therefore, it is enough to check check those triples where $k \geq 0$.

Here is the list of all such dangerous triples:

$(1,0,0)$,	$(0,1,0)$,	$(-1,0,0)$,	$(0,-1,0)$,	$(1,1,0)$,	$(-1,1,0)$,
$(-1,-1,0)$,	$(1,-1,0)$,	$(2,0,0)$,	$(0,2,0)$,	$(-2,0,0)$,	$(0,-2,0)$,
$(0,0,1)$,	$(-1,0,1)$,	$(-1,-1,1)$,	$(0,-1,1)$,	$(1,0,1)$,	$(0,1,1)$,
$(1,-1,1)$,	$(0,-2,1)$,	$(-1,-2,1)$,	$(-2,-1,1)$,	$(-2,0,1)$,	$(-1,1,1)$,
$(0,0,2)$,	$(-1,0,2)$,	$(-2,0,2)$,	$(-2,-1,2)$,	$(-1,-1,2)$,	$(-2,-2,2)$,
$(-1,-2,2)$,	$(0,-2,2)$,	$(0,-1,2)$			
$(-1,-1,3)$,	$(-1,-2,3)$,	$(-2,-1,3)$,	$(-2,-2,3)$,	$(-2,-2,4)$	

Finally, it is easy to check that none of the triples satisfy $5i+3j+k \not\equiv 0 \pmod{15}$. □

Return to the proof of the Theorem. Suppose that there are two points, p and p' of the same color and at unit distance, $p \in \mathcal{T}_{i,j,k}$ and $p' \in \mathcal{T}_{i',j',k'}$. Since the unit distance is not realized in $\mathcal{T}$, $(i,j,k) \neq (i',j',k')$. Let $I = i'-i$, $J = j'-j$, $K = k'-k$. Since p and p' have the same color, $5I+3J+K \equiv 0 \pmod{15}$. Let $q = p - i\vec{a} - j\vec{b} - k\vec{c}$, $q' = p' - i\vec{a} - j\vec{b} - k\vec{c}$. Then $q \in \mathcal{T}_{0,0,0}$ and $q' \in \mathcal{T}_{I,J,K}$, q and q' have the same color they are unit distance apart. But then (I,J,K) is a dangerous triple so by the Claim q and q' have different colors, a contradiction. This concludes the proof of the Theorem. □

Remarks. 1. If we add the missing vertices to each $\mathcal{T}_{i,j,k}$, then we have two problems. The unit distance would be realized within each tile, and we would get an additional set of dangerous neighbors:

$(1,2,0)$,	$(2,1,0)$,	$(2,-1,0)$,	$(1,-2,0)$,	$(-1,-2,0)$,	$(-2,-1,0)$,
$(-2,1,0)$,	$(-1,2,0)$,	$(1,-1,2)$,	$(-1,-3,2)$,	$(-3,-1,2)$,	$(-1,1,2)$,
$(-1,1,-2)$,	$(1,3,-2)$,	$(3,1,-2)$,	$(1,-1,-2)$,	$(-1,-2,4)$,	$(-3,-2,4)$,
$(-2,-1,4)$,	$(-2,-3,4)$,	$(1,2,-4)$,	$(3,2,-4)$,	$(2,1,-4)$,	$(2,3,-4)$

If we want a modular coloring where these tiles also have different colors than $\mathcal{T}_{0,0,0}$, we have to use 24 colors. In other words, our construction is "rigid", the tiling can not be scaled even a little bit.

2. Since the permutahedron used in the construction has 14 facets, we can not have a similar proper coloring with 14 colors.

3. We conjecture that there is a proper modular coloring based on the lattice tiling of $\Re^d$ with d-dimensional permutahedra, that uses asymptotically fewer than $(3 + o(1))^d$ colors. With this method we found a proper 54-coloring of $\Re^4$.

Added in proof. Very recently, Coulson [C02] has independently found a very similar 15-coloring of 3-space.

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