A Ramsey-type problem on right-angled triangles in space

Miklós Bóna
Massachusetts Institute of Technology, Cambridge, Massachusetts
Géza Tóth[1]
Courant Institute, New York University, New York, NY

February 24, 2003

Abstract

It is proved, that for any 3-coloring of $\mathbf{R}^3$ and for any right-angled triangle T, one can find a congruent copy of T, all of whose vertices are of the same color.

1 Introduction

In a series of papers, Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus [1, 2, 3, see also 4] have examined a variety of problems, adapting Ramsey theory to set systems defined by geometric means. In particular, they showed that for any 2-coloring of the 3-space and for any triangle T, there exists a congruent copy of T, all of whose vertices are of the same color. The question arises, whether the same result remains true for all 3-colorings of $\mathbf{R}^3$. The answer is known to be in the affirmative if T is a triangle with angles $\pi/6$, $\pi/3$ and $\pi/2$ (see [5]).

In the present note we extend this result to all right-angled triangles.

Theorem: Let T be any right-angled triangle. Then, for any 3-coloring of $\mathbf{R}^3$, there exists a congruent copy of T, all of whose vertices are of the same color.

[1]Partially supported by NSF grant CCR-91-22103 and OTKA-4269

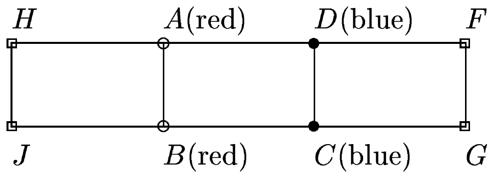
2 Proof of Theorem:

Let T be a right-angled triangle whose perpendicular sides are of length a and b. Fix a coloring of $\mathbf{R}^3$ with three colors (red, blue, and green, say). A triangle congruent to T is said to be *good* if its vertices are of the same color.

Assume in order to obtain a contradiction that there are no good triangles. Using this assumption, one can deduce a number of simple but useful properties of the coloring.

Lemma 1 *Let $ABCD$ be a rectangle, $AB = CD = a$, $BC = AD = b$, A and B are red, and C and D are blue. Then the points F, G, H, J shown in Fig.1 are all green.*

Fig. 1



Proof: It is obvious that F cannot be blue. Suppose that it is red. Let v be a vector of length b, perpendicular to AB such that the angle between v and the plane determined by $ABCD$ is $\pi/3$. Let $A'B'C'D'$ denote the translate of $ABCD$ by v. It is easy to see that $ABA'B'$, $A'B'CD$, $CDC'D'$ and $C'D'FG$ are rectangles congruent to $ABCD$, so A' and B' must be green which forces C' and D' to be red, making the good triangle $C'D'F$. In the same way we can prove that G, H and J are green. □

Note that A, B, C, D, A', B' induce a triangular prism, whose rectangular faces have sides a and b. We use this type of prism throughout the proof.

Now we return to the proof of our theorem. We distinguish two different cases. The first one is when there is a line segment either of length $2a$ or of $2b$, whose endpoints and midpoint are of the same color. (We will call it a *good segment of type (a)* or *(b)*, respectively). The second one is when there are no good segments.

A rectangle with sides a and b is called *normal.*

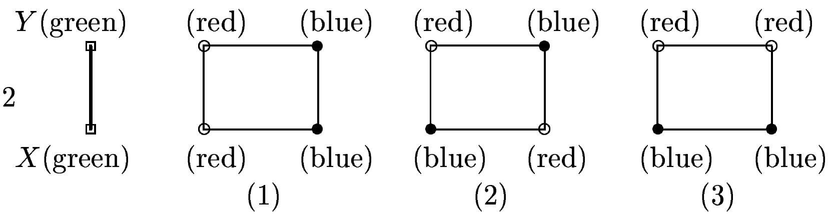
Suppose $a \geq b$. Let $ABCD$ be a normal rectangle so that $AB = CD = a$, $BC = AD = b$. Let E and F be two points above the plane of $ABCD$ such that $AE = DE = b$, $BE = CE = a$, $AF = DF = a$, and $BF = CF = b$. Let G and H denote the reflections of E and F, respecively, about the plane of $ABCD$. Clearly $AG = DG = b$, $BG = CG = a$, $AH = DH = a$, and $BH = CH = b$, and easy calculations show that $EH = FG = AC = BD$. We call the points E, F, G, H the *special exterior points* of our normal rectangle, and we say that E and H (resp. F and G) are *opposite* to each other.

Note that two opposite special exterior points together with any vertex of the normal rectangle form a right-angled triangle of sides a and b.

Case 1: Suppose there is a good segment XZ of type (a). Let Y denote the midpoint of XZ and suppose that X, Y, Z are green.

Let $c(X)$, $c(Y)$, $c(Z)$ denote the circles of radius b centered at X, Y and Z, respectively, whose planes are perpendicular to XZ. Clearly, these circles can have no green points. Consider a normal rectangle with two vertices on $c(X)$ and two vertices on $c(Y)$. Notice, that it can be colored in three essentially different ways. (See Fig.2)

Fig. 2



We claim that the coloring of no such rectangle has coloring of type (3). Indeed, Lemma 1 shows that both points of the circle $c(Z)$ in the plane of our rectangle are green, which is a contradiction.

We claim that either there exists a normal rectangle having two vertices on both circles $c(X)$ and $c(Y)$, whose coloring is of type (1), or there exists a normal rectangle having two vertices on both circles $c(Y)$ and $c(Z)$, whose coloring is of type (1). Indeed, suppose that all of these normal rectangles are of type (2). It is clear that any normal rectangle of type (2) forces the corresponding four special exterior points to be green. Consequently, the circles formed by these exterior points are completely green circles. However, in this case we obtain infinitely many good green triangles.

Lemma 2 *There is no rectangle of side–lengths a and $b\sqrt{3}$, whose vertices are colored with exactly two different colors and whose sides of length a have monochromatic vertices.*

Proof: Suppose that there exists such a rectangle $ABCD$. Say, A and

B are red, and C and D are blue. Consider the rhombuses $AEDF$ and $BGCH$, whose planes are perpendicular to the plane $ABCD$, and whose shorter diagonals are of length b. Then E, F, G and H must be green, which is a contradiction because these points form a normal rectangle, and hence four good triangles. $\square$

Now we are in a position to prove the theorem in Case 1. Take a good segment XZ of type a with midpoint Y, define $c(X)$, $c(Y)$ and $c(Z)$ as before. Consider a normal rectangle $KLMN$ of type (1) and having two vertices on both of $c(X)$ and $c(Y)$. Reflect XY about the plane of $KLMN$, and get the green points S and T. Let us rotate the polyhedron $XYKLMNST$ (in fact, it is a rhombohedron and four of its faces are normal rectangles) around XY in the positive direction to the polyhedron $XYK'L'M'N'S'T'$, so that $STS'T'$ be a normal rectangle. Applying Lemma 2 to the rectangle $XYS'T'$, we obtain that S' and T' have different colors. So one of them (say, S') is red and the other one is blue.

Consider the coloring of the rectangle $K'L'M'N'$. It is easy to see that it cannot be of type (1) (because of S' and T'), so it is of type (2). So either $K'L'S'T'$ or $M'N'S'T'$ forms a normal rectangle whose coloring is of type (3). Suppose without loss of generality that $K'L'S'T'$ is such a rectangle. Applying Lemma 1, we find that there exists a green point on the circle $c(Z)$ ("above" K'). This is a contradiction completes the proof in Case 1.

Note that the proof of Lemma 2 does not use the assumption that there exists a good segment, so it will remain valid in Case 2.

Case 2: Suppose that no good segment exists.

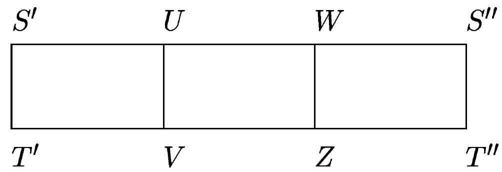
Claim: We can choose XY, a segment of length a with monochromtic vertices, so that we have a normal rectangle of type (1) with two vertices on both of $c(X)$ and $c(Y)$. (The circles (X) and $c(Y)$ are defined in the same way as in Case 1.)

Proof: We can clearly find a segment XY of length a with monochromatic vertices. If there is not a normal rectangle of type (1) with two vertices on both of $c(X)$ and $c(Y)$, rotate a normal rectangle with two vertices on $c(X)$, two on $c(Y)$, around XY. If we find a position of type (3) then we are done, since two monochromatic vertices of that rectangle can play the role of X and Y. (Their distance is b instead of a, but one can interchange the notation for a and b.)

So we may assume that all of these rectangles are of type (2). As we have proved above, all points of the circles formed by the special exterior points of these normal rectangles are green. (There can be two or four such circles, depending on the relationship between a and b.) Consider one of them which has the bigger radius. Then we can take any two points of this circle at distance b and make them play the role of X and Y. If for every choice of these points and for every choice of the normal rectangle, we obtain a coloring (2), then we can repeat the previous argument, which brings an entirely green torus around this circle, obviously large enough to accomodate a good triangle. Once a coloring other than (2) occurs then we are also done. □

So choose XY as above, let $KLMN$ be a normal rectangle with coloring type (1), and define S, T, S', T', K', L', M' and N', as above. We know that either S' or T' is red and the other one is blue. Suppose without loss of generality that S' is red. Let us rotate all of our points around XY in the positive direction by $2\pi/3$. Denote the images of S' and T' by S'' and T''. It is easy to see that one of them is red and the other is blue (because the image of a normal rectangle of type (1) by this rotation is of the same type). Moreover, the segments $S'S''$ and $T'T''$ are of length $3b$, and the points that divide them in the ratios $1:2$ and $2:1$ are on the circles $c(X)$ and $c(Y)$, respectively. Hence these points cannot be green. Denote them by U, V, W and Z as it is shown in Fig.3.

Fig. 3

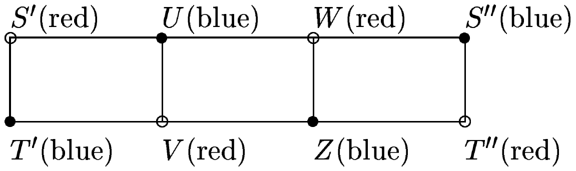


Our proof will consist of the examination of all possible colorings of the eight points, $S', T', S'', T'', U, V, W$ and Z, each of which is either red or blue. There are four different subcases up to symmetry). We have already supposed that S' is red and T' is blue. In the first and the second subcases, we suppose that S'' and T'' have different colors (say, S'' is blue and T'' is red). These subcases will be very easy. The third and the fourth subcases will be more complicated.

Observe that any special exterior point forms a good triangle with the vertices of any diagonal of its basic rectangle if they are monochromatic. Furthermore, two opposite special exterior points also form a good triangle with any vertex of their basic rectangle.

- Subcase 2.1: See Fig.4

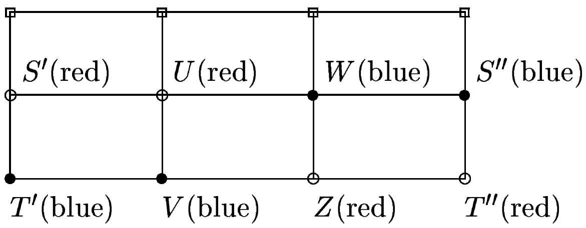
Fig. 4



It is clear that all special exterior points of the three normal rectangles must be green, contradicting our assumption that there are no good segments.

- Subcase 2.2: See Fig.5

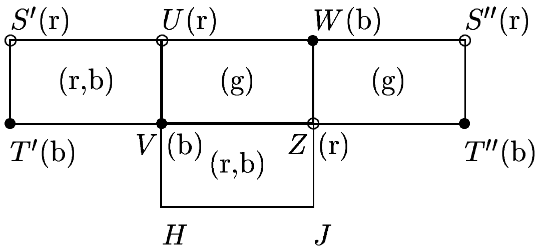
Fig. 5



Apply lemma 1 to the rectangles $S'UVT'$ and $WS''T''Z$. We get four collinear green points, each at distance a from its neighbors. Hence, there are good segments, which is a contradiction.

- Subcase 2.3: See Fig.6

Fig. 6



In the interior of some rectangles we have listed all possible colors of their special exterior points. Since the special exterior points of $S''T''WZ$ and $UVWZ$ are green, it is clear that no special exterior point of $S'UVT'$ and $VZHJ$ can be green. (Otherwise we would have a good segment or a good triangle.) Since both of $S'UVT'$ and $VZHJ$ have red and blue vertices, their opposite special exterior points must be red and blue. Let B_1 and B_2 be two opposite special exterior points of the rectangle $S'UVT'$, and let C_1 and C_2 two opposite special exterior points of $VZHJ$ such that B_1B_2 is a translate of C_1C_2. Let B (resp. C) denote the blue element of B_1, B_2 (resp. C_1, C_2). If B and C are on the same side of the plane $S'T'S''$, then BCV is a good triangle. If they are on opposite sides of this plane, then BVC is a good segment. So we obtain a contradiction in both cases.

- Subcase 2.4: See Fig.7

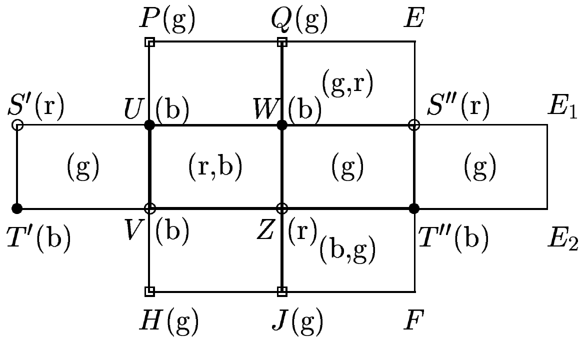


Fig. 7

Since the special exterior points of $WZT''S''$ and $S'T'VU$ are green, it is easy to see that one of the opposite special exterior points of $UVWZ$ is red, and the other is blue. Applying Lemma 1 to $UVWZ$ we obtain that H, J, P and Q are all green. Repeating the argument at the end of Subcase 3, we get that the opposite special exterior points of the rectangle $ZT''FJ$ are blue and green. Moreover, the rectangle $QWS''E$ has red-green pairs. (For if not, then we can complete the proof in the same way as in the previous subcase). For the same reason, all special exterior points of $S''T''E_1E_2$ must be green. Hence, neither E_1 nor E_2 is green, and they have different colors, because otherwise they would form a good triangle with S'' or T''. If E_1 is red and E_2 is blue, then apply Lemma 1 to the rectangle $E_1E_2S''T''$ and get the good segment HF wth midpoint J. If E_1 is blue and E_2 is red, then observe that $V, Z, T'', E_2, E_1, S'', W, U$ are colored in exactly the same way as the points $S', U, W, S'', T'', Z, V, T'$ in the previous subcase. Since we have already obtained a contradiction in Subcase 3 based on these 8 points, we can conclude that subcase 4 cannot occur, either.

This completes the discussion of Case 2, and hence the proof of the theorem.

Remarks:

It is natural to ask that if T is any right-angled triangle then how many colors are needed to color the space so that there is no good triangle. A general way to ensure that there is no right-angled triangle with monochromatic vertices and with hypotenuse of unit length is to define a coloring in which there is no unit segment with monochromatic vertices. This can be done with 21 colors as follows: let us consider a regular hexagonal lattice H of the plane with side length $1/2$. It is well known that the plane can be 7-colored so that all interior points of each of the hexagons of the lattice have the same color and there is no unit segment in the plane with monochromatic vertices.

Now let us consider the planes $z = 0$, $z = 3/4$, $z = 3/2 \dots z = (3/4)n$, where n takes all integer values. Let us take a copy of H in the plane $z = 0$ and take a translated copy of this parallel to the z–axis on each of the planes $z = n(3/4)$. Now if $n = 3k$, then let us 7-color the plane $z = (3/4)n$ with colors $a_1, a_2, \dots a_7$ according to its copy of H, if $n = 3k + 1$, then let us 7-color the plane $z = (3/4)n$ with colors $b_1, b_2, \dots b_7$, and if $n = 3k + 2$, then let us 7-color the plane $z = (3/4)n$ with colors $c_1, c_2, \dots c_7$, according to its translated copy of H. Finally, give to any yet uncolored point P of the space the color of its z-projected image on the closest plane $z = (3/4)n$ which is below P.

It is clear that this 21-coloring cannot have any unit segment of monochromatic vertices, as it consists of monochromatic regular hexagonal prisms, each of which is too small to contain such a segment, and the distance of any two of them having the same color is larger than 1.

Of course, if we ask how many colors are needed to exclude *just one* triangle T, then a smaller number of colors might be enough. For example, if T is the isosceles right-angled triangle, then one easily sees that there exists a 9-coloring of the space without good triangles.

REFERENCES

1. P. Erdős, R. L. Graham, P. Montgomery, B. L. Rothschild, J. H. Spencer and E. G. Straus: *Euclidean Ramsey theorems I*, Journal of Combinatorial Theory, Series A, 14 (1973), pp. 341-363.

2. P. Erdős, R. L. Graham, P. Montgomery, B. L. Rothschild, J. H. Spencer and E. G. Straus: *Euclidean Ramsey theorems II*, in: Infi-

nite and Finite Sets, (A. Hajnal et al, eds), North Holland, (1975), pp. 529-558.

3. P. Erdős, R. L. Graham, P. Montgomery, B. L. Rothschild, J. H. Spencer and E. G. Straus: *Euclidean Ramsey theorems III*, in: Infinite and Finite Sets, (A. Hajnal et al, eds), North Holland, (1975), pp. 559-584.

4. R. L. Graham, B. L. Rothschild, J. H. Spencer: *Ramsey Theory* John Wiley & Sons, New York, 1980.

5. M. Bóna: *A Euclidean Ramsey theorem*, Discrete Mathamatics, 122 (1993), pp. 349-352

6. L. E. Shader: *All right triangles are Ramsey in the plane*, Journal of Combinatorial Theory, Series A, 20 (1976), pp 385-390.