Thirteen Problems on Crossing Numbers

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Abstract

The crossing number of a graph G is the minimum number of crossings in a drawing of G. We introduce several variants of this definition, and present a list of related open problems. The first item is Zarankiewicz's classical conjecture about crossing numbers of complete bipartite graphs, the last ones are new and less carefully tested. In Section 5, we state some conjectures about the expected values of various crossing numbers of random graphs, and prove a large deviation result.

1 Introduction

Let G be a graph, whose vertex set and edge set are denoted by $V(G)$ and $E(G)$, respectively. A *drawing* of G is a representation of

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G in the plane such that its vertices are represented by distinct points and its edges by simple continuous arcs connecting the corresponding point pairs. For simplicity, we assume that in a drawing (a) no edge passes through any vertex other than its endpoints, (b) no two edges touch each other (i.e., if two edges have a common interior point, then at this point they properly cross each other), and (c) no three edges cross at the same point.

Turán [23] defined the *crossing number* of G, $\text{CR}(G)$, as the smallest number of edge crossings in any drawing of G. Clearly, $\text{CR}(G) = 0$ if and only if G is planar.

Problem 1. (Zarankiewicz's Conjecture [11]) *The crossing number of the complete bipartite graph $K_{n,m}$ with n and m vertices in its classes satisfies*

$$\text{CR}(K_{n,m}) = \left\lfloor \frac{m}{2} \right\rfloor \cdot \left\lfloor \frac{m-1}{2} \right\rfloor \cdot \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{n-1}{2} \right\rfloor .$$

Kleitman [13] verified this conjecture in the special case when $\min\{m, n\} \leq 6$ and Woodall [25] for $m = 7$, $n \leq 10$.

Problem 2. *Is it true that the crossing number of the complete graph K_n satisfies*

$$\text{CR}(K_n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{n-1}{2} \right\rfloor \cdot \left\lfloor \frac{n-2}{2} \right\rfloor \cdot \left\lfloor \frac{n-3}{2} \right\rfloor .$$

Of course, the best known upper bounds for both $\text{CR}(K_{n,m})$ and $\text{CR}(K_n)$ are the conjectured values in Problems 1 and 2, respectively. The best known lower bounds, $\text{CR}(K_{m,n}) \geq n^2m^2\,(1/20 - o(1))$ and $\text{CR}(K_n) \geq n^4\,(1/80 - o(1))$, can be deduced from Kleitman's result by an easy counting argument.

Garey and Johnson [10] proved that the determination of the crossing number is an *NP-complete* problem. In the past twenty years, it turned out that crossing numbers play an important role in various fields of discrete and computational geometry, and they can also be used, e.g.,to obtain lower bounds on the chip area required for the VLSI circuit layout of a graph [14].

2 Lower Bounds

The following general lower bound on crossing numbers was discovered by Ajtai–Chvátal–Newborn–Szemerédi [1] and, independently, by Leighton [14]. For any graph G with n vertices and $e \geq 7.5n$ edges, we have

$$\text{CR}(G) \geq \frac{1}{33.75}\frac{e^3}{n^2}. \tag{1}$$

This estimate is tight up to a constant factor. The best known constant, $1/33.75$, in (1) is due to Pach and Tóth [20], who also showed that the result does not remain true if we replace $1/33.75$ by roughly 0.06.

For any positive valued functions $f(n), g(n)$, we write $f(n) \gg g(n)$ if $\lim_{n\to\infty} f(n)/g(n) = \infty$. It was shown by Pach, Spencer, and Tóth [19] that

$$\lim_{\substack{n \to \infty \\ n \ll e \ll n^2}} \frac{\min\{\text{CR}(G) : |V(G)| = n, |E(G)| = e\}}{e^3/n^2} = K_0 \tag{2}$$

exists and is positive. It follows from what we have said before that $0.029 < 1/33.75 \leq K_0 \leq 0.06$.

Problem 3. *Determine the precise value of K_0.*

Problem 4. (Erdős-Guy [8]) *Do there exist suitable constants C_1, $C_2 > 0$ such that*

$$\lim_{\substack{n \to \infty \\ C_1 n \leq e \leq C_2 n^2}} \frac{\min\{\text{CR}(G) : |V(G)| = n, |E(G)| = e\}}{e^3/n^2} = K_0?$$

If the answer to the last question were in the affirmative, then, clearly, $C_1 > 3$. We would also have that $C_2 < 1/2$, because, by [12], for $e = \binom{n}{2}$, $\text{CR}(K_n) > \left(\frac{1}{10} - \varepsilon\right)\frac{e^3}{n^2}$ holds for any $\varepsilon > 0$, provided that n is sufficiently large.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *bisection width* of G, $b(G)$, is defined as the minimum number of edges, whose removal splits the graph into two roughly equal subgraphs. More precisely, $b(G)$ is the minimum number of edges running between V_1 and V_2, over all partitions of the vertex set of G into two disjoint parts $V_1 \cup V_2$ such that $|V_1|, |V_2| \geq |V(G)|/3$.

Leighton observed that there is an intimate relationship between the bisection width and the crossing number of a graph [15], which is based on the Lipton–Tarjan separator theorem for planar graphs [16]. The following version of this relationship was obtained by Pach, Shahrokhi, and Szegedy [18]. Let G be a graph of n vertices with degrees $d_1, d_2, \ldots, d_n$. Then

$$b(G) \leq 10\sqrt{\mathrm{CR}(G)} + 2\sqrt{\sum_{i=1}^{n} d_i^2}. \tag{3}$$

The example of a *star* (i.e., a tree consisting of a vertex connected to all other vertices) shows that (3) does not remain true if we remove the last term on its right-hand side. However, it is possible that the dependence of the bound on the degrees of the vertices can be improved.

Problem 5. *Does there exist a constant $t < 2$ such that*

$$b(G) = O\left(\sqrt{\mathrm{CR}(G)} + \left(\sum_{i=1}^{n} d_i^t\right)^{1/t}\right)$$

holds for every graph of n vertices with degrees $d_1, d_2, \ldots, d_n$?

3 Extensions

For every $g \geq 0$, one can define a new crossing number, $\mathrm{CR}_g(G)$, as the minimum number of crossings in any drawing of G on the *torus with g holes.* It was shown in [19] that (2) remains true with exactly the same constant K_0, if we replace $\mathrm{CR}(G)$ by $\mathrm{CR}_g(G)$ and keep g fixed. What happens if g tends to infinity with n?

Problem 6. *Find a function $g = g(n)$ tending to infinity such that*

$$\lim_{\substack{n \to \infty \\ n \ll e \ll n^2}} \frac{\min\{\mathrm{CR}_g(G) : |V(G)| = n, |E(G)| = e\}}{e^3/n^2}$$

exists and is positive.

M. Simonovits suggested that the lower bound (1) for crossing numbers may be substantially improved, if we restrict our attention to some special classes of graphs, e.g., to graphs not containing some fixed, so-called *forbidden* subgraph. Indeed, this turned out to be true.

A graph property $\mathcal{P}$ is said to be *monotone* if (i) for any graph G satisfying $\mathcal{P}$, every subgraph of G also satisfies $\mathcal{P}$; and (ii) if G_1 and G_2 satisfy $\mathcal{P}$, then their disjoint union also satisfies $\mathcal{P}$.

For any monotone property $\mathcal{P}$, let $\mathrm{ex}(n, \mathcal{P})$ denote the maximum number of edges that a graph of n vertices can have if it satisfies $\mathcal{P}$. In the special case when $\mathcal{P}$ is the property that the graph does not contain a subgraph isomorphic to a fixed forbidden subgraph H, we write $\mathrm{ex}(n, H)$ for $\mathrm{ex}(n, \mathcal{P})$.

Let $\mathcal{P}$ be a monotone graph property with $\mathrm{ex}(n, \mathcal{P}) = O(n^{1+\alpha})$ for some $\alpha > 0$. In [19], we proved that there exist two constants $c, c' > 0$ such that the crossing number of any graph G with property $\mathcal{P}$, which has n vertices and $e \geq cn \log^2 n$ edges, satisfies

$$\mathrm{CR}(G) \geq c' \frac{e^{2+1/\alpha}}{n^{1+1/\alpha}}. \tag{4}$$

This bound is asymptotically tight, up to a constant factor.

In some interesting special cases when we know the precise order of magnitude of the function $\mathrm{ex}(n, \mathcal{P})$, we obtained a slightly stronger result: we proved that (4) is valid for every $e \geq 4n$. For instance, if $\mathcal{P}$ is the property that G does not contain C_4, a cycle of length 4, as a subgraph, then $\mathrm{ex}(n, \mathcal{P}) = \mathrm{ex}(n, C_4) = O(n^{3/2})$, and we know that the crossing number of any graph with n vertices and $e \geq 4n$ edges, which satisfies this property, is at least constant times e^4/n^3. This bound is asymptotically tight.

If the answer to the following question were in the affirmative, we could extend this stronger result to many further graph properties $\mathcal{P}$.

Problem 7. *Let G be a bipartite graph, and let G' be a graph that can be obtained from G by identifying two vertices whose distance is at least three. Is it true that*

$$\mathrm{ex}(n, G) = O\left(\mathrm{ex}(n, G')\right)?$$

4 Three Other Crossing Numbers

We define three variants of the notion of crossing number.

(1) The *rectilinear crossing number*, $\textsc{lin-cr}(G)$, of a graph G is the minimum number of crossings in a drawing of G, in which every edge is represented by a straight-line segment.

(2) The *pairwise crossing number* of G, $\textsc{pair-cr}(G)$, is the minimum number of crossing pairs of edges over all drawings of G. (Here the edges can be represented by arbitrary continuous curves, so that two edges may cross more than once, but every pair of edges can contribute at most one to $\textsc{pair-cr}(G)$.)

(3) The *odd-crossing number* of G, $\textsc{odd-cr}(G)$, is the minimum number of those pairs of edges which cross an odd number of times, over all drawings of G.

It readily follows from the definitions that

$$\textsc{odd-cr}(G) \leq \textsc{pair-cr}(G) \leq \textsc{cr}(G) \leq \textsc{lin-cr}(G).$$

Bienstock and Dean [6] exhibited a series of graphs with crossing number 4, whose rectilinear crossing numbers are arbitrary large. The following is perhaps the most exciting unsolved problem in the area.

Problem 8. *Is it true that*

$$\textsc{odd-cr}(G) = \textsc{pair-cr}(G) = \textsc{cr}(G),$$

for every graph G?

According to a remarkable theorem of Hanani (alias Chojnacki) [7] and Tutte [24], if a graph G can be drawn in the plane so that any pair of its edges cross an even number of times, then it can also be drawn without any crossing. In other words, $\textsc{odd-cr}(G) = 0$ implies that $\textsc{cr}(G) = 0$. Note that in this case, by a theorem of Fáry [9], we also have that $\textsc{lin-cr}(G) = 0$.

The main difficulty in this problem is that a graph has so many essentially different drawings that the computation of any of the above crossing numbers, for a graph of only 15 vertices, appears to be a hopelessly difficult task even for a very fast computer [22].

As we mentioned at the end of the Introduction, Garey and Johnson [10] showed that the determination of the crossing number is an *NP-complete* problem. Analogous results hold for the rectilinear crossing number [5] and for the odd-crossing number [21]. However, for the pairwise crossing number, we could prove only that it is *NP-hard.*

Problem 9. *Given a graph G of n vertices and an integer K, can one check in polynomial time that* $\text{PAIR-CR}(G) \leq K$*? In other words, is the problem of finding the pairwise crossing number of a graph in NP?*

Concerning Problem 8, all we can show is that the parameters $\text{CR}(G)$, $\text{PAIR-CR}(G)$, and $\text{ODD-CR}(G)$, are not completely unrelated. More precisely, we proved in [21] that $\text{CR}(G) \leq 2(\text{ODD-CR}(G))^2$, for every graph G. The next step would be to answer the following question.

Problem 10. *Does there exist a constant C such that*

$$\text{CR}(G) \leq C \cdot \text{ODD-CR}(G)$$

holds for every graph G?

5 Crossing Numbers of Random Graphs

Let $G = G(n,p)$ be a *random* graph with n vertices, whose edges are chosen independently with probability $p = p(n)$. Let e denote the *expected number* of edges of G, i.e., $e = p\binom{n}{2}$. It is not hard to see that if $e > 10n$, almost surely $b(G) \geq e/10$. Therefore, it follows from (3) that almost surely we have

$$\text{CR}(G) \geq \frac{e^2}{4000}.$$

Evidently, the order of magnitude of this bound cannot be improved. We do not have a formula analogous to (3) for the other two crossing numbers.

Problem 11. *Do there exist suitable constants C_1, $C_2 > 0$ such that every graph G satisfies*

$$\text{(i)} \qquad b(G) = C_1\left(\sqrt{\text{PAIR-CR}(G)} + \sqrt{\sum_{i=1}^{n} d_i^2}\right),$$

$$\text{(ii)} \qquad b(G) = C_2\left(\sqrt{\text{ODD-CR}(G)} + \sqrt{\sum_{i=1}^{n} d_i^2}\right)\ ?$$

We cannot determine the right order of magnitude of the *expected value* of ODD-CR(G) and PAIR-CR(G) for a random graph $G = G(n,p)$.

Problem 12. *Let $G = G(n,p)$ be a random graph with n vertices, with edge probability $0 < p < 1$, and let $e = p\binom{n}{2} > 4n$. Do there exist suitable positive constants c_1 and c_2 such that*

$$\text{(i)} \qquad \mathrm{E}\left[\text{PAIR-CR}(G)\right] \geq c_1 e^2,$$

$$\text{(ii)} \qquad \mathrm{E}\left[\text{ODD-CR}(G)\right] \geq c_2 e^2\ ?$$

Although we are far from knowing the expectations of crossing numbers of random graphs, it is not hard to argue that the crossing numbers are sharply concentrated in very short intervals around these values.

To show this, we need a simple observation.

Lemma. *Let G be a graph with edge set $E = E(G)$, and let G' be another graph obtained from G by adding an edge. Then*

$$\text{(i)} \qquad \text{CR}(G') \leq \text{CR}(G) + |E|,$$

$$\text{(ii)} \qquad \text{PAIR-CR}(G') \leq \text{PAIR-CR}(G) + |E|,$$

(iii) $$\text{ODD-CR}(G') \le \text{ODD-CR}(G) + |E|.$$

Proof: Parts (ii) and (iii) are obviously true, because we can arbitrarily add to any optimal drawing of G an arc representing the new edge.

To prove part (i), fix a drawing of G, which minimizes the number of crossings. It is easy to see that in such a drawing any two edges have at most one point in common [22]. Add a continuous arc a representing the new edge so as to minimize the number of crossings in the resulting drawing of G'.

In this new drawing, a cannot have two points, p and q, in common with any arc b representing an edge of $E = E(G)$. Otherwise, we could replace the piece of a between p and q by an arc running very close to the piece of b between p and q. By the minimality of the initial drawing of G, this replacement would strictly decrease the number of crossings, because at least one crossing between a and b would be eliminated. This would contradict the minimality of the drawing of G'. $\Box$

Theorem. *Let $G(n,p)$ be a random graph with n vertices, with edge probability $0 < p = p(n) < 1$, and let $e = p\binom{n}{2}$. Then*

$$\Pr\left[|\text{CR}(G) - \text{E}[\text{CR}(G)]| > 3\alpha e^{3/2}\right] < 3\exp(-\alpha^2/4)$$

holds for every α satisfying $(e/4)^3 \exp(-e/4) \le \alpha \le \sqrt{e}$.

The same result holds for PAIR-CR(G) *and* ODD-CR(G).

Proof: Let $e_1, e_2, \dots, e_{\binom{n}{2}}$ be the edges of the complete graph on $V(G)$. Define another random graph G^* on the same vertex set, as follows. If G has at most $2e$ edges, let $G^* = G$. Otherwise, there is an $i < \binom{n}{2}$ so that $|\{e_1, e_2, \dots, e_i\} \cap E(G)| = 2e$, and set $E(G^*) = \{e_1, e_2, \dots, e_i\} \cap E(G)$. Finally, let $f(G) = \text{CR}(G^*)$.

According to the Lemma, the addition of any edge to G can modify the value of f by at most $2e$. Following the terminology of Alon–Kim–Spencer [2], we say that the *effect* of every edge is at most $2e$. The *variance* of any edge is defined as $p(1-p)$ times the square of its effect. Therefore, the *total variance* cannot exceed

$$\sigma^2 = \binom{n}{2} p(2e)^2 = 4e^3.$$

Applying the Martingale Inequality of [2], which is a variant of Azuma's Inequality [4] (see also [3]), we obtain that for any positive $\alpha \le \sigma/e = 2\sqrt{e}$,

$$\Pr\left[|f(G) - \mathrm{E}[f(G)]| > \alpha\sigma = 2\alpha e^{3/2}\right] < 2\exp(-\alpha^2/4).$$

Our goal is to establish a similar bound for $\textsc{cr}(G)$ in place of $f(G)$. Obviously,

$$\Pr\left[f(G) \ne \textsc{cr}(G)\right] \le \Pr\left[G \ne G^*\right] < \exp(-e/4).$$

Thus, we have

$$|\mathrm{E}\left[f(G)\right] - \mathrm{E}\left[\textsc{cr}(G)\right]| \le \Pr\left[f(G) \ne \textsc{cr}(G)\right] \max \textsc{cr}(G) \le$$

$$\exp\left(-e/4\right)\frac{n^4}{8} \le \alpha e^{3/2},$$

whenever $\alpha \ge (e/4)^3 \exp(-e/4)$ (say). Therefore,

$$\Pr\left[|\textsc{cr}(G) - \mathrm{E}[\textsc{cr}(G)]| > 3\alpha e^{3/2}\right] \le$$

$$\Pr\left[\textsc{cr}(G) \ne f(G)\right] + \Pr\left[|f(G) - \mathrm{E}[f(G)]| > 2\alpha e^{3/2}\right] \le$$

$$\exp(-e/4) + 2\exp(-\alpha^2/4).$$

If $\alpha \le \sqrt{e}$, the last some is at most $3\exp(-\alpha^2/4)$, as required.

The same argument works for $\textsc{pair-cr}(G)$ and $\textsc{odd-cr}(G)$ in place of $\textsc{cr}(G)$. $\Box$

6 Even More Crossing Numbers

We can further modify each of the above crossing numbers, by applying one of the following rules:

Rule $+$ **:** Consider only those drawings where two edges with a common endpoint do not cross each other.

Rule 0 **:** Two edges with a common endpoint are allowed to cross and their crossing counts.

Rule $-$: Two edges with a common endpoint are allowed to cross, but their crossing does not count.

In the previous definitions we have always used Rule 0. If we apply Rule $+$ (Rule $-$) in the definition of the crossing numbers, then we indicate it by using the corresponding subscript, as shown in the table below. This gives us an array of nine different crossing numbers. It is easy to see that in a drawing of a graph, which minimizes the number of crossing points, any two edges have at most one point in common (see e.g. [22]). Therefore, $\textsc{cr}_+(G) = \textsc{cr}(G)$, which slightly simplifies the picture.

Rule $+$	$\textsc{odd-cr}_+(G)$	$\textsc{pair-cr}_+(G)$	$\textsc{cr}(G)$
Rule 0	$\textsc{odd-cr}(G)$	$\textsc{pair-cr}(G)$	
Rule $-$	$\textsc{odd-cr}_-(G)$	$\textsc{pair-cr}_-(G)$	$\textsc{cr}_-(G)$

Moving from left to right or from bottom to top in this array, the numbers do not decrease. It is not hard to generalize (1) to each of these crossing numbers. We obtain (as in in [21]) that

$$\textsc{odd-cr}_-(G) \geq \frac{1}{64}\frac{e^3}{n^2},$$

for any graph G with n vertices and with $e \geq 4n$ edges. We cannot prove anything else about $\textsc{odd-cr}_-(G)$, $\textsc{pair-cr}_-(G)$, and $\textsc{cr}_-(G)$. We conjecture that these values are very close to $\textsc{cr}(G)$, if not the same. That is, we believe that by letting pairs of *incident* edges cross an arbitrary number of times, we cannot effectively reduce the total number of crossings between *independent* pairs of edges. The weakest open questions are the following.

Problem 13. *Do there exist suitable functions f_1, f_2, f_3 such that every graph G satisfies*

(i) $$\textsc{odd-cr}(G) \leq f_1(\textsc{odd-cr}_-(G)),$$

(ii) $$\textsc{pair-cr}(G) \leq f_2(\textsc{pair-cr}_-(G)),$$

(iii) $$\textsc{cr}(G) \leq f_3(\textsc{cr}_-(G))\ ?$$

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