Geometric graphs with no self-intersecting path of length three*

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Abstract

Let G be a *geometric graph* with n vertices, i.e., a graph drawn in the plane with straight-line edges. It is shown that if G has no self-intersecting path of length 3, then its number of edges is $O(n \log n)$. This result is asymptotically tight. Analogous questions for longer forbidden paths and for graphs drawn by not necessarily straight-line edges are also considered.

1 Introduction

A *geometric graph* is a graph drawn in the plane so that its vertices are points and its edges are possibly crossing straight-line segments. We assume, for simplicity, that the points are in *general position*, i.e., no three points are on a line and no three edges pass through the same point. *Topological graphs* are defined similarly, except that now the edges are not necessarily rectilinear; every edge can be represented by an arbitrary continuous arc which does not pass through any vertex different from its endpoints. Throughout this paper, we also assume that any two edges have a finite number of common interior points and that they properly cross at each of them. Clearly, every geometric graph is also a topological graph.

Using this terminology, the fact that every planar graph with n vertices has at most $3n-6$ edges can be rephrased as follows: any topological graph with n vertices and more than $3n-6$ edges must have two edges that cross each other. This result is tight even for geometric graphs.

In the mid-sixties Avital and Hanani [AH66], Erdős, and Perles initiated, later Kupitz [K79] and many others continued the systematic study of extremal problems for geometric

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graphs. In particular, they proposed the following general question. Let H be a so-called *forbidden* geometric configuration or a *class* of forbidden configurations. *What is the maximum number of edges that a geometric graph with n vertices can have without containing any forbidden subconfiguration?* If H consists of $k = 2$ (pairwise) crossing edges, then, according to the previous paragraph, the answer is $3n - 6$. For $k = 3$, this maximum is linear in n (see [AAPPS97]), but for larger values of k the best known bound due to Valtr is only $O(n \log n)$ [V98]. It is an exciting open problem to decide whether one can get rid of the logarithmic factor here. If H is the class of all configurations consisting of $k + 1$ edges, one of which crosses all the others, then the maximum number of edges is equal to $(k + 2)(n - 2)$, provided that $k = 1, 2, 3$, and the maximum is $O(\sqrt{k}n)$ for large values of k (cf. [PT97]). For surveys on Geometric Graph Theory, consult [P99], [P03], and [PRT03].

The above questions can also be regarded as geometric analogues of the fundamental problem of Extremal Graph Theory [B78]: determine the maximum number of edges of all K-*free* graphs on n vertices, i.e., all graphs which do not contain a subgraph isomorphic to a fixed graph K. Denote this maximum by $\mathrm{ex}(n, K)$.

In the present note, we consider the special instance of the above question when H consists of all *self-intersecting* straight-line drawings of a fixed graph K. In other words, what is the maximum number $\mathrm{ex}_{\mathrm{cr}}(n, K)$ of edges that a geometric graph with n vertices can have, if it contains no self-intersecting copy of K? Obviously, we have $\mathrm{ex}_{\mathrm{cr}}(n, K) \geq \mathrm{ex}(n, K)$, because if a graph contains no copy of K, then it cannot contain a self-intersecting copy either. Therefore, if K is not a bipartite graph, then $\mathrm{ex}_{\mathrm{cr}}(n, K)$ is quadratic in n. On the other hand, if K is not planar then $\mathrm{ex}_{\mathrm{cr}}(n, K) = \mathrm{ex}(n, K)$, since if a graph contains a copy of K, then it is a self-intersecting copy. The question is more exciting for bipartite planar graphs. What happens if $K = P_k$ (or $K = C_k$), a path (or a cycle) of (an even) length k? The case where $K = C_4$ is discussed in [PR03].

We analyze the case when $K = P_3$. The corresponding graph property is a relaxation of planarity: the geometric graphs satisfying the condition are allowed to have two crossing edges, but if this is the case, no endpoint of one of these edges can be joined to an endpoint of the other. Is it still true that the number of edges of such geometric graphs is $O(n)$? The following theorem provides a negative answer to this question.

Theorem 1. *The maximum number of edges of a geometric graph with n vertices, containing no self-intersecting path of length 3, satisfies*

$$\mathrm{ex}_{\mathrm{cr}}(n, P_3) \leq cn \log n,$$

for a suitable constant c. Apart from the value of the constant, this bound cannot be improved.

The proof of this result (presented in three different versions in the next three sections) applies to a slightly more general situation. Theorem 1 remains true for *topological* graphs whose edges are continuous functions defined on subintervals of the x-axis, i.e., every line perpendicular to the x-axis intersects each edge in at most one point. The topological graphs satisfying this condition are usually called x-*monotone.*

On the other hand, a construction in Section 3 shows that Theorem 1 cannot be improved even for geometric graphs all of whose edges are crossed by a straight line.

What happens if we drop the requirement of x-monotonicity? We do not have any example of a topological graph with n vertices and more than constant times $n \log n$ edges, in which every path of length 3 is *simple*, i.e., non-self-intersecting. The best *upper* bound we have is the following.

Theorem 2. *The maximum number of edges of a topological graph with n vertices, containing no self-intersecting path of length 3, is $O(n^{3/2})$.*

As was pointed out by Tutte [T70], *parity* plays an important role in determining the possible crossing patterns between the edges of a topological graph. This may well be a consequence of the Jordan Curve Theorem: every Jordan arc connecting an interior point and an exterior point of a simple closed Jordan curve must cross this curve an *odd* number of times. In particular, Tutte showed that every topological graph with n vertices and more than $3n-6$ edges has two edges that not only cross each other, but (i) they cross an *odd* number of times, and (ii) they do not share an endpoint. (See also [H34].)

This may suggest that Theorem 2 and perhaps any other bound of this type can be sharpened as follows.

Theorem 3. *The maximum number of edges of a topological graph with n vertices, containing no path of length 3 whose first and last edges cross an odd number of times, is $O(n^{3/2})$.*

In Section 5 we prove this stronger statement. Somewhat surprisingly (to the authors), it turns out that this last result is asymptotically tight. More precisely, in Section 6 we establish

Theorem 4. *Let G be a bipartite graph on n vertices, containing no cycle of length 4. Then G can be drawn in the plane as an x-monotone topological graph with the property that any two edges belonging to a path of length 3 cross an even number of times.*

It is well known that there are C_4-free bipartite graphs of n vertices and at least constant times $n^{3/2}$ edges (see e.g. [B78]).

In Section 7, we consider geometric and x-monotone topological graphs with no self-intersecting path of length *five*. In this case, Theorem 9 provides a slightly stronger upper bound on the number of edges than those obtained for graphs with no self-intersecting P_3. We do not believe that Theorem 9 is tight. However, a recent construction of Tardos [T03] shows that $\mathrm{ex}_{\mathrm{cr}}(n, P_k)$ is superlinear in n, for any fixed value $k \geq 3$.

In the final section, we discuss a few related results and open problems.

2 A Davenport-Schinzel bound for double arrays

In this section, we discuss the special case of Theorem 1 when G is a *bipartite* geometric (or x-monotone topological) graph, whose vertices are divided by the y-axis into two classes, A and B, and all edges of G run between these classes. We assume, for simplicity, that no two edges of G cross the y-axis at the same point.

Let $a_1b_1, a_2b_2, \ldots, a_mb_m$ be the edges of G listed from top to bottom, in the order of their intersections with the y-axis, where $a_i \in A$ and $b_i \in B$ for every i. Consider the corresponding *double array* ($2 \times m$ matrix)

$$M = \begin{pmatrix} a_1 & a_2 & \dots & a_m \\ b_1 & b_2 & \dots & b_m \end{pmatrix}$$

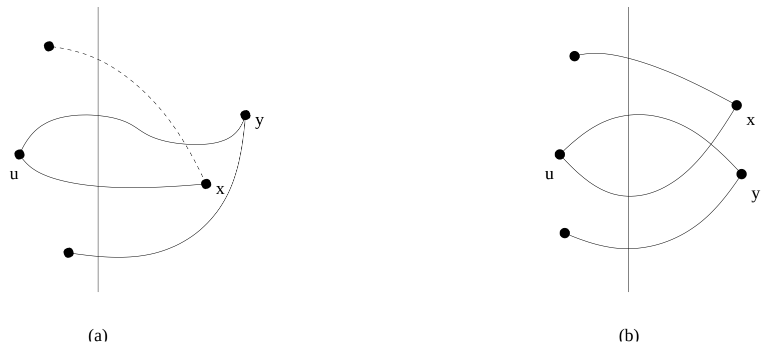

Fig. 1. *(a) F_2 is forbidden, (b) not necessarily forbidden if adjacent edges may cross*

It is easy to verify that if G is a geometric graph (or an x-monotone topological graph) without any self-intersecting path of length three, then the corresponding matrix M does not contain any submatrix of the form $F_1 = \begin{pmatrix} u & v & u & v \\ * & x & x & * \end{pmatrix}$ or $F_2 = \begin{pmatrix} * & u & u & * \\ x & y & x & y \end{pmatrix}$, where $u \neq v$, $x \neq y$ and $*$ stands for an unspecified entry (see Fig. 1(a)).

In what follows, we show that if a $2 \times m$ matrix M having at most n distinct entries does not contain any forbidden submatrix of the above two types, then its number of columns is $O(n \log n)$. Therefore, the number of edges of G is at most $O(n \log n)$, as required by Theorem 1.

If G is an x-monotone topological graph whose adjacent edges are allowed to cross, and we only require that the first and last edges of every path of length three must be disjoint, then the situation is slightly more complicated, because M may contain submatrices of the above forms (see Fig. 1(b)). However, in this case the following 2×6 submatrices are forbidden:

$$\begin{pmatrix} v & \leftrightarrow & u & v & u & v & \leftrightarrow & u \\ * & & * & x & x & * & & * \end{pmatrix} \tag{1}$$

and

$$\begin{pmatrix} * & & * & u & u & * & & * \\ y & \leftrightarrow & x & y & x & y & \leftrightarrow & x \end{pmatrix}. \tag{2}$$

Here the signs $\leftrightarrow$ indicate that the order of the first two columns and the order of the last two columns are not specified.

Theorem 5. *Let M be a $2 \times m$ matrix with at most n distinct entries, all of whose columns are different. If M has no 2×6 submatrix of types (1) or (2), then $m \leq 17n \log_2 n$.*

It follows from the construction at the end of Section 3, that the bound in Theorem 5 is tight apart from the value of the constant. In fact, for any n there exist a $2 \times m$ matrix with at most n distinct entries having neither F_1 nor F_2 as a submatrix with $m \geq n \log_2 n/4$.

Proof. We need some definitions. Let

$$M = \begin{pmatrix} a_1 & a_2 & \dots & a_m \\ b_1 & b_2 & \dots & b_m \end{pmatrix}$$

For any $1 \le i \le m$, we say that a_i is a *leftmost* (or *rightmost*) entry if $a_k \neq a_i$ for every $k < i$ (or $k > i$, resp.). Accordingly, a_i is called a *second leftmost* (or *second rightmost*) entry if $a_k = a_i$ for precisely one index $k < i$ (or precisely one index $k > i$, resp.). Analogous terms are used for the entries b_i in the second row of M.

A set of consecutive columns of M is called a *block*. A block is said to be *pure* if all elements in the first row of the block are distinct and the same is true for the elements in the second row.

Assume the columns of M are partitioned into l pure blocks. Consider now two consecutive pure blocks, B_1 and B_2, consisting of the columns $i+1, i+2, \dots, j$ and $j+1, j+2, \dots, k$, resp., for some $0 \le i < j < k \le n$. Suppose that there is an element which appears in the first row of B_1 as well as in the first row of B_2. That is, $a_p = a_q$ for some $i < p \le j$ and $j < q \le k$. We claim that either b_q is a leftmost, second leftmost or rightmost entry, or b_p is a rightmost, second rightmost or leftmost entry. Indeed, otherwise, using the fact that b_q is neither a leftmost nor a second leftmost entry, we obtain that there exists an index $r \le i$ such that $b_r = b_q$. Since b_q is not a rightmost entry, there is an index $s > k$ such that $b_s = b_q$. Similarly, in view of the fact that b_p is neither a rightmost nor a second rightmost entry, we can conclude that $b_{s'} = b_p$ for some $s' > k$. Since b_p is not leftmost, there is a $r' \le i$ such that $b_{r'} = b_p$. Observe that now the columns $r, r' < p < q < s, s'$ form a forbidden submatrix of type

$$\begin{pmatrix} * & & * & u & u & * & & * \\ y & \leftrightarrow & x & y & x & y & \leftrightarrow & x \end{pmatrix},$$

a contradiction.

A symmetric argument shows that if $b_p = b_q$ for some $i < p \le j$ and $j < q \le k$, then either a_q is a leftmost, second leftmost or rightmost entry, or a_p is a rightmost, second rightmost or leftmost entry. Thus, if we delete from M (and from its block decomposition) every column whose upper or lower element is a leftmost, second leftmost, rightmost, or second rightmost entry, the union of the remainders of any two consecutive blocks becomes pure.

There are at most n distinct entries, each may appear in the first row and in the second row, so the number of deleted columns is at most $8n$. The resulting matrix M' can be decomposed into $\lceil l/2 \rceil$ pure blocks. Repeating this process at most $\lceil \log_2 l \rceil$ times, we end up with a matrix consisting of at least $m - 8n\lceil \log_2 l \rceil$ columns that form a single pure block. Thus, we have

$$m - 8n\lceil \log_2 l \rceil \le n.$$

Applying the above procedure to the initial partition of M into $l = m$ pure blocks, each consisting of a single column, the upper bound follows. ■

For many other extremal result on excluded submatrices (in somewhat different settings) consult [FH92], [AGS97], and [AFS01].

As we have pointed out before, the last theorem implies that every geometric or x-monotone topological graph with n vertices and no path of length three whose first and last edges cross each other, has at most constant times $n \log n$ edges, provided that all of its edges can be stabbed by a vertical line. Thus, we immediately obtain

Corollary 1. *The maximum number of edges of an x-monotone topological graph with n vertices, containing no path of length 3 whose first and last edges cross, is $O(n \log^2 n)$.*

This bound is slightly weaker than the bound in Theorem 1.

3 Proof of Theorem 1

First, we prove the following more general statement.

Theorem 6. *Let G be an x-monotone topological graph of n vertices, which has no self-intersecting path of length 3. Then G has at most constant times $n \log n$ edges.*

Proof. Assume without loss of generality that no two edges that share an endpoint cross each other. Otherwise, the two non-common endpoints of these edges must be of degree 1 or 2, because G has no self-intersecting path of length 3. So we can delete these endpoints, and complete the argument by induction on the number of vertices.

It will be convenient to use the following terminology. If a vertex v is the left (resp. right) endpoint of an edge e, then e is said to be a *right* (resp. *left*) edge at v. It follows from our assumption on adjacent edges that the left and the right edges at a given vertex can be ordered from bottom to top.

Let $e_1 = vu_1$ and $e_2 = vu_2$ be two right edges at a vertex v such that the x-coordinate of u_1 is at most as large as the x-coordinate of u_2. We define the *right triangle* determined by them as the bounded *closed* region bounded by e_1, a segment of e_2,, and a segment of the vertical line passing through u_1. The vertex v is called the *apex* of this triangle. Analogously, we can introduce the notion of *left triangle.*

Construct a sequence of subgraphs $G_0, G_1, G_2, \ldots$ of G, as follows. Let $G_0 = G$. If G_i has already been defined for some i, then let G_{i+1} be the topological graph obtained from G_i by deleting at each vertex the bottom 2 and the top 2 left and right edges (if they exist). We delete at most 8 edges per vertex.

Claim. *For any $k \geq 0$, every triangle determined by two edges of G_k contains at least 2^k pairwise different triangles of G.*

Proof. By induction on k. Obviously, for $k = 0$, the Claim is true, because every triangle contains itself. Assume that the Claim holds for $k - 1$ ($k > 0$). Consider, e.g., a *right* triangle T in G_k, determined by the edges $e_1 = vu_1$ and $e_2 = vu_2$, where the x-coordinate of u_1 is at most as large as the x-coordinate of u_2. Suppose without loss of generality that e_1 lies below e_2. Using the fact that $e_1 \in E(G_k)$, we obtain that at u_1 there are at least two left edges $f_1, f_2 \in E(G_{k-1})$ which lie above e_1. Both of these edges must be entirely contained in T, otherwise we could find a self-intersecting path of length 3. Suppose that f_1 lies below f_2.

Let T_1 and T_2 denote the left triangles with apex u_1, determined by e_1 and f_1, and by f_1 and f_2, resp. Clearly, T_1 and T_2 both belong to G_{k-1}, and they have disjoint interiors.

By the induction hypothesis, both T_1 and T_2 contain 2^{k-1} pairwise different triangles. It follows that T contains 2^k pairwise different triangles, as required. ■

Now we can easily complete the proof of Theorem 6. Since every triangle is specified by a pair of edges meeting at its apex, the total number of different triangles is at most n^3. Hence, for $k > 3\log_2 n$, the graph G_k cannot determine any triangle, and its number of edges is smaller than n. On the other hand, we have that $|E(G_k)| \geq |E(G_0)| - 8kn$. Therefore, $|E(G)| = |E(G_0)| \leq 25n\log_2 n$, completing the proof of Theorem 6. ■

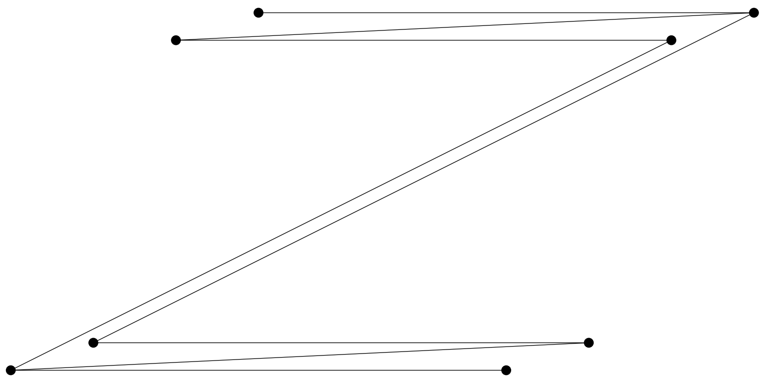

Fig. 2. *The construction of* G_i $(i = 3)$

We close this section by showing that, up to the value of the constant c, Theorem 1 (and hence Theorem 6, too) is best possible. Let $n = 2^k$ be fixed. We will recursively construct a sequence of bipartite geometric graphs $G_i = G_i^{(k)}$, $i = 1, 2, \ldots, k$, such that G_i has 2^i vertices, $(i+1)2^{i-2}$ edges, and contains no self-intersecting path of length 3. Furthermore, we will maintain the following properties for every i.

1. The vertices of G_i have distinct x-coordinates, which are all integers in the closed intervals $[-2^k, -2^k + 2^i - 1]$ and $[0, 2^i - 1]$. Vertices with x-coordinates in the first (resp. second) interval are called *left* (resp. *right*).

2. Every edge of G_i connects a left vertex to a right vertex, and hence it must cross the vertical line $(x = -\frac{1}{2})$.

3. The horizontal edges of G_i are of length 2^k and form a perfect matching. If two vertices of $u, v \in V(G_i)$, are connected by a horizontal edge, than they are said to form a *pair*.

4. For any vertex v of G_i, the order of the edges incident to v according to their slopes coincides with the order according to the lengths of their projections to the x-axis.

Let G_1 consist of two vertices, $(-2^k, 0)$ and $(0, 0)$, connected by an edge. Obviously, this meets the requirements.

Assuming that we have already constructed G_i for some i, we show how to obtain G_{i+1}. Let G'_i denote the translate of G_i by a vector $(2^{i-1}, Y_i)$, where Y_i is a very large positive integer to be specified later. Let G_{i+1} be the union of G_i and G'_i, together with the following 2^{i-1} "new" edges: connect every left vertex $v \in V(G_i)$ to the right vertex $v + (2^k + 2^{i-1}, Y_i) \in V(G'_i)$, that is, to the right vertex forming a pair with the translate of v. See Fig. 2.

Choose Y_i so large that the slope of the new edges exceeds the slope of any line induced by the points of G_i (or by the points of G'_i).

We have to check that G_{i+1} has the required properties. We have $|V(G_{i+1})| = 2|V(G)| = 2^{i+1}$ and $|E(G_{i+1})| = 2|E(G_i)| + 2^{i-1} = (i+2)2^{i-1}$. Properties 1, 2, 3 and 4 are all inherited from G_i. To see that property 4 is maintained, it is sufficient to recall that both the slope and length of the x-projection of every new edge between G_i and G'_i is larger than the corresponding values for the old edges.

It remains to verify that G_{i+1} does not contain a self-intersecting path of length 3. Assume to the contrary that there is such a path P in G_{i+1}, and denote its edges by $e_1 = uv$, $e_2 = vw$, and $e_3 = wz$. Since G_i (and thus G'_i) does not contain a self-intersecting path of length 3, at least one of these edges must run between G_i and G'_i. Note that there cannot be two such edges, because all edges of G_{i+1} running between G_i and G'_i are parallel. It is also clear that e_2 is not such an edge.

Assume, without loss of generality, that e_1 runs between G_i and G'_i, and that we have $u \in V(G_i)$ and $v \in V(G'_i)$. Thus, e_2 and e_3 belong to G'_i. As v is a right vertex, w must be a left vertex, and both e_2 and e_3 are to the right of w. Since e_3 crosses e_1, the slope of e_3 must be smaller than that of e_2. In view of property 4, we conclude that the x-coordinate of z is smaller than the x-coordinate of v. This implies that the slope of the line connecting z and v is larger than the slope of e_2, contradicting our assumption.

4 A strengthening of Theorem 6

The aim of this section is to establish the following stronger form of Theorem 6.

Theorem 7. *The maximum number of edges of an x-monotone topological graph with n vertices, containing no path of length 3 whose first and last edges cross, is $O(n \log n)$.*

Proof. Let G be an x-monotone topological graph with n vertices and m edges, containing no path of length 3 whose first and last edges cross. Our goal is to construct another topological graph G' with $n' = 2n$ vertices and $m' \geq m/2 - n$ edges, with the property that G' has no path of length 3 whose first and last edges cross, and no two adjacent edges of G' cross each other. Applying Theorem 6, the statement follows.

First, we split each vertex of G into into two vertices, one of them just a bit left to the other, so that every original edge e becomes an edge connecting the right copy of the left endpoint of e to the left copy of its right endpoint. The resulting x-monotone topological graph G_0 has $n' = 2n$ vertices and m edges, it has no self-intersecting path of length three whose first and last edges cross, and the right endpoint of any edge of G_0 is distinct from the left endpoint of any other edge.

In the rest of this section, the *length* of an edge means the length of its projection to the x-axis, and the terms *shorter* and *longer* will be used in the same sense. We write $e = uv$

for an edge of G_0, whose left and right endpoints are u and v, resp. We call an edge $e = uv$ *long* if it is the longest either among all edges $uv' \in E(G_0)$ or among all edges $u'v \in E(G_0)$. Clearly, G_0 has fewer than n' long edges. Let e and e' be two edges of G_0, where e is shorter than e', and either we have $e = uv$ and $e' = uw$, or we have $e = vu$ and $e' = wu$. We say that e is *higher than* e' if v is above e'. Similarly, we say e is *lower than* e' if v is below e'. Note that the relations "higher than" and "lower than" are not partial orders, and they are not inverse to each other. Also note that if e is higher or lower than e' then e is shorter, but e and e' may cross several times.

Let $e = uv$ be an edge of G_0 which is not long. By definition, there exist two edges, $e' = uw$ and $e'' = zv \in E(G_0)$, such that both of them are longer than e. So e is either higher or lower than e' and e is also higher or lower than e''. However, e cannot be higher than both e' and e''. Indeed, otherwise u is above e'' while v is above e', so e' and e'' cross, contradicting our assumption on G. Similarly, e cannot be lower than both e' and e''. Thus, each edge $e = uv \in E(G_0)$ which is not long either satisfies that e is higher than every longer edge uw and lower than every longer edge zv, or it satisfies that e is lower than every longer edge uw and higher than every longer edge zv. We can assume, by symmetry, that the former condition (which will be referred to as the *monotonicity condition*) holds for $m' \geq (m-n')/2 = m/2 - n$ edges. Let G_1 be the subgraph of G_0 formed by these edges.

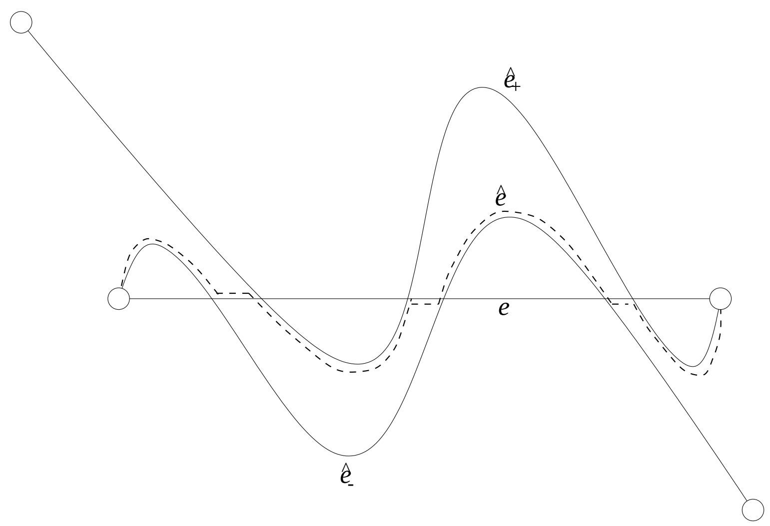

Fig. 3. *The construction of the edge $\widehat{e}$ in G'*

We are now in a position to define the x-monotone topological graph G'. As an abstract graph, G' is identical to G_1. The locations of the vertices will coincide, too. For any edge $e \in E(G_1)$, denote by $\widehat{e}$ the corresponding edge of G'. We draw the edges of G' one by one, in decreasing order of length. If e in G_1 is neither higher nor lower than any other edge, set $\widehat{e} = e$. If $e = uv$ is higher (lower) than at least one other edge, let e_- be the shortest edge such that e is higher than e_- (resp. let e_+ be the shortest edge such that e is lower than e_+). Draw $\widehat{e}$ in such a way that all of its internal points lie strictly above $\widehat{e}_-$ and below $\widehat{e}_+$ (if these edges exist). Notice that, if they exist, e_+ and e_- are longer than e, so $\widehat{e}_+$ and $\widehat{e}_-$

are already defined. We make sure during the construction that, if e_+ exists, it passes above u, if e_- exists, it passes below v (see property 2 below), and if both of them exist, they are disjoint (see property 4 below). We define $\widehat{e}$ to follow e, except in the intervals where $\widehat{e}_+$ is below e or $\widehat{e}_-$ is above e. In these intervals, let $\widehat{e}$ run just below $\widehat{e}_+$ or just above $\widehat{e}_-$, close enough not to intersect any further edges and going on the same side of every vertex. See Fig. 3.

We claim that the resulting graph G' has the following four properties.

1. If e is lower (higher) than e' in G_1, then every interior point of $\widehat{e}$ is below (resp. above) $\widehat{e}'$.
2. If e' is lower (higher) than e in G_1, then the endpoint of e' which is not an endpoint of e is below (resp. above) $\widehat{e}$.
3. If e, e', and e'' form a path in G_1 and e is longer than e', then $\widehat{e}$ and e'' do not cross.
4. If e, e', and e'' form a path in G_1 then $\widehat{e}$ and $\widehat{e''}$ do not cross.

We verify these properties by showing that if they hold for the partially drawn graph, they do not get violated when we add an extra edge $\widehat{e}$.

(1) By the monotonicity, if there exists at least one edge f such that e is lower than f, then the shortest among them, e_+, must be lower than all others. Similarly, e_- (if exists) must be higher than all other edges f with e higher than f. Therefore, as property 1 has been satisfied so far, it does not get violated now, provided that $\widehat{e}$ is in between $\widehat{e}_-$ and $\widehat{e}_+$, which is the case.

(2) Let $e = uv$ and assume that $e' = uw$ is above e. By definition, w is above e and, by the monotonicity condition, w is above e_-, if the latter exists. As property 2 has been satisfied so far, w is above $\widehat{e}_-$, so w must be above $\widehat{e}$. Similarly, if $e' = zv$ is below e, then z is below $\widehat{e}$.

(3) Note that e' is higher or lower than e. By symmetry, we can assume that e' is lower than e. By monotonicity, this means that they share their right endpoints. Here e and e'' do not cross, as they are first and last edges of a path of length 3, and the left endpoint of e'' is below e. So every point of e'' must be below e or to the right of the right endpoint of e. If e_+ exists, we can apply property 3 to the edges e_+, e', e'', and find that $\widehat{e}_+$ does not cross e''. By the construction, wherever $\widehat{e}$ runs below e, it follows $\widehat{e}_+$, so $\widehat{e}$ is disjoint from e''.

(4) We consider two cases.

If both e and e'' are shorter than e', then one of them is lower and the other one is higher than e' (by the monotonicity). Thus, by property 1, $\widehat{e'}$ (drawn before the other two) separates $\widehat{e}$ from $\widehat{e''}$, so they cannot cross.

We may assume that e is shorter than e'', so in the remaining case e'' is longer than e'. The edge e' is lower or higher than e'', and we can again assume, by symmetry, that e' is lower than e''. Applying property 3 to the path formed by e'', e', and e, we find that e is disjoint from $\widehat{e''}$. By property 2, the left endpoint of e lies below $\widehat{e''}$. Thus, all points of e must be below $\widehat{e''}$ or to the right of its right endpoint. As $\widehat{e}$ follows $\widehat{e}_-$ wherever it runs above e, it is enough to show that if e_- exists, $\widehat{e}_-$ is disjoint from $\widehat{e''}$. If $e_- = e'$, this follows from property 1, otherwise, from property 4 of the initial configuration (before $\widehat{e}$ has been drawn).

Observe that, by property 1, no two adjacent edges of G' cross each other and, by property 4, the same is true for second neighbors. Hence, we can indeed apply Theorem 6 to G', and Theorem 7 follows. ■

5 Forbidden subgraphs – Proof of Theorem 3

For any $k \geq 2$, let F_k denote a graph with vertex set

$$V(F_k) = \{x, y\} \cup \{b_i : 1 \leq i \leq k\} \cup \{c_{ij} : 1 \leq i < j \leq k\}$$

and edge set

$$E(F_k) = \{xb_i, yb_i : 1 \leq i \leq k\} \cup \{c_{ij}b_i, c_{ij}b_j : 1 \leq i < j \leq k\}.$$

We need the following theorem, which can be obtained by a straightforward generalization of a result of Füredi [F91].

Theorem 8. *For any fixed integer $k \geq 2$, let $\mathrm{ex}(n, F_k)$ denote the maximum number of edges of an F_k-free graph with n vertices. Then we have $\mathrm{ex}(n, F_k) = O(n^{3/2})$.* ■

Let G be a topological graph with n vertices, containing no path of length 3 whose first and last edges cross an *odd* number of times. To establish Theorem 3, it is sufficient to verify that the *abstract* graph obtained from G by disregarding how the edges are drawn does not have a subgraph isomorphic to F_4. In fact, it is enough to concentrate to a the subgraph F_4' of F_4 induceed by the vertex set $\{x, y\} \cup \{b_i : 1 \leq i \leq 4\} \cup \{c_{ij} : 1 \leq i < j \leq 3\}$. Notice that F_4' is a *subdivision* of K_5: it can be obtained from K_5 by replacing four of its edges (a triangle and an edge not incident to the triangle) by paths of length two. This means that a topological graph isomorphic to F_4' can be also considered as a topological graph isomorphic to K_5 (simply remove the subdividing points). As K_5 is not a planar graph, any topological graph isomorphic to it must have at least one crossing. Furthermore, by Tutte's theorem [T70], there must exist two non-adjacent edges that cross an *odd* number of times. Thus, any topological graph isomorphic to F_4' has two edges that cross an odd number of times and they are either non-adjacent edges of the underlying K_5 or portions of two such edges. However, any two edges with this property can be extended to a self-intersecting path of length 3. Consequently, F_4' is not isomorphic to a subgraph of G, and Theorem 3 follows.

6 Drawing C_4-free graphs – Proof of Theorem 4

Let G be a C_4-free bipartite graph with vertex set $V(G) = A \cup B$, where $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. The edge set of G is denoted by $E(G)$.

We now construct a drawing of G. Pick $2n$ points, $a_1, \ldots, a_n, b_1, \ldots b_n$, on the x-axis, from left to right in this order. These points will be identified with the vertices of G. For every edge $a_ib_j \in E(G)$, draw an x-monotone arc e_{ij} connecting a_i to b_j, according to the following rules:

(i) for any $k > i$, the arc e_{ij} passes above a_k if and only if $a_k b_j \notin E(G)$;

(ii) for any $l < j$, the arc e_{ij} passes above b_l if and only if $a_i b_l \in E(G)$;

(iii) no two distinct arcs "touch" each other (internal crossings are proper).

Notice that, unless two arcs share an endpoint, the *parity* of their number of intersections is determined by these rules.

Take two non-adjacent edges $a_i b_j, a_k b_l \in E(G)$ that belong to a path of length 3. We have to distinguish four different cases:

1. $i < k$, $j < l$, and $a_k b_j \in E(G)$;
2. $i < k$, $j < l$, and $a_i b_l \in E(G)$;
3. $i < k$, $l < j$, and $a_i b_l \in E(G)$;
4. $i < k$, $l < j$, and $a_k b_j \in E(G)$.

Consider the first case. By drawing rule (i), the arc e_{ij} passes below a_k. By rule (ii), e_{kl} passes above b_j. In view of rule (iii), this implies that e_{ij} and e_{kl} cross an even number of times, as required. The second case can be treated similarly and is left to the reader.

In the third case, applying rule (i), we obtain that a_k lies above e_{ij}. It is sufficient to show that the same is true for b_l. At this point, we use that G is C_4-free: since $a_i b_j$, $b_j a_k$, $a_k b_l \in E(G)$, we have $a_i b_l \notin E(G)$. By rule (ii), this implies that b_l is above e_{ij}, as required. The last case follows in the same way, by symmetry.

So far we have checked that in our drawing any two non-adjacent edges cross an even number of times. It is not hard to extend the same property to *all* pairs of edges, even if they share endpoints. To this end, we slightly modify the arcs e_{ij} in some very small neighborhoods of their endpoints. Clearly, this will not effect the crossing patterns of non-adjacent pairs.

Fix a vertex a_i. Redraw the arcs e_{ij} incident to a_i so that the counter-clockwise order of their initial pieces in a small neighborhood of a_i will be the same as the order of x-coordinates of their right endpoints. Consider now two arcs, e_{ij}, e_{il}, $(l < j)$, incident to a_i. By rule (ii), b_l lies below e_{ij}. On the other hand, after performing the local change described above, the initial piece of e_{il} will also lie below e_{ij}. This guarantees that e_{ij} and e_{il} cross an even number of times. Repeating this procedure for each vertex a_i, and its symmetric version for each b_j, we obtain a drawing which meets the requirements of Theorem 4.

7 Longer paths

If we exclude longer self-intersecting paths, the upper bounds on the number of edges can be improved. The next theorem represents a very modest improvement, but in the special case when all edges of an x-monotone topological graph cross the y-axis we have stronger results (see Theorem 10). We do not think that any of these results would be best possible.

Theorem 9. *Let G be an x-monotone topological graph of n vertices with no self-intersecting path of length 5. Then G has at most constant times $n \log n / \log \log n$ edges.*

Proof. We modify the proof of Theorem 6, and use the same notation. We call an edge a *left edge* at its right endpoint and a *right edge* at its left endpoint.

Suppose that G has nm edges with $m \geq 8$. Construct a sequence of subgraphs G_0, G'_0, G''_0, G_1, G'_1, G''_1, $G_2, \dots$ of G, as follows. Let G_0 be the topological graph obtained from G by deleting each vertex of degree at most $\frac{m}{2}$.

1. If G_i has already been defined for some i, let G'_i denote the topological graph obtained from G_i by deleting each vertex of degree at most $\frac{m}{2}$.
2. If G'_i has already been defined for some i, let G''_i denote the topological graph obtained from G'_i by deleting the bottom and the top left and right edges at each vertex (if they exist). We delete at most four edges per vertex.
3. If G''_i has already been defined for some i, let G_{i+1} be the topological graph obtained from G''_i by deleting the bottom and the top left and right edges at every vertex (if they exist). We delete at most four edges per vertex.

Notice that no two adjacent edges of G'_0 cross each other, and similarly, no path of length 3 or 4 is self-intersecting in G'_0. Otherwise, the self-intersecting path could be extended to a self-intersecting P_6 in G, a contradiction. As adjacent edges do not cross, all left (respectively, right) edges at a vertex are naturally ordered top to bottom, so our choices for edges to be deleted from G'_i and G''_i are well defined.

Let a_i denote the average degree in G_i. It is easy to see that if $a_i \geq m$, then the average degree of G'_i is at least a_i, the average degree of G''_i is at least $a_i - 8i$, and we have $a_{i+1} \geq a_i - 16$. So, we have $a_{\lfloor \frac{m}{16} \rfloor} \geq m$. Therefore, $G_{\lfloor \frac{m}{16} \rfloor}$ still determines at least one triangle (actually, several triangles).

Recall from the proof of Theorem 6 in Section 3 that a *left (right) triangle* at a vertex is determined by two left (resp., right) edges at this vertex, and it is the region bounded by one of the edges, a piece of the other edge, and a vertical interval.

It is sufficient to establish the following.

Claim. *For any $0 \leq k \leq \frac{m}{16}$, every triangle determined by two edges of G_k contains at least $\left(\frac{m}{2} - 2\right)^k$ pairwise different triangles in G.*

We postpone the proof of the Claim and finish the proof of the Theorem, assuming that the Claim is true. A triangle determined by $G_{\lfloor \frac{m}{16} \rfloor}$ contains at least $\left(\frac{m}{2} - 2\right)^{\lfloor \frac{m}{16} \rfloor}$ triangles, and this number is at most n^3. It follows that $m \leq c \log n / \log \log n$, as required by the theorem. ■

Proof of Claim. By induction on k. Obviously, for $k = 0$, the assertion is true, because every triangle contains itself. Assume that the Claim holds for $k-1$ ($k > 0$). Consider a *right* triangle T in G_k, determined by the edges $e_1 = vu_1$ and $e_2 = vu_2$, where the x-coordinate of u_1 is at most as large as the x-coordinate of u_2. Suppose without loss of generality that e_1 lies below e_2. Since $e_1 \in E(G_k)$, there is at least one left edge, $f_1 \in E(G''_{k-1})$, at u_1 above e_1. This edge, $f_1 = w_1 u_1$, must be entirely contained in T, otherwise we could find a self-intersecting path of length 3. Since $f_1 \in E(G''_{k-1})$, there is at least one right edge, $f_2 \in E(G'_{k-1})$, at w_1 below f_1. Similarly, this edge, $f_2 = w_1 w$, must be entirely contained in the triangle determined by e_1 and f_1. Therefore, f_2 must also lie in T. See Fig. 4. As $w \in V(G'_{k-1})$, its degree in G_{k-1} is at least $\frac{m}{2}$. In view of the fact that there is no self-intersecting path of length 5 or shorter, none of the at least $m/2$ edges of G_{k-1} incident to w crosses e_1, e_2, or f_1. Therefore, all of them lie inside T. They determine at least $\frac{m}{2} - 2$ triangles with pairwise disjoint interiors, each of which contains at least $\left(\frac{m}{2} - 2\right)^{k-1}$ further triangles in G, by the induction hypothesis. This finishes the proof of the Claim. ■

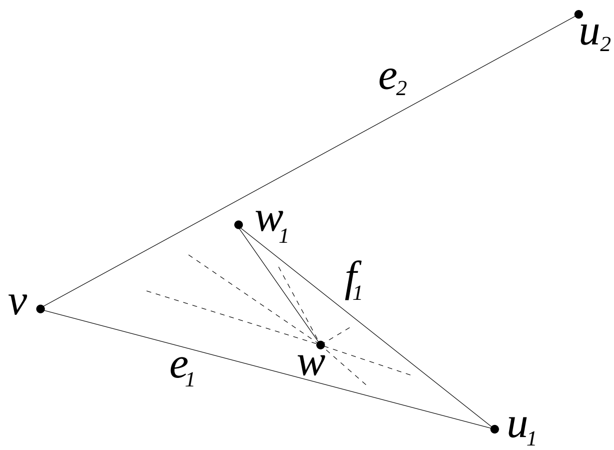

Fig. 4. *The edges at w are all in T*

Theorem 10. *Let G be an x-monotone topological graph of n vertices, all of whose edges cross the y-axis. Let $k \geq 2$ and suppose G has no self-intersecting path of length at most $2k$. Then G has at most $ckn \log^{1/k} n$ edges for some absolute constant $c > 0$.*

Proof. Let G be the topological graph satisfying the conditions in the theorem. Assume without loss of generality that no two vertices have the same x-coordinate. Just like in the proof of Theorem 7, the *length* of an edge is defined as the length of its projection to the x-axis. We call every vertex to the left of the y-axis a *left vertex*, the remaining vertices are *right vertices*. As in the proof of Theorem 7, first we ensure a monotonicity condition (see below). For each vertex, delete the longest edge incident to it. Since there is no self-intersecting path of length 3, for each remaining edge $e = xy$, one of the following two conditions are satisfied: Either (i) all longer edges incident to x are *above* e and all longer edges incident to y are *below* e, or (ii) all longer edges incident to x are *below* e and all longer edges incident to y are *above* e. (The terms "above" and "below" make perfect sense for adjacent edges, because, in contrast to Theorem 7, here two adjacent edges are not allowed to cross.) So, by deleting at most half of the edges, we can assume by symmetry that among any two adjacent edges incident to a left (right) vertex, the longer one passes above (resp., below) the other. For simplicity, the resulting graph will also be denoted by G.

We can assume without loss of generality that the edges of G intersect the y-axis at distinct points, and let $e_1, \ldots, e_m$ be the list of all edges in the bottom-to-top order of these intersections. Let x and y denote the left and right endpoints of e_i, respectively. Let e_{i+}

be the shortest edge in G incident to x which is longer than e_i. If no such edge exists, set $i^+ = m+1$. Similarly, let e_{i^-} be the shortest edge in G incident to y which is longer than e_i, where $i^- = 0$ if no such edge exists. By the monotonicity, we have $i^- < i < i^+$. Certainly, either $i^+ - i \geq i - i^-$ or $i - i^- \geq i^+ - i$ holds. Assume without loss of generality that $i^+ - i \geq i - i^-$ is true for at least half of the edges $e_i \in E(G)$. Let G' denote the subgraph of G formed by these edges.

For a left vertex x, let i_x denote the smallest index of an edge of G incident to x, provided that such an edge exists. Define the *rank* of an edge $e_i \in E(G)$ incident to a left vertex x as $r(e_i) = i - i_x + 1$. Clearly, we have $1 \leq r(e_i) \leq m$.

Claim. *Let e and e' be two edges of G' with a common left endpoint, and assume that e' passes above e. Then we have $r(e') \geq 2r(e)$.*

Proof of Claim. Let $e = e_i$ and let x be its left endpoint. Obviously, we have $r(e) = i - i_x + 1$, $r(e') \geq i^+ - i_x + 1$, and, as the path formed by e_{i^-}, e, and e_{i_x} does not cross itself, we also have that $i^- \leq i_x - 1$. This last inequality also holds if e_{i^-} does not exist and $i^- = 0$. Since e belongs to G', we have $i^+ - i \geq i - i^-$. Combining the above inequalities, the Claim immediately follows. ■

Notice that for the proof of the Claim we only needed the fact that no path of length three intersects itself, and this readily implies that the degree of each left vertex in G' is at most $\log m + 1$. Thus, the number of all edges in G' (which is obviously at least $m/2$) cannot exceed $n(\log m + 1)$. This can be regarded as another proof of the special case of Theorem 1 settled in Section 2.

Recursively, construct a sequence of graphs $G' = G_0, G_1, \ldots G_{k-1}$, as follows. Assuming that G_i has already been defined, we obtain G_{i+1} from G_i by deleting the $l = 2\lceil \log^{1/k} n \rceil$ longest edges incident to each vertex. In each step, we decrease the number of edges by at most nl. So, if G' has more than kln edges, on either side of G_{k-1} we can find a vertex of degree at least l. However, as is shown in the next paragraph, no such left (right) vertex can exist, provided that k is odd (resp., even). Hence, the number of edges of G' (which is at least $m/2$) cannot exceed kln, and this implies Theorem 10.

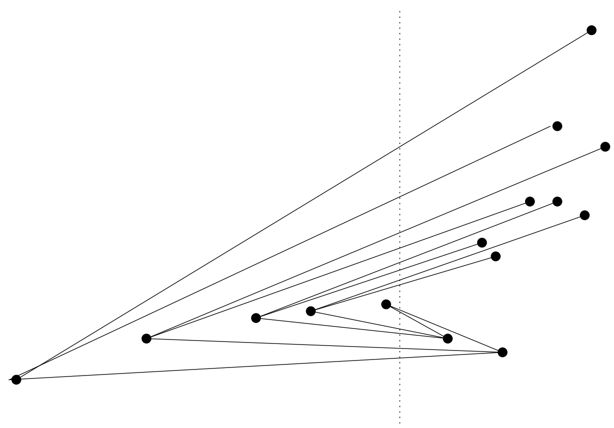

Fig. 5. *Illustration to the proof of Theorem 10 for k=6*

To complete the proof, suppose to the contrary that G_{k-1} has a left (right) vertex x of degree at least l and that k is odd (resp., even). By the construction of the sequence of graphs, G' has at least l^{k-1} distinct paths of length $k-1$, starting at x and extendible in at least l different ways to paths of length k consisting of edges of monotone increasing lengths. The l possible extensions at each vertex are said to form a *star*. Notice that, according to our assumption that there is no self-intersecting path of length at most $2k$, no two edges participating in these paths can cross each other. It follows from the monotonicity condition that these paths form a subtree of G' (i.e., no vertex can be reached in more than one ways) and that there is a topologically unique way to draw all of these paths (see Figure 5). Order the edges participating in the stars according to their intersections with the y-axis. We find that the edges of the individual stars form subintervals in this order and that the ranks of these edges increase from bottom to top. Furthermore, according to the Claim, inside a star, the ranks increase by a factor of two (here we use the fact that the common vertex of the edges of a star is a left vertex). Therefore, the rank of the highest edge of a star is at least 2^{l-1} times the rank of its lowest edge. Consequently, the rank of the highest edge of the highest star is at least $(2^{l-1})^{l^{k-1}}$ times the rank of the lowest edge of the lowest star. This ratio is larger than $n^2 > m$, which is a contradiction. ■

As in Section 2, the above proof can be easily rephrased in terms of forbidden submatrices of a double array. Suppose that in a double array containing n distinct symbols, all columns are distinct, and there are no submatrices of the following type:

For $k = 2$, the forbidden submatrices are

$$F_1 = \begin{pmatrix} u & v & u & v \\ * & x & x & * \end{pmatrix}$$

and

$$M_2 = \begin{pmatrix} * & * & u & u \\ x & y & x & y \end{pmatrix},$$

and one more matrix obtained from F_1 by a top-bottom flip, and three others obtained from M_2 by top-bottom and left-right flips.

In general, for $k \geq 2$, the forbidden submatrices are $F_1, M_2, \ldots, M_k$, and all other matrices that can be obtained from them by top-bottom or by left-right flips. Here

$$M_3 = \begin{pmatrix} v & w & u & u & v & w \\ * & * & x & y & y & x \end{pmatrix},$$

and, in general, each M_k corresponds to a specific spiral path of length $2k$ that cannot be drawn without crossing.

With the above assumptions, we can conclude that the double array has at most $O(kn \log^{1/k} n)$ columns.

Theorem 10 is not known to be tight for any $k \geq 2$. However, a recent construction of Tardos [T03] shows that, even if all paths of a given length are non-selfintersecting, the number of edges can still be superlinear.

8 Related problems

A. Theorems 1 and 6 easily imply

Corollary 2. *For any tree T other than a star, there exists a constant $c(T)$ such that every geometric (or x-monotone topological) graph G with n vertices and more than $c(T)n \log n$ edges contains a self-intersecting copy of T. That is, we have*

$$\text{ex}_{\text{cr}}(n, T) \leq c(T)n \log n.$$

Indeed, deleting one-by-one every vertex of G whose degree is smaller than $|V(T)|$, we end up with a graph G' having at most n vertices and at least $(c(T) \log n - |V(T)|)n$ edges. If $c(T)$ is sufficiently large, then G' has a self-intersecting path of length 3. Using the fact that the degree of every vertex in G' is at least $|V(T)|$, this path can be extended to a copy of T in G' (and hence in G).

B. A slight modification of the proof of Theorem 1 gives

Corollary 3. *For any positive integer k, there exists a constant c_k with the property that every geometric graph with n vertices and at least $c_k n \log n$ edges has two adjacent vertices, u and v, and $2k$ edges incident to them, $uu_1, uu_2, \ldots, uu_k$ and $vv_1, vv_2, \ldots, vv_k$, such that uu_i crosses vv_j for every pair $1 \leq i, j \leq k$.*

C. We conjecture that Theorem 1 (and Theorem 10) can be generalized to *every* topological graph with no self-intersecting path of length 3 (resp., length $2k$). In particular, we believe that every topological graph without a self-intersecting path of length 4 has $O(n \log^{1/2} n)$ edges. It is interesting to note that one cannot guarantee the existence of any *specific* crossing pattern of a path of length 4, even in a geometric graph with $\Omega(n \log n)$ edges, each intersecting the y-axis. Indeed, the construction in Section 3 provides such a geometric graph with no self-intersecting path of length 3. On the other hand, a convex, balanced, complete bipartite geometric graph, all of whose edges cross the y-axis, has no path of length 4, whose only self-intersection occurs between its first and last edges.

D. Any drawing of $K_{3,3}$, a complete bipartite graph with 3 vertices in each of its classes, has two non-adjacent edges that cross each other. Clearly, any two edges belong to a cycle of length 4, so

$$\text{ex}_{\text{cr}}(n, C_4) \leq \text{ex}(n, K_{3,3}) = O(n^{5/3}).$$

This bound has been recently improved to $O(n^{8/5})$ by Pinchasi and Radoičić [PR03]. It seems likely that the best possible bound is close to $n^{3/2}$.

It also follows from Theorem 8 that $\text{ex}_{\text{cr}}(n, C_6) = O(n^{3/2})$, and it generalizes to topological graphs. On the other hand, we have $\text{ex}_{\text{cr}}(n, C_6) \geq \text{ex}(n, C_6) \geq cn^{4/3}$, for a suitable constant $c > 0$ (see [BS74]). For C_4-free graphs this bound is almost tight.

Theorem 11. *Let G be a C_4-free geometric (or x-monotone topological) graph on n vertices. If G has no self-intersecting cycle of length 6, then G has $O(n^{4/3} \log^{2/3} n)$ edges.*

Proof. Assume without loss of generality that the left end of an edge is not the right end of another edge in G. This can be achieved by splitting the vertices in two as in the proof of Theorem 7. Let G have n vertices and $|E(G)| = m > c'n^{4/3} \log^{2/3} n$ edges. For $p = \frac{2cn \log n}{|E(G)|} < 1$, color randomly and independently with probability p each vertex of G *red*. Let G' be the subgraph of G induced by the red vertices.

Let $i(G')$ denote the number of self-intersecting paths of length 3 in G'. Deleting one edge from each such path, we obtain a graph with no self-intersecting path of length 3. Thus, in view of Theorem 1, we have

$$|E(G')| - i(G') < c|V(G')| \log |V(G')|,$$

for some positive c. Taking expected values, this yields

$$p^2|E(G)| - p^4 i(G) < cpn \log n.$$

We obtain $i(G) > \frac{|E(G)|^3}{8c^2n^2\log^2 n}$. If c' is large enough, then $i(G) > \binom{n}{2}$, and there must exist two self-intersecting paths of length 3 connecting the same pair of vertices. These paths cannot share an internal vertex as that would lead to a C_4. Therefore, putting them together, we get a C_6 which intersects itself at least twice. ■

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