

NP-completeness of list coloring and precoloring extension on the edges of planar graphs

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Abstract

In the edge precoloring extension problem we are given a graph with some of the edges having a preassigned color and it has to be decided whether this coloring can be extended to a proper k -edge-coloring of the graph. In list edge coloring every edge has a list of admissible colors, and the question is whether there is a proper edge coloring where every edge receives a color from its list. We show that both problems are NP-complete on (a) planar 3-regular bipartite graphs, (b) bipartite outerplanar graphs, and (c) bipartite series-parallel graphs. This improves previous results of Easton and Parker [6], and Fiala [8].

1 Introduction

In graph vertex coloring we have to assign colors to the vertices such that neighboring vertices receive different colors. Starting with [7] and [28], a generalization of coloring was investigated: in the *list coloring* problem each vertex can receive a color only from its prescribed list of admissible colors. In the *precoloring extension* problem a subset W of the vertices have preassigned colors and we have to extend this precoloring to a proper coloring of the whole graph, using only colors from a given color set C . It can be viewed as a special case of list coloring: the list of a precolored vertex consists of a single color, while the list of every other vertex is C . A thorough survey on list coloring, precoloring extension, and list chromatic number can be found in [26, 1, 12, 13].

In this paper we consider the edge coloring version of list coloring and precoloring extension. Colbourn [3] has shown that edge precoloring extension is NP-complete for complete bipartite graphs. Easton and Parker [6], and independently Fiala [8] have shown that the problem is NP-complete for 3-regular bipartite graphs. In Section 2 we strengthen this result by showing that the problem remains NP-complete for planar 3-regular bipartite graphs. The main

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difficulty of the proof is to find gadgets that are planar and 3-regular at the same time. As the first step of the proof, we show that list edge coloring is also NP-complete on the same class of graphs.

The unrestricted version of edge coloring is known to be NP-complete [11, 18]. However, the problem is easy for bipartite graphs: by König's Line Coloring Theorem [16], every bipartite graph with maximum degree Δ can be edge colored with Δ colors. Vizing [27] has shown that every planar graph with maximum degree $\Delta \geq 8$ can be edge colored with Δ colors, and conjectured that the same holds for every $\Delta \geq 6$. The case $\Delta = 7$ was proved by Sanders and Zhao [25]. For $\Delta = 3, 4, 5$ there are planar graphs with maximum degree Δ that cannot be edge colored with Δ colors. Hence edge coloring planar graphs is easy for $\Delta \geq 7$, expected to be easy for $\Delta = 6$, and the complexity is open for $\Delta = 3, 4, 5$. (See [30] for an overview of algorithmic results on edge coloring.) Our results show that the more general list edge coloring and edge precoloring extension problems are NP-hard even if the graph is both planar and bipartite.

In [10, 9, 15, 29] the list edge coloring problem is considered for series-parallel graphs, sufficient conditions are given for some special cases. In Section 3 we investigate the computational complexity of the problem. An easy argument shows that precoloring edge extension and list edge coloring can be solved in polynomial-time for bounded degree outerplanar graphs. However, we prove that the problems are NP-complete for outerplanar and series-parallel graphs if the maximum degree can be arbitrary.

Outerplanar and series-parallel graphs have treewidth at most 2, thus the results of Section 3 show that precoloring edge extension and list edge coloring are NP-complete for partial 2-trees. However, both problems can be solved in polynomial time for trees (see [19]). This is somewhat surprising, since there are very few problems that are polynomial-time solvable for trees but NP-hard for partial 2-trees. Usually it is expected that if a dynamic programming approach works for trees, then it can be generalized to partial k -trees. A recent example where the problem is easy for trees but NP-hard for partial 2-trees is the edge disjoint paths problem [23].

2 Planar bipartite graphs

In this section we prove that the precoloring extension problem is NP-complete on the edges of planar 3-regular bipartite graphs. First we prove that the more general list edge coloring problem is NP-complete for such graphs, then we present a reduction from list coloring to precoloring extension.

The proof is by reduction from the 1-in-3 Satisfiability problem, which is the following: given a formula in conjunctive normal form, every clause contains exactly 3 literals, decide if a variable assignment exists such that *exactly one* literal is true in every clause. The 1-in-3 Satisfiability problem remains NP-complete even with the following restrictions:

Theorem 2.1 ([22]). *1-in-3 Satisfiability is NP-complete even if*

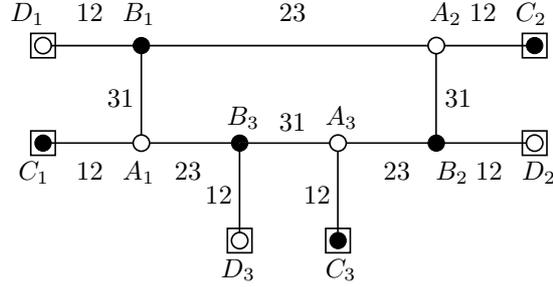


Figure 1: The variable setting gadget. The two numbers on each edge show the list of available colors.

- every variable appears in exactly 3 clauses,
- there is no negation in the formula, and
- the underlying bipartite graph of the formula (where the vertex representing a clause is connected to vertices representing the variables appearing in the clause) is planar.

Theorem 2.2. *List edge coloring is NP-complete for planar 3-regular bipartite graphs.*

Proof. We construct variable setting gadgets and satisfaction testing gadgets, and connect them in such a way that the resulting graph can be colored if and only if the given formula is satisfiable (in 1-in-3 sense). If the original formula satisfies the requirements of Theorem 2.1, then the resulting graph is planar and 3-regular.

Figure 1 shows the variable setting gadget. It is easy to verify that it has only two colorings: the coloring that assigns the first (resp. second) color of the list to each edge. Therefore in every coloring of the gadget, the pendant edges receive the same color, either 1 or 2. The coloring that assigns color 1 to the pendant edges corresponds to setting the variable to “true,” and the coloring that assigns 2 to these edges corresponds to “false.”

The satisfaction testing gadget is shown on Figure 2. We claim that it has only three colorings, the numbers in the frames are the colors assigned to the edges in the three colorings. Let ψ be a coloring of this gadget. First, let us verify that $\psi(A_i C_i) = \psi(B_i D_i)$ for $i = 1, 2, 3$ in every coloring ψ . For this purpose, it is sufficient to follow the implications of say, $\psi(A_1 C_1) = 1$ and $\psi(B_1 D_1) = 2$, to arrive to a contradiction. The case $i = 3$ is somewhat more complicated to verify: $\psi(A_3 C_3) = 1$ and $\psi(B_3 D_3) = 2$ imply $\psi(C_3 D_3) = \psi(E_3 F_3) = 3$, $\psi(C_3 E_3) = 2$, $\psi(D_3 F_3) = 1$; edges $D_3 F_3$ and $E_3 F_3$ force $\psi(F_3 E_2) = 2$, $\psi(F_2 E_2) = 1$; edges $C_3 E_3$ and $E_3 F_3$ force $\psi(J_1 E_3) = 1$, $\psi(J_1 I_1) = 3$, $\psi(H_1 J_1) = 2$, $\psi(H_1 G_1) = 3$, $\psi(E_1 G_1) = 1$, and $\psi(G_1 I_1) = 2$. Now there is no color left for edge $I_1 F_2$, since edges $F_2 E_2$, $G_1 I_1$, and $J_1 I_1$ use all three colors. Furthermore, it can be shown

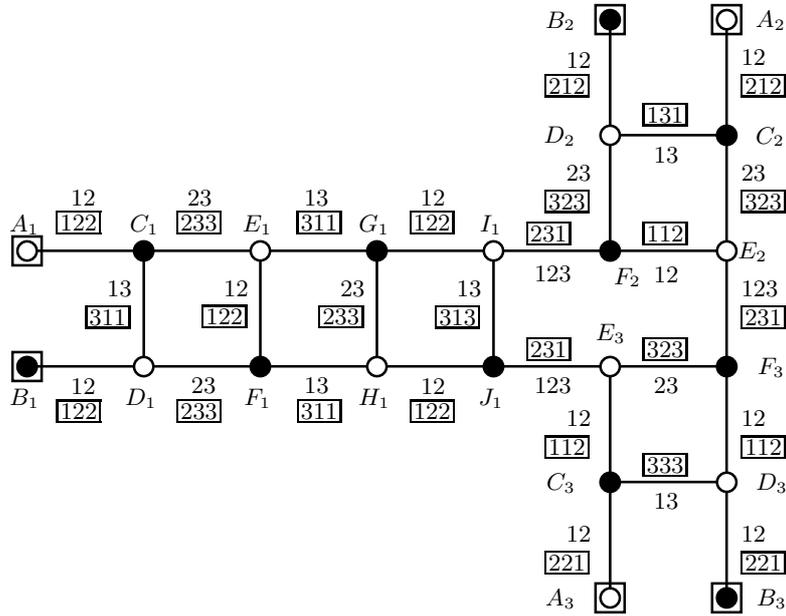


Figure 2: The satisfaction-testing gadget. The numbers on the edges show the list of available colors on the edge. The three numbers in each frame show the three colors assigned to the edge in the three possible colorings of the graph.

that $\psi(A_1C_1) = \psi(B_1D_1) = 1$ implies that ψ is the first coloring defined on Figure 2. Similarly, $\psi(A_2C_2) = \psi(B_2D_2) = 1$ (resp. $\psi(A_3C_3) = \psi(B_3D_3) = 1$) implies that ψ is the second (resp. third) coloring. Thus in every coloring of the gadget, exactly one of the pairs A_iC_i and B_iD_i is colored with color 1, the others with color 2. A coloring that assigns color 1 to A_iC_i and B_iD_i corresponds to a variable assignment where the clause is satisfied by its i th literal.

Figure 3 shows the overview of the construction. Take a copy of the variable setting gadget G_x for each variable x and a satisfaction-testing gadget G_C for each clause C of the formula. If x appears in clause C , then G_x and G_C are connected by a pair of edges. Fix a planar embedding of the formula, and fix an ordering of the occurrences of each variable in such a way that the clauses of a variable are located in clockwise order around the variable. Similarly, for each clause fix an ordering of the literals in such a way that the variables of a clause are in clockwise order around the clause. Assume that x is the i th variable in clause C , and C is the j th clause where x appears, then connect G_x and G_C by identifying the edges A_jC_j and B_jD_j of G_x with A_iC_i and B_iD_i of G_C , respectively.

The resulting graph is bipartite since the gadgets are bipartite (Figure 1 and 2 show the two color classes) and when we identified two edges, we only identified vertices that belong to the same color class. Since every variable

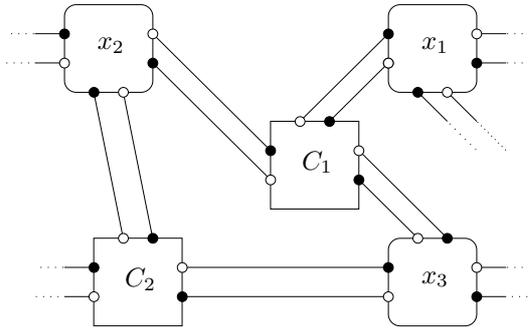


Figure 3: The reduction from 1-in-3 SAT to the list edge coloring problem. We connect the variable setting components and the satisfaction testing components with pairs of edges.

appears in exactly three clauses and every clause has exactly three literals, the resulting graph is 3-regular. The gadgets are planar, and because of the planarity of the formula, the graph can be embedded in the plane such that two pairs of edges do not cross each other. Note that in the variable setting gadget, edge $B_j D_j$ is in clockwise direction from the edge $A_j C_j$, while in the satisfaction testing gadget $B_i D_i$ is in counterclockwise direction from $A_i C_i$. Thus the two edges connecting a variable setting gadget with a satisfaction testing gadget do not cross each other, as shown on Figure 3. Therefore the resulting graph G is planar, bipartite and 3-regular.

It is clear from the construction that G has a proper list edge coloring if and only if the formula has a satisfying variable assignment (in 1-in-3 sense). Obviously, the graph can be built in polynomial time, thus we have proved that the problem is NP-complete. ■

List coloring is reduced to precoloring extension by locally replacing the edges by some partially precolored graphs.

Theorem 2.3. *Precoloring extension is NP-complete on the edges of planar 3-regular bipartite graphs.*

Proof. We reduce list edge coloring to precoloring extension as follows. By the proof of Theorem 2.2, it can be assumed that only the three colors 1, 2, 3 appear in the lists. If only a single color appears in the list of an edge, then the edge is precolored with that color. If all three colors appear in the list of an edge, then the edge is not precolored. If edge uv has a 2 element list, say color i is not allowed ($1 \leq i \leq 3$), then we replace uv by the gadget shown on Figure 4. The edges $x_1 x_3$ and $x_2 x_4$ are precolored with color i .

We claim that the list edge coloring problem has a solution in the original graph G if and only if the precoloring in the constructed graph G' can be extended to a proper 3-edge-coloring. First, a coloring of G can be used to

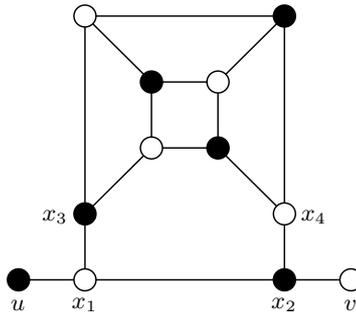


Figure 4: The gadget used to replace those edges that have 2 element lists.

obtain a precoloring extension of G' : if edge uv has color c , then the gadget corresponding to uv can be colored such that both ux_1 and x_2v receive color c . On the other hand, in every 3-edge-coloring of G' edges ux_1 and x_2v receive the same color, a color different from the forbidden color i of edge uv : since both x_1x_3 and x_2x_4 are precolored to i , only two colors are available for the edges ux_1, x_1x_2, x_2v . Therefore a precoloring extension of G' determines a list coloring of G . ■

3 Outerplanar graphs

A graph is *outerplanar* if it has a planar embedding such that all the vertices lie on the exterior face. A graph is *series-parallel* if it can be created from K_2 by repeatedly duplicating and subdividing the edges. If an outerplanar graph is 2-connected, then it is series-parallel. Some authors give a more general definition of series-parallel graphs by allowing to start from an arbitrary tree instead of a K_2 . We are giving complexity results here, thus using the more restrictive definition makes our results stronger.

Edge precoloring extension can be solved in linear time for bounded degree outerplanar and series-parallel graphs as follows. It is well-known that these graphs have treewidth at most 2. We show that if both the treewidth and the maximum degree of a graph are bounded by a constant, then precoloring extension can be solved in linear time. First, if the graph has maximum degree d , and the number of available colors is greater than $2(d-1)$, then the problem becomes trivial: every edge can receive a color that is not used on the at most $2(d-1)$ edges adjacent to it. Therefore it can be assumed that there are at most $2(d-1)$ available colors. If a graph has treewidth at most w and maximum degree d , then the treewidth of its line graph is at most $(w+1)d-1$ (see [2, Lemma 32]). Edge precoloring extension is the same as precoloring extension in the line graph. By [14], precoloring extension can be solved in linear time if both the treewidth and the number of available colors are constant. (This result also follows from the fact that precoloring extension with a fixed number

of colors can be expressed in monadic second order logic, see [4, 17].) A similar argument works in the case of list edge coloring.

Thus Theorem 2.3 cannot be strengthened to outerplanar graphs. However, if we drop 3-regularity, then the problem remains NP-complete for bipartite outerplanar graphs and series-parallel graphs. In the proof we follow the same path as in Section 2: first it is shown that list coloring is NP-complete with the given restrictions, then list coloring is reduced to precoloring extension.

Theorem 3.1. *List edge coloring is NP-complete for bipartite outerplanar graphs.*

Proof. The proof is by reduction from 3SAT. Given a formula ϕ in conjunctive normal form with n variables and m clauses, we construct an instance of the list edge coloring problem in such a way that the graph can be colored if and only if ϕ is satisfiable.

We assume that every variable occurs exactly twice positively and exactly twice negated in ϕ , and every clause contains exactly 3 literals. This can be achieved as follows. It is well-known that 3SAT remains NP-complete if every variable occurs exactly twice positively, exactly once negated, and every clause contains two or three literals (see e.g., [24, Section 9]). Let us assume that the number of variables is even, if not, then duplicate every variable and every clause. Let x_1, x_2, \dots, x_t be the variables of ϕ . We add $t/2$ new variables $y_1, y_2, \dots, y_{t/2}$ and t new clauses $(\bar{x}_1 \vee y_1 \vee \bar{y}_1), (\bar{x}_2 \vee y_1 \vee \bar{y}_1), (\bar{x}_3 \vee y_2 \vee \bar{y}_2), (\bar{x}_4 \vee y_2 \vee \bar{y}_2), \dots, (\bar{x}_{t-1} \vee y_{t/2} \vee \bar{y}_{t/2}), (\bar{x}_t \vee y_{t/2} \vee \bar{y}_{t/2})$ to the formula. Now every variable occurs exactly twice positively and twice negated. These new clauses are satisfied in every variable assignment, hence the new formula is satisfiable if and only if the original is satisfiable. Furthermore, if there is a clause $(x \vee y)$ containing only two literals, then add a new variable z , and replace this clause with $(x \vee z \vee z) \wedge (\bar{z} \vee \bar{z} \vee y)$. It is easy to see that this transformation does not change the satisfiability of the formula.

The set of colors C contains $4n$ colors: there is one color corresponding to each occurrence of a variable. For $1 \leq i \leq n$, color $4i$ corresponds to the first positive occurrence of x_i , color $4i - 1$ corresponds to the second positive occurrence of x_i , color $4i - 2$ corresponds to the first negated occurrence of x_i , and color $4i - 3$ corresponds to the second negated occurrence of x_i .

We construct the list edge coloring instance as follows. Let v be a vertex. For every $1 \leq i \leq n$, attach 5 new vertices a_i, b_i, c_i, d_i, e_i to v as shown on Figure 5. These edges correspond to variable x_i . The lists are as shown on the figure. For every $1 \leq j \leq m$, a new vertex u_j is attached to v . The list of vu_j contains three colors: the colors that correspond to the three literals in the j th clause of ϕ .

Given a satisfying assignment of ϕ , we construct a coloring ψ of G . If variable x_i is true, then the 6 edges corresponding to x_i receive the first color from their lists. If x_i is false, then the edges receive the second color from the lists. Notice that with this coloring the colors used on edges incident to v are exactly those colors that correspond to the *false* literals in the assignment. By assumption, the assignment satisfies every clause of the formula, every clause contains at

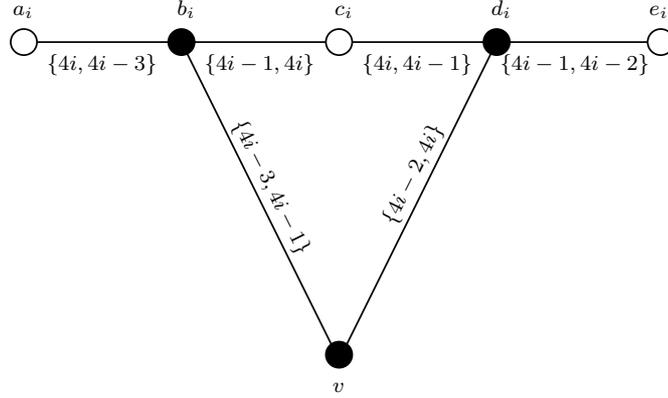


Figure 5: The edges corresponding to variable x_i of the formula.

least one true literal, hence the list of every edge vu_j contains at least one color not used on v . Therefore we can extend ψ to the edges vu_j , and the graph can be colored.

To prove the other direction assume that there is a coloring ψ of G . We show that for every $1 \leq i \leq n$, either

- $\psi(vb_i) = 4i - 3$ and $\psi(vd_i) = 4i - 2$, or
- $\psi(vb_i) = 4i - 1$ and $\psi(vd_i) = 4i$ hold.

If $\psi(vb_i) = 4i - 3$, then $\psi(a_ib_i) = 4i \Rightarrow \psi(b_ic_i) = 4i - 1 \Rightarrow \psi(c_id_i) = 4i \Rightarrow \psi(vd_i) = 4i - 2$. A similar argument shows that if $\psi(vb_i) = 4i - 1$, then $\psi(vd_i) = 4i$ follows. We set variable x_i to true in the first case, and to false in the second case. This yields a satisfying variable assignment of ϕ : if the j th clause is not satisfied, then all of its literals are false, which implies that the colors corresponding to these literals appear at v . However, this means that the three colors in the list of vu_j are already used at v , therefore edge vu_j cannot receive any color, contradicting the assumption that ψ is a list coloring of G . ■

Theorem 3.2. *Precoloring extension is NP-complete on the edges of bipartite outerplanar graphs.*

Proof. The proof is by reduction from the list edge coloring problem. Consider a bipartite outerplanar graph constructed by the reduction of Theorem 3.1. Notice that every edge lies on the outer face. Let the color set C be the union of the lists. If the list of an edge xy contains 2 colors, then this edge is replaced as follows. We add two new vertices x' , y' , and 3 new edges xx' , $x'y'$, $y'y'$. Furthermore, we attach $|C| - 2$ new edges $x'x'_1, \dots, x'x'_{|C|-2}$ to vertex x' , and another $|C| - 2$ new edges $y'y'_1, \dots, y'y'_{|C|-2}$ to y' . Denote by $c_1, \dots, c_{|C|-2}$ the $|C| - 2$ colors in C that do not appear in the list of xy . The edges $x'x'_i$ and $y'y'_i$ are precolored with color c_i . These precolored edges ensure that in every edge

coloring xx' , $x'y'$, $y'y$ receive a color from the list of xy , which implies that the colors of xx' and $y'y$ are the same. Therefore edges xx' and yy' effectively act as a single edge with the same list as xy . Moreover, since xy is on the outer face, the graph remains outerplanar and bipartite.

Consider an edge vu_j having list size 3. Here we use the fact that u_j has degree 1. We attach $|C| - 3$ new edges $u_ju_{j,1}, \dots, u_ju_{j,|C|-3}$ to u_j , and precolor them with the colors not in the list of vu_j . It is clear that now edge vu_j has to receive a color from the list of vu_j . Therefore we have constructed an instance of the edge precoloring extension problem that has a solution if and only if the list edge coloring problem has a solution. ■

Corollary 3.3. *Precoloring extension is NP-complete on the edges of bipartite series-parallel graphs.*

Proof. We show how to make a bipartite outerplanar graph 2-connected without changing the solvability of the edge precoloring extension instance. Together with Theorem 3.2 and the fact that 2-connected outerplanar graphs are series-parallel, this proves that edge precoloring extension is NP-complete for series-parallel graphs.

The graph is outerplanar, hence by traversing the boundary of the outer face we visit every vertex at least once. Since there can be cutvertices in the graph, there might be vertices that are visited more than once. Let v_0, v_1, \dots, v_{n-1} the order of vertices as they are first encountered while traversing the boundary of the outer face. We add new vertices and edges to the graph to make it Hamiltonian, and therefore 2-connected. If v_i and v_{i+1} (indices are taken modulo n) belong to the same bipartition class, then add a new vertex w_i and two new edges v_iw_i and w_iv_{i+1} . Edge v_iw_i is precolored with a new color α , and w_iv_{i+1} is precolored with a new color β . If v_i and v_{i+1} are in different classes, then add two new vertices w'_i, w''_i , and three edges $v_iw'_i$ (precolored to α), $w'_iw''_i$ (not precolored), w''_iv_{i+1} (precolored to β). It can be shown that the graph remains outerplanar after these modifications. The two new colors α and β appear at every vertex of the original graph, hence they cannot be used for the original edges. Thus the solvability of the instance did not change. ■

4 Conclusions

In this paper we have considered the edge precoloring extension and list edge coloring problems on certain restricted graph classes. First we proved hardness results for list edge coloring, since building gadgets is more convenient in this problem. With simple local replacements, we reduced the list coloring problem to precoloring extension. It turned out that both problems are NP-complete for planar 3-regular bipartite graphs. This result can be useful in showing the NP-completeness of other planar graph problems. For example, this result is used in [5] to show the NP-hardness of weighted edge coloring of planar 3-regular

bipartite graphs, and in [20] to show that minimum sum edge coloring of planar bipartite graphs is NP-hard.

List edge coloring is polynomial-time solvable for trees [19]. In fact, for trees the more general problem of list edge multicoloring is polynomial-time solvable as well [19, 21]. However, by Theorem 3.1 these algorithms cannot be generalized for bounded treewidth graphs, since the problem is NP-complete already for partial 2-trees.

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