Fixed Parameter Algorithms

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Parameterized complexity

Main idea: Instead of expressing the running time as a function $T(n)$ of n, we express it as a function $T(n, k)$ of the input size n and some parameter k of the input.

In other words: we do not want to be efficient on all inputs of size n, only for those where k is small.

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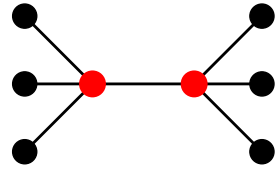
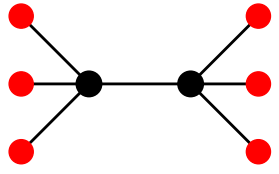
Main idea: Instead of expressing the running time as a function $T(n)$ of n, we express it as a function $T(n, k)$ of the input size n and some parameter k of the input.

In other words: we do not want to be efficient on all inputs of size n, only for those where k is small.

What can be the parameter k?

- The size k of the solution we are looking for.
- The maximum degree of the input graph.
- The diameter of the input graph.
- The length of clauses in the input Boolean formula.
- …

Parameterized complexity

Problem:	VERTEX COVER	INDEPENDENT SET
Input:	Graph G, integer k	Graph G, integer k
Question:	Is it possible to cover the edges with k vertices? 	Is it possible to find k independent vertices? 
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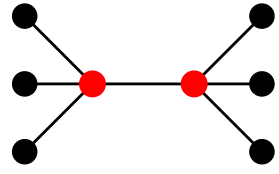
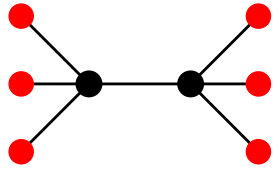
Question:	Is it possible to cover the edges with k vertices? 	Is it possible to find k independent vertices? 
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	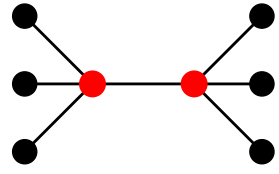	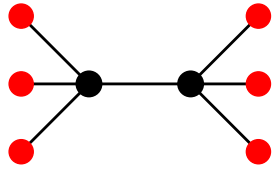
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	$O(2^k n^2)$ algorithm exists	No $n^{o(k)}$ algorithm known
		

Bounded search tree method

Algorithm for VERTEX COVER:

$$e_1 = x_1 y_1$$

Bounded search tree method

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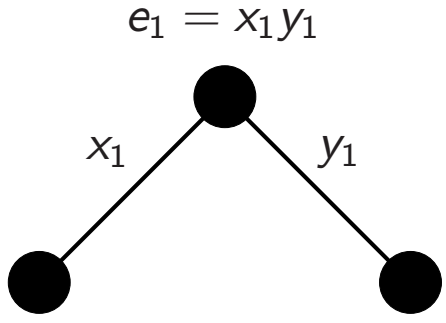

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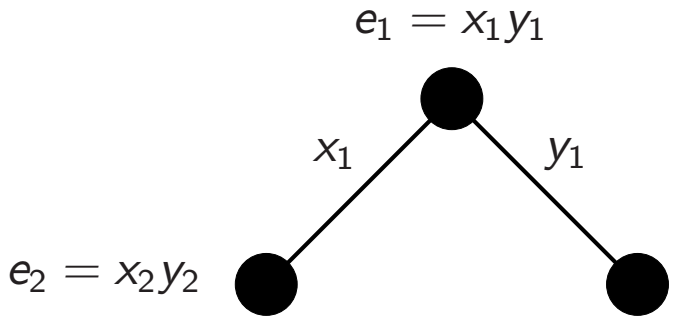

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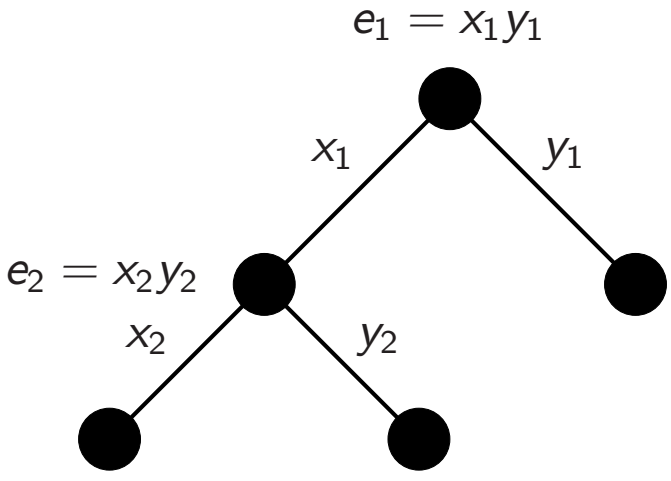

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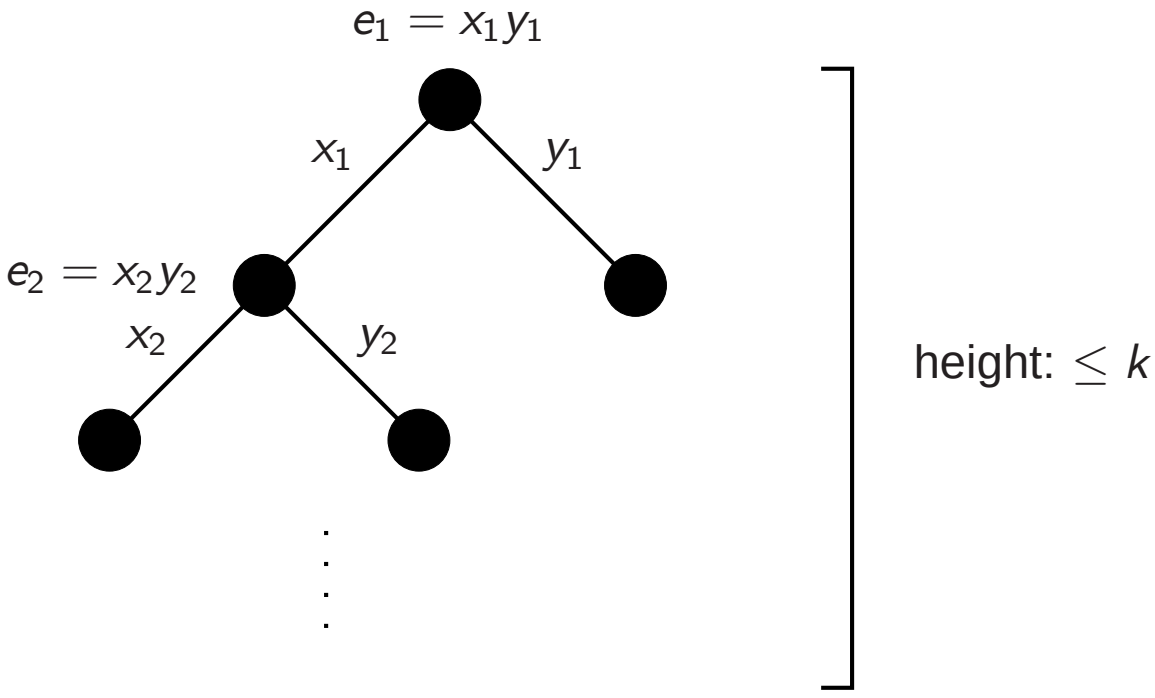

Height of the search tree is $\leq k \Rightarrow$ number of leaves is $\leq 2^k \Rightarrow$ complete search requires $2^k \cdot$ poly steps.

Fixed-parameter tractability

Definition: A **parameterization** of a decision problem is a function that assigns an integer parameter k to each input instance x.

The parameter can be

- explicit in the input (for example, if the parameter is the integer k appearing in the input (G, k) of VERTEX COVER), or
- implicit in the input (for example, if the parameter is the diameter d of the input graph G).

Main definition:

> A parameterized problem is **fixed-parameter tractable (FPT)** if there is an $f(k)n^c$ time algorithm for some constant c.

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Example: VERTEX COVER parameterized by the required size k is FPT:
we have seen that it can be solved in time $O(2^k \cdot n^2)$.

Better algorithms are known: e.g, $O(1.2832^k k + k|V|)$.

Main goal of parameterized complexity: to find FPT problems.

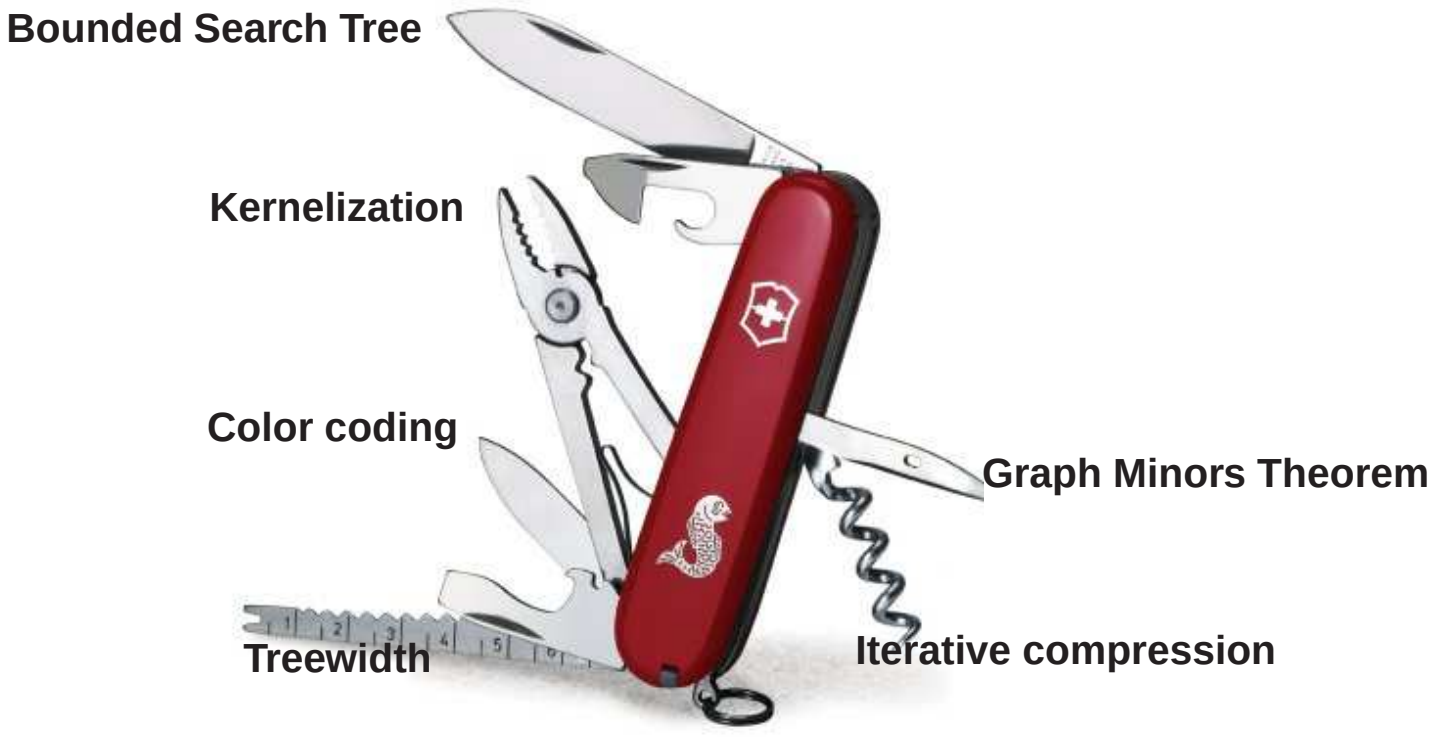
FPT problems

Examples of NP-hard problems that are FPT:

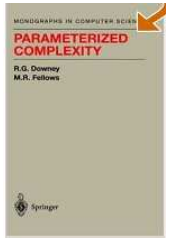
- Finding a vertex cover of size k.
- Finding a path of length k.
- Finding k disjoint triangles.
- Drawing the graph in the plane with k edge crossings.
- Finding disjoint paths that connect k pairs of points.
- …

FPT algorithmic techniques

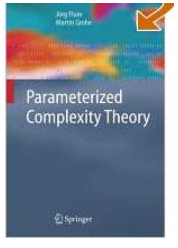
- Significant advances in the past 20 years or so (especially in recent years).
- Powerful toolbox for designing FPT algorithms:



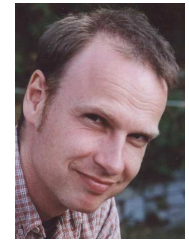
Books



Downey-Fellows: Parameterized Complexity, Springer, 1999



Flum-Grohe: Parameterized Complexity Theory, Springer, 2006

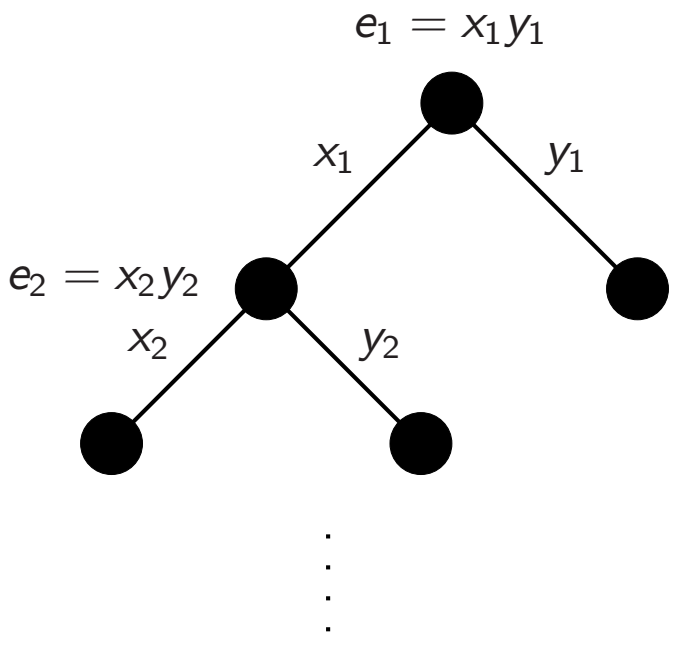


Niedermeier: Invitation to Fixed-Parameter Algorithms, Oxford University Press, 2006.

Bounded search trees

Bounded search tree method

Recall how we solved VERTEX COVER:



height: $\leq k$

If we branch into a bounded number of directions a bounded number of times, then we get an FPT algorithm!

MIN ONES SAT

VERTEX COVER can be interpreted as a CSP problem:

> "Given constraints $x \vee y$ on Boolean variables, find a satisfying assignment of weight at most k."

More generally:

MIN ONES SAT: Given an r-SAT formula and an integer k, find a satisfying assignment of weight at most k.

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The bounded search tree approach gives:

Fact: MIN ONES SAT on r-SAT formulas can be solved in time $O(r^k \cdot n)$, i.e., problem is FPT parameterized jointly by r and k.

Fact: MIN ONES SAT(Γ) is FPT parameterized by k for every Γ.

Color coding

k-PATH

Task: Given a graph G, an integer k, two vertices s, t, find a **simple** s-t path with exactly k internal vertices.

Note: Finding such a **walk** can be done easily in polynomial time.

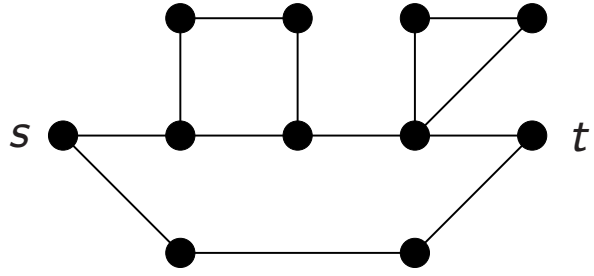
Note: The problem is clearly NP-hard, as it contains the s-t HAMILTONIAN PATH problem.

The k-PATH algorithm can be used to check if there is a cycle of length exactly k in the graph.

Fact: [Alon-Yuster-Zwick 1995] k-PATH can be solved in time $c^k \cdot n^{O(1)}$ for some c.

k-PATH

- Assign colors from $\{1, \ldots, k\}$ to vertices $V(G) \setminus \{s, t\}$ uniformly and independently at random.



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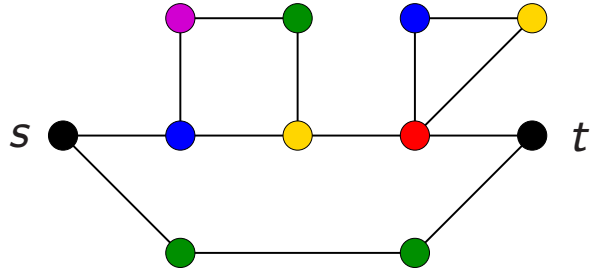

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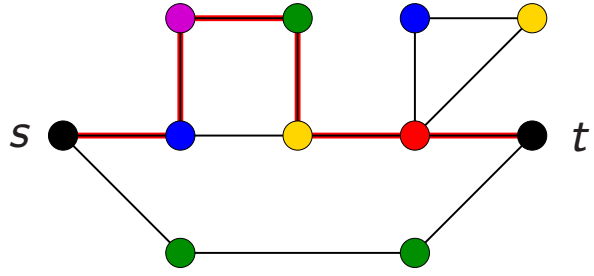

- Check if there is a **colorful** s-t path: a path where each color appears exactly once on the internal vertices; output “YES” or “NO”.

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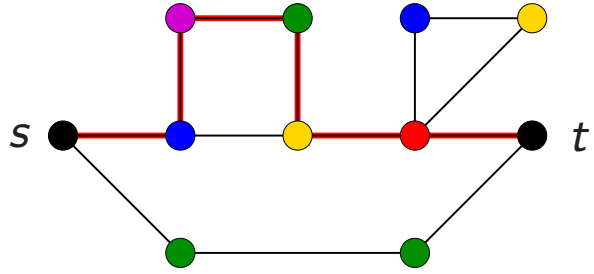

- Check if there is a **colorful** s-t path: a path where each color appears exactly once on the internal vertices; output "YES" or "NO".
  - If there is no s-t k-path: no such colorful path exists $\Rightarrow$ "NO".
  - If there is an s-t k-path: the probability that such a path is colorful is

$$\frac{k!}{k^k} > \frac{\left(\frac{k}{e}\right)^k}{k^k} = e^{-k},$$

  thus the algorithm outputs "YES" with at least that probability.

Error probability

- **Useful fact:** If the probability of success is at least p, then the probability that the algorithm **does not** say "YES" after $1/p$ repetitions is at most

$$(1-p)^{1/p} < (e^{-p})^{1/p} = 1/e \approx 0.38$$

- Thus if $p > e^{-k}$, then error probability is at most $1/e$ after e^k repetitions.
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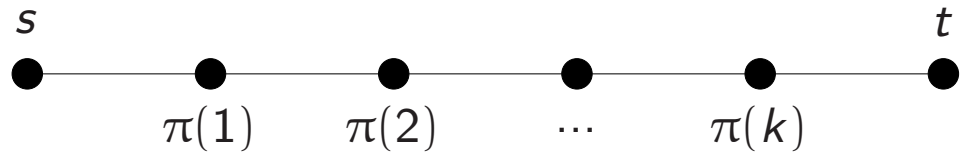
It remains to see how a colorful s-t path can be found.

Finding a colorful path

The colors encountered on a colorful s-t path form a permutation π of $\{1, 2, \ldots, k\}$:



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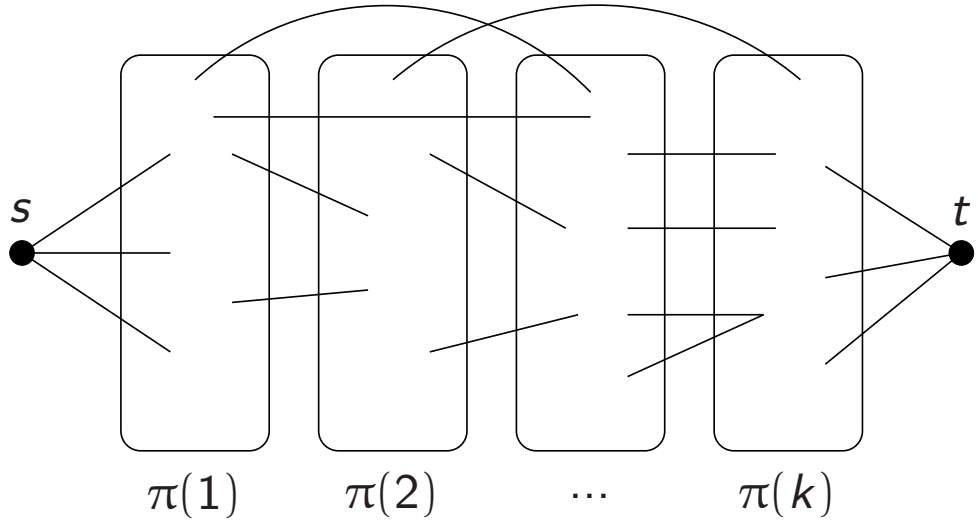

- Edges connecting nonadjacent color classes are removed.
- The remaining edges are directed.
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- Running time is $O(k! \cdot |E(G)|)$ (can be improved to $O(2^k \cdot |E(G)|)$).

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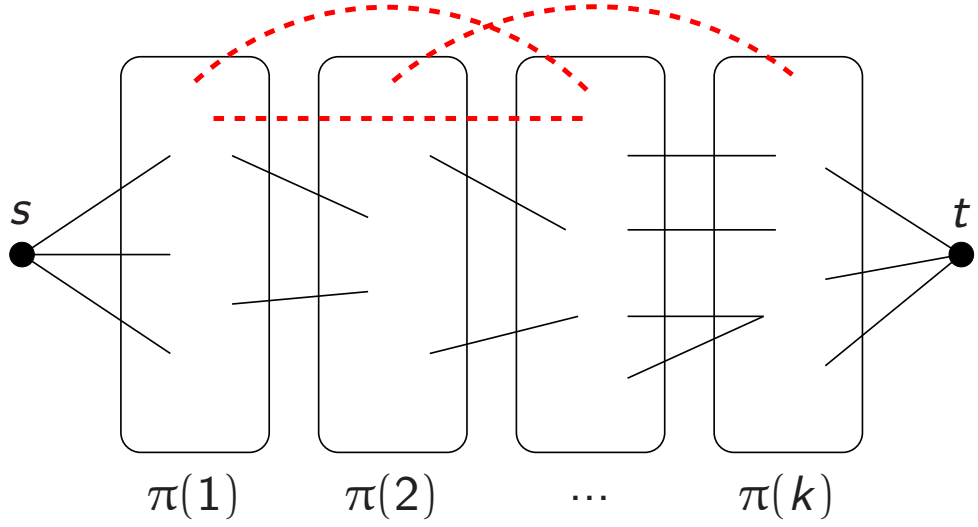

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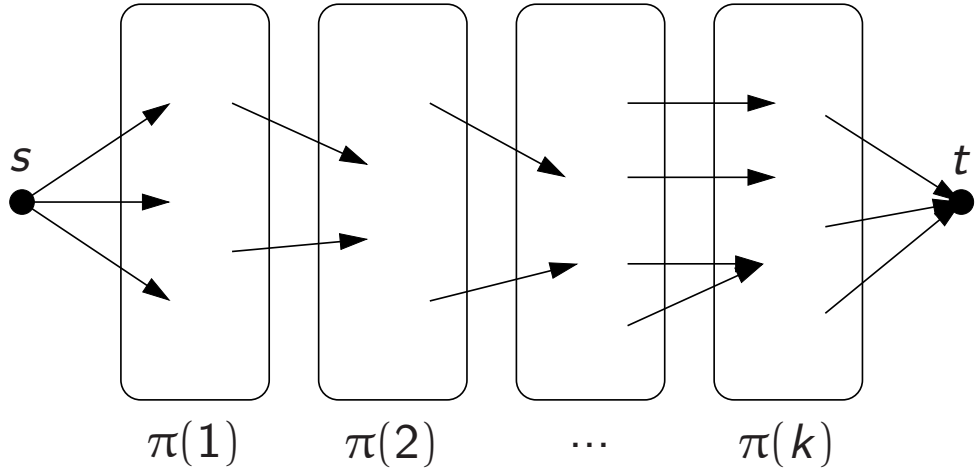

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Color coding

- Algorithm has small probability of error, but can be derandomized using k-perfect hash functions.
- More generally: finding graphs of low treewidth.
- Color coding is useful to ensure that certain objects are disjoint.

Iterative compression

BIPARTITE DELETION

BIPARTITE DELETION: Given a graph G and an integer k, delete k vertices to make G bipartite (2-colorable).

Fixed-parameteter tractability was open until Reed, Smith, and Vetta showed it in 2004 using the technique of **iterative compression.**

BIPARTITE DELETION

Solution based on iterative compression:

- **Step 1:**

  Solve the **annotated problem** for bipartite graphs:

  Given a bipartite graph G, two sets $B, W \subseteq V(G)$, and an integer k, find a set S of at most k vertices such that $G \setminus S$ has a 2-coloring where $B \setminus S$ is black and $W \setminus S$ is white.

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- **Step 3:**

  Apply the magic of iterative compression...

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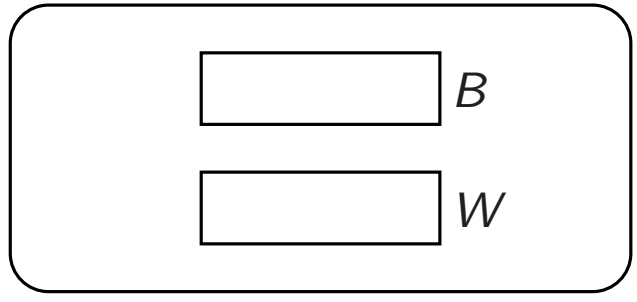

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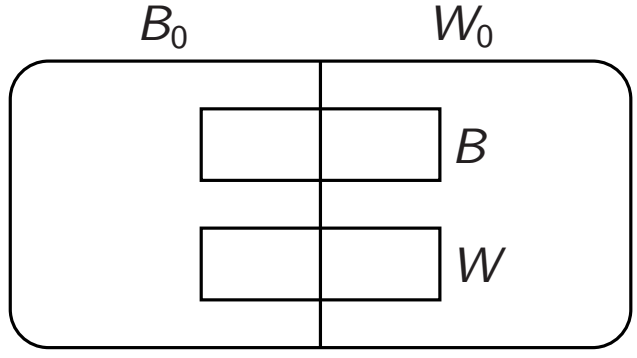

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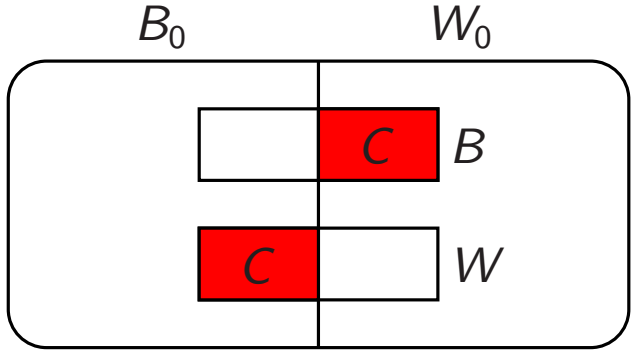

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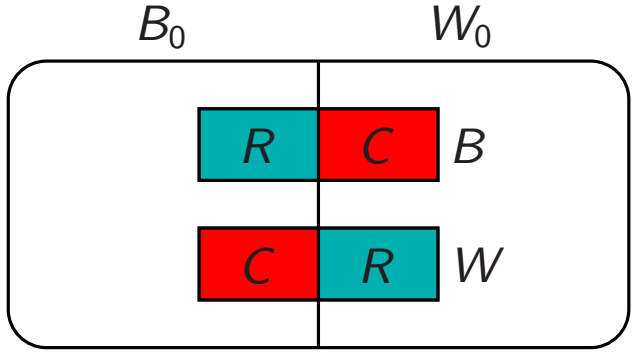

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Lemma: $G \setminus S$ has the required 2-coloring if and only if S separates C and R, i.e., no component of $G \setminus S$ contains vertices from both $C \setminus S$ and $R \setminus S$.

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Lemma: $G \setminus S$ has the required 2-coloring if and only if S separates C and R, i.e., no component of $G \setminus S$ contains vertices from both $C \setminus S$ and $R \setminus S$.

Proof:

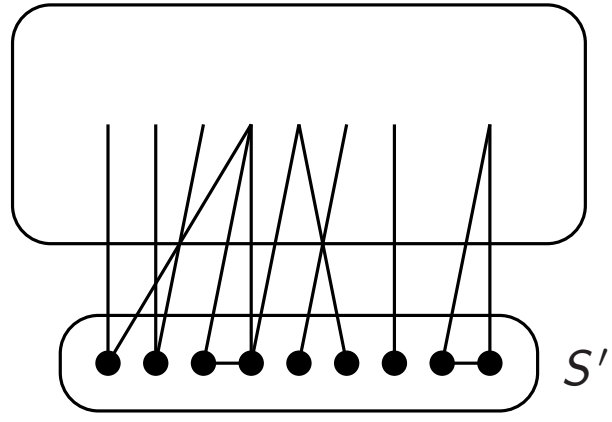
$\Rightarrow$ In a 2-coloring of $G \setminus S$, each vertex either remained the same color or changed color. Adjacent vertices do the same, thus every component either changed or remained.

$\Leftarrow$ Flip the coloring of those components of $G \setminus S$ that contain vertices from $C \setminus S$. No vertex of R is flipped.

Algorithm: Using max-flow min-cut techniques, we can check if there is a set S that separates C and R. It can be done in time $O(k|E(G)|)$ using k iterations of the Ford-Fulkerson algorithm.

Step 2: The compression problem

Given a graph G, an integer k, and a set S' of $k+1$ vertices such that $G \setminus S'$ is bipartite, find a set S of k vertices such that $G \setminus S$ is bipartite.



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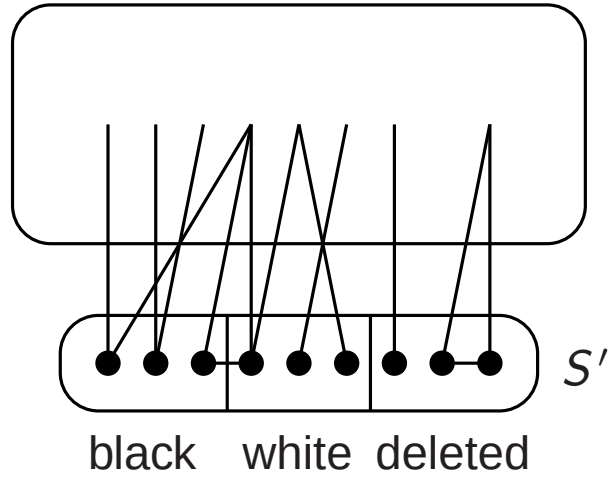

Branch into 3^{k+1} cases: each vertex of S' is either black, white, or deleted.

Trivial check: no edge between two black or two white vertices.

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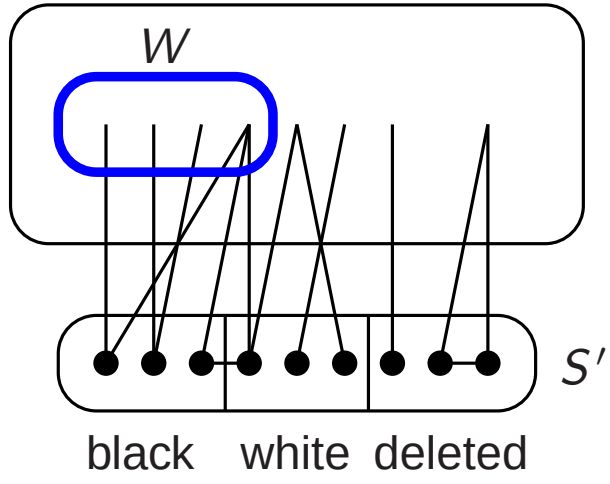

Branch into 3^{k+1} cases: each vertex of S' is either black, white, or deleted.

Trivial check: no edge between two black or two white vertices.

Neighbors of the black vertices in S' should be white and the neighbors of the white vertices in S' should be black.

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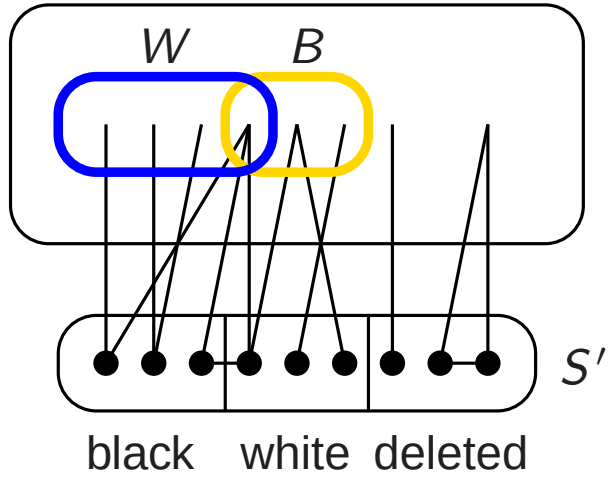

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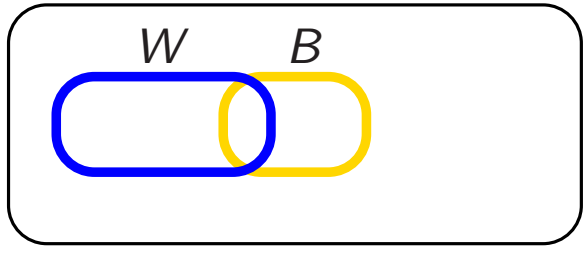

The vertices of S' can be disregarded. Thus we need to solve the annotated problem on the bipartite graph $G \setminus S'$.

Running time: $O(3^k \cdot k|E(G)|)$ time.

Step 3: Iterative compression

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Dirty trick : we get it for free!

Let $V(G) = \{v_1, \ldots, v_n\}$ and let G_i be the graph induced by $\{v_1, \ldots, v_i\}$.

For every i, we find a set S_i of size k such that $G_i \setminus S_i$ is bipartite.

- For G_k, the set $S_k = \{v_1, \ldots, v_k\}$ is a trivial solution.
- If S_{i-1} is known, then $S_{i-1} \cup \{v_i\}$ is a set of size $k+1$ whose deletion makes G_i bipartite $\Rightarrow$ We can use the compression algorithm to find a suitable S_i in time $O(3^k \cdot k|E(G_i)|)$.

Step 3: Iterative Compression

Bipartite-Deletion(G, k)

1. $S_k = \{v_1, \ldots, v_k\}$
2. for $i := k + 1$ to n
3. Invariant: $G_{i-1} \setminus S_{i-1}$ is bipartite.
4. Call Compression$(G_i, S_{i-1} \cup \{v_i\})$
5. If the answer is “NO” $\Rightarrow$ return “NO”
6. If the answer is a set $X \Rightarrow S_i := X$
7. Return the set S_n

Running time: the compression algorithm is called n times and everything else can be done in linear time

$\Rightarrow O(3^k \cdot k|V(G)| \cdot |E(G)|)$ time algorithm.

Almost 2-SAT

BIPARTITE EDGE DELETION: Delete k edges to make the graph bipartite.

Fact: BIPARTITE EDGE DELETION can be solved in time $O(2^k \cdot k|V(G)| \cdot |E(G)|)$.

BIPARTITE EDGE DELETION can be seen as a CSP problem:

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More generally:

ALMOST 2-SAT: Given a 2SAT formula and an integer k, find an assignment that satisfies all but at most k clauses.

Fact: [O'Sullivan and Razgon 2008] ALMOST 2-SAT is FPT.

Next: A simplified $7^k \cdot n^{O(1)}$ time algorithm.

Almost 2-Sat

Solution based on iterative compression:

- **Step 1:**

  Solve the **annotated problem** for 0-valid formulas

  Given a 0-valid formula φ, two sets V_0, V_1 of variables, and an integer k, find a set S of at most k clauses such that $\varphi \setminus S$ has a satisfying assignment where V_0 is 0 and V_1 is 1.

- **Step 2:**

  Solve the **compression problem** for general formulas:

  Given a formula φ, an integer k, and a set S' of $k+1$ clauses such that $G \setminus S$ is satisfiable, find a set S of k clauses such that $G \setminus S$ is satisfiable.

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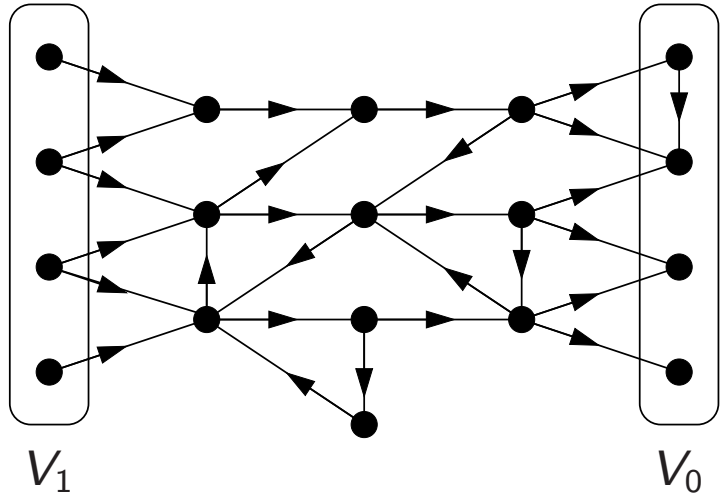
Given a 0-valid formula φ, two sets V_0, V_1 of variables, and an integer k, find a set S of at most k clauses such that $\varphi \setminus S$ has a satisfying assignment where V_0 is 0 and V_1 is 1.

If φ is 0-valid, then clauses are of form $x \rightarrow y$ and $\overline{x} \vee \overline{y}$. Graph of implications:



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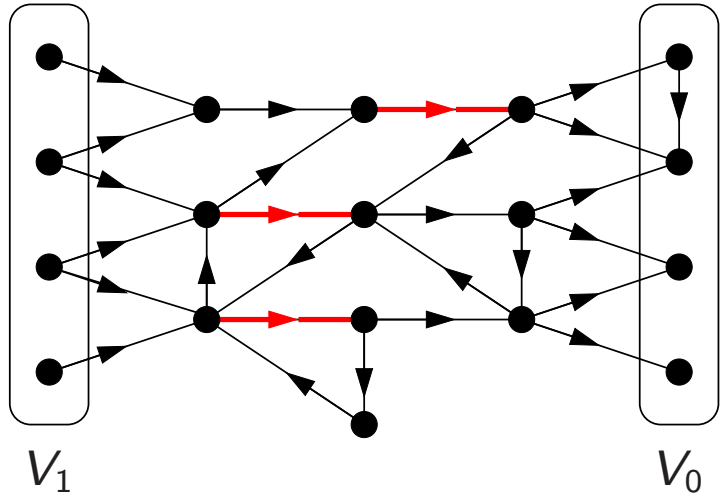
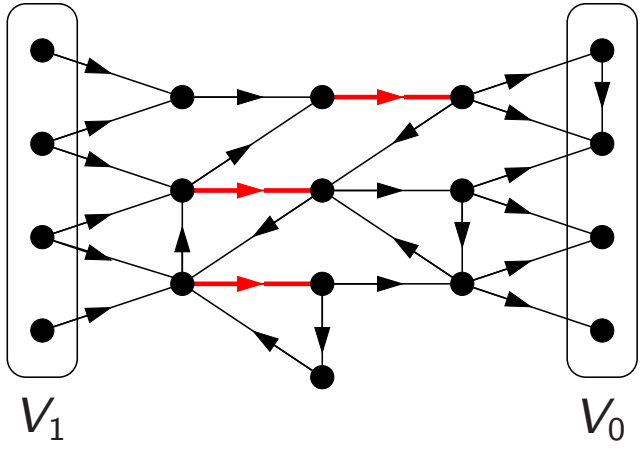


Observation: Any solution S is a (V_1, V_0)-cut (necessary, but not sufficient).

Step 1: The annotated problem

Let λ be the size of the minimum (V_1, V_0)-cut.

Fact: There is a unique cut of size λ such that the set of vertices reachable from V_1 has to be reachable in every cut of size λ.



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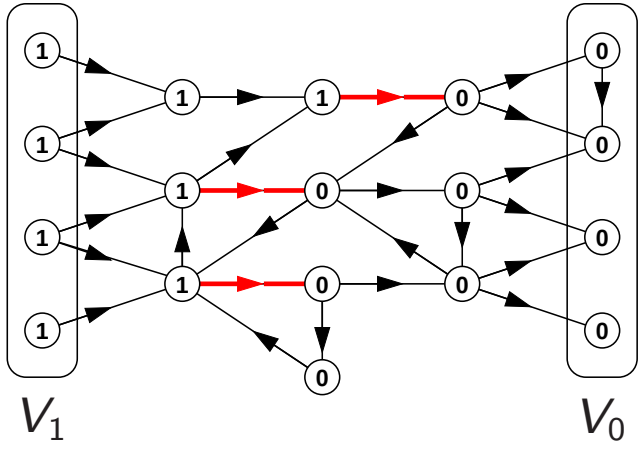

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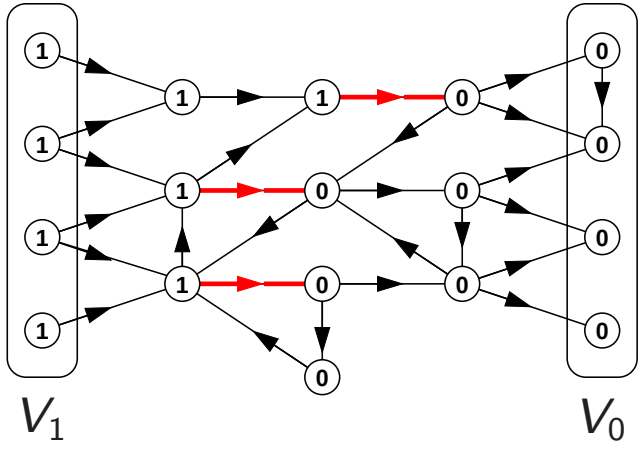

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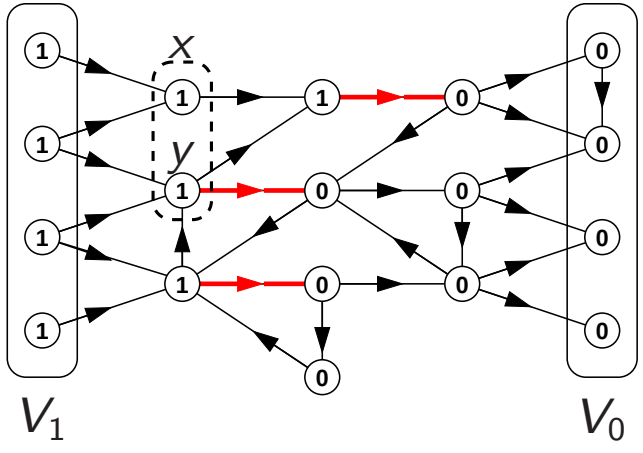

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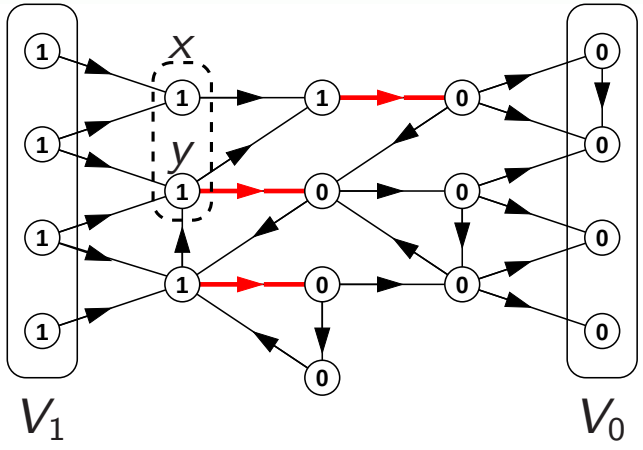

If this assignment satisfies every $\overline{x} \vee \overline{y}$ clause, then we are done. Otherwise we branch into 3 directions.

1. Remove the clause $\overline{x} \vee \overline{y}$ and decrease k.
2. Put x into V_0 (increases λ).
3. Put y into V_0 (increases λ).

We can branch at most k times: λ cannot be larger than k

$\Rightarrow O(3^k \cdot m)$ time algorithm for the annotated problem.

Step 2: The compression problem

Given a formula φ, an integer k, and a set S' of $k+1$ clauses such that $\varphi \setminus S'$ is satisfiable, find a set S of k clauses such that $\varphi \setminus S'$ is satisfiable.

Small trick: by negating variables, we can assume that $\varphi \setminus S'$ is 0-valid.

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- Let V be the variables involved in S'. Let us guess the values of these variables in the solution ($\leq 2^{2(k+1)}$ possibilities).
- Suppose that $V = V_0 \cup V_1$. Remove S' and decrease k by the number of clauses in S' that are unsatisfied by (V_0, V_1).
- Task: find a solution of $\varphi \setminus S'$ that is compatible with V_0 and V_1
  $\Rightarrow$ can be solved in time $O(3^k \cdot k \cdot m)$ by the annotated problem.

Running time: $O(2^{2(k+1)} \cdot 3^k \cdot k \cdot m) = O(12^k \cdot k \cdot m)$.

Slightly better analysis gives a $O(7^k \cdot k \cdot m)$ bound.

Step 3: Iterative compression

We get for free the solution of size $k+1$.

Let $C_1, \ldots, C_m$ be the clauses and let φ_i be the formula containing only the first i clauses.

For every i, we find a set S_i of size k such that $\varphi_i \setminus S_i$ is satisfiable.

- For φ_k, the set $S_k = \{C_1, \ldots, C_k\}$ is a trivial solution.
- If S_{i-1} is known, then $S_{i-1} \cup \{C_i\}$ is a set of size $k+1$ whose deletion makes φ_i satisfiable $\Rightarrow$ We can use the compression algorithm to find a suitable S_i in time $O(7^k \cdot km)$.

Running time: $O(7^k \cdot k \cdot m^2)$ for a formula with m clauses.

Kernelization

Kernelization for VERTEX COVER

General strategy: We devise a list of reduction rules, and show that if none of the rules can be applied and the size of the instance is still larger than $s(k)$, then the answer is trivial.

$\Rightarrow$ We can assume that the instance is of size at most $s(k)$ and can solved by brute force $\Rightarrow$ FPT parameterized by k.

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Reduction rules for VERTEX COVER instance (G, k):

Rule 1: If v is an isolated vertex $\Rightarrow (G \setminus v, k)$

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If neither Rule 1 nor Rule 2 can be applied:

- If $|V(G)| > k(k+1) \Rightarrow$ There is no solution (every vertex should be the neighbor of at least one vertex of the cover).
- Otherwise, $|V(G)| \leq k(k+1)$ and we have a $k(k+1)$ vertex kernel.

Kernelization

Definition: **Kernelization** is a polynomial-time transformation that maps an instance (I, k) to an instance (I', k') such that

- (I, k) is a yes-instance if and only if (I', k') is a yes-instance,
- $k' \leq k$, and
- $|I'| \leq f(k)$ for some function $f(k)$.

In other words: a polynomial-time preprocessing technique with some provable guarantee.

We are especially interested in kernelizations where $f(k)$ is polynomial in k: which problems have **polynomial kernels?**

Kernelization for MIN ONES SAT

We have seen that VERTEX COVER is a special case of MIN ONES SAT and MIN ONES SAT(Γ) is FPT for every Γ.

Do we have a polynomial kernelization for MIN ONES SAT(Γ) for every Γ?

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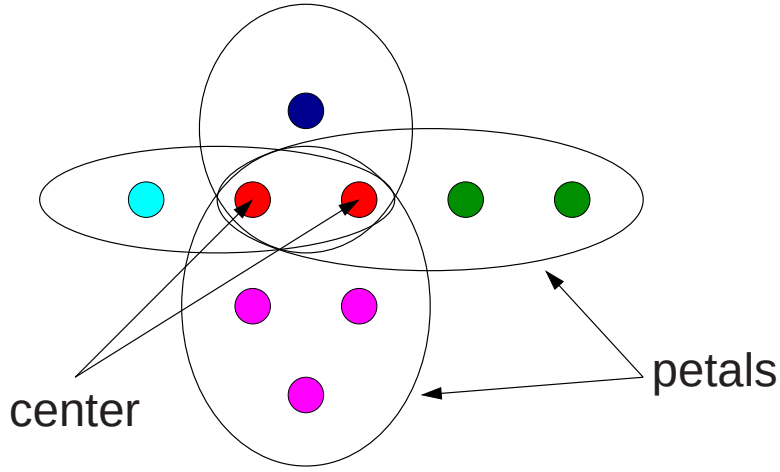
First another special case:

d-HITTING SET: Given a collection $\mathcal{S}$ of sets of size at most d and an integer k, find a set S of k elements that intersects every set of $\mathcal{S}$.

Fact: d-HITTING SET has a polynomial kernel for every fixed d.

Sunflower lemma

Definition: Sets $S_1, S_2, \ldots, S_k$ form a **sunflower** if the sets $S_i \setminus (S_1 \cap S_2 \cap \cdots \cap S_k)$ are disjoint.



Lemma: [Erdős and Rado, 1960] If the size of a set system is greater than $(p-1)^d \cdot d!$ and it contains only sets of size at most d, then the system contains a sunflower with p petals. Furthermore, in this case such a sunflower can be found in polynomial time.

Sunflowers and d-HITTING SET

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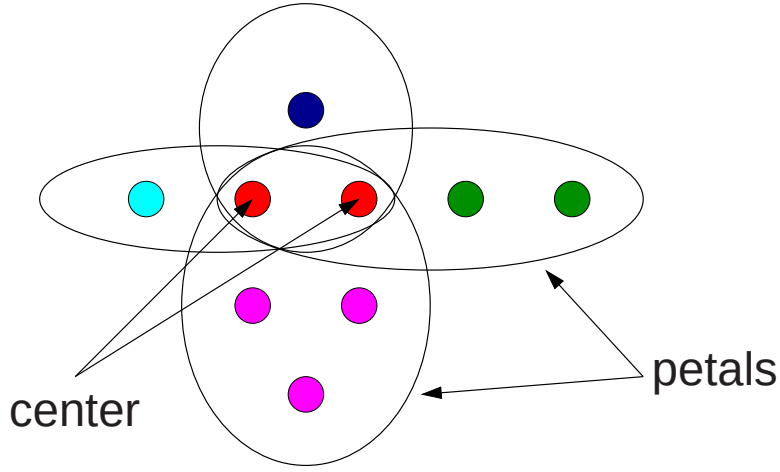

Reduction Rule: If $k+1$ sets form a sunflower, then remove these sets from $\mathcal{S}$ and add the center C to $\mathcal{S}$ (S does not hit one of the petals, thus it has to hit the center).

Note: if the center is empty (the sets are disjoint), then there is no solution.

If the rule cannot be applied, then there are at most $O(k^d)$ sets.

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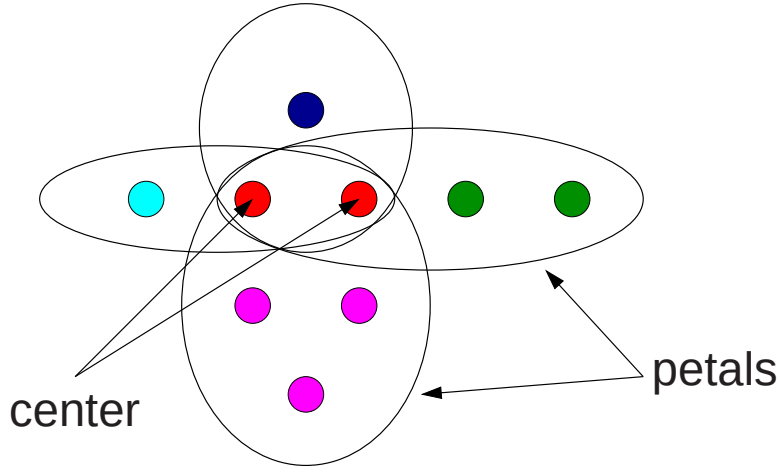

Reduction Rule (variant): Suppose more than $k+1$ sets form a sunflower.

- If the sets are disjoint $\Rightarrow$ No solution.
- Otherwise, keep only $k+1$ of the sets.

If the rule cannot be applied, then there are at most $O(k^d)$ sets.

Kernelization for MIN ONES SAT

The story very briefly:

- We have seen that MIN ONES SAT(Γ) is FPT for every Γ.
- Not every FPT problem admits a polynomial kernel and there is a technology for proving that (modulo some complexity assumption).
- Some cases of MIN ONES SAT(Γ), including d-HITTING SET, have polynomial kernels.
- MIN ONES SAT(Γ) has a polynomial kernel if and only if Γ is mergable [Kratsch and Wahlström 2010].

Lots of possibilities for Schaefer-style dichotomy theorems in parameterized complexity!

Conclusions

- Nice techniques for fixed parameter algorithms.
- Lots of interesting NP-hard problems are FPT.
- Bad news: W[1]-hardness theory.
- New kind of questions.