Finding topological subgraphs is fixed-parameter tractable

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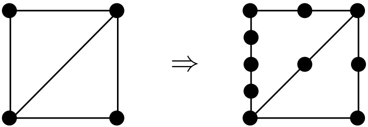
Topological subgraphs

Definition

Subdivision of a graph: replacing each edge by a path of length 1 or more.

Graph H is a **topological subgraph** of G (or **topological minor** of G, or $H \leq_T G$) if a subdivision of H is a subgraph of G.



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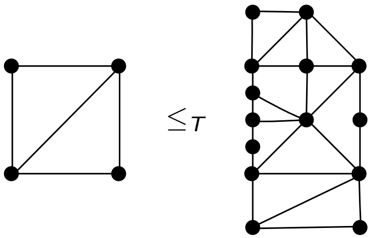

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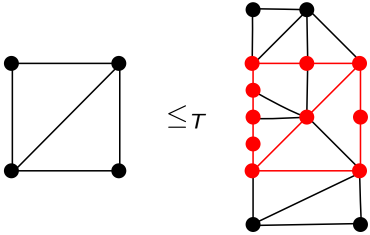

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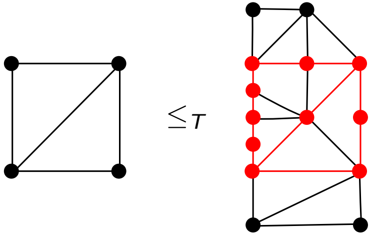
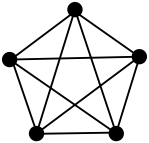


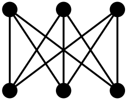
Equivalently, $H \leq_T G$ means that H can be obtained from G by removing vertices, removing edges, and dissolving degree two vertices.

A classical result

Theorem [Kuratowski 1930]

A graph G is planar if and only if $K_5 \not\leq_T G$ and $K_{3,3} \not\leq_T G$.





Algorithms

Theorem [Robertson and Seymour]

Given graphs H and G, it can be tested in time $f(|V(H)|) \cdot |V(G)|^{O(V(H))}$ if $H \leq_T G$ (for some function f).

$\Rightarrow$ Polynomial-time algorithm for every fixed H.

Main result

Given graphs H and G, it can be tested in time $f(|V(H)|) \cdot |V(G)|^3$ if $H \leq_T G$ (for some function f).

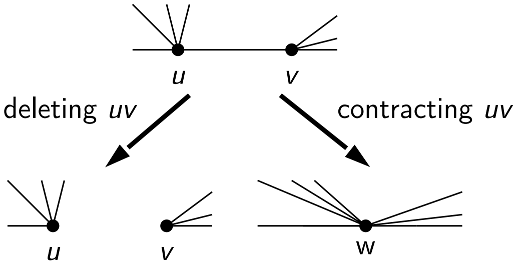
$\Rightarrow$ Cubic algorithm for every fixed H.
$\Rightarrow$ Topological subgraph testing is fixed-parameter tractable.

Answers an open question of [Downey and Fellows 1992].

Minors

Definition

Graph H is a minor G ($H \leq G$) if H can be obtained from G by deleting edges, deleting vertices, and contracting edges.



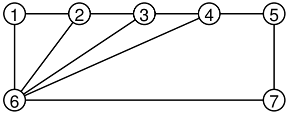
Note: $H \leq_T G \Rightarrow H \leq G$, but the converse is not necessarily true.

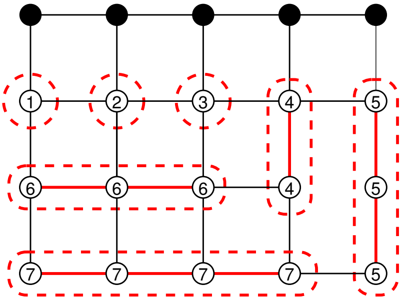
Minors

Equivalent definition

Graph H is a minor of G if there is a mapping ϕ (the minor model) that maps each vertex of H to a connected subset of G such that

- $\phi(u)$ and $\phi(v)$ are disjoint if $u \neq v$, and
- if $uv \in E(G)$, then there is an edge between $\phi(u)$ and $\phi(v)$.





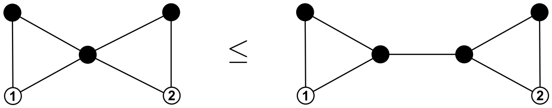
Algorithm for minor testing

Theorem [Robertson and Seymour]

Given graphs H and G, it can be tested in time $f(|V(H)|) \cdot |V(G)|^3$ if $H \le G$ (for some function f).

In fact, they solve a more general rooted problem:

- H has a special set $R(H)$ of vertices (the roots),
- for every $v \in R(H)$, a vertex $\rho(v) \in V(G)$ is specified, and
- the model ϕ should satisfy $\rho(v) \in \phi(v)$.



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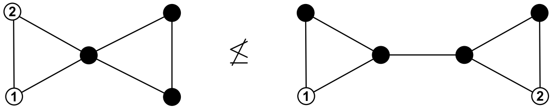


Algorithm for minor testing

Special case of rooted minor testing: k-Disjoint Paths problem (connect $(s_1, t_1), \ldots, (s_k, t_k)$ with vertex-disjoint paths).

Corollary [Robertson and Seymour]

k-Disjoint Paths can be solved in time $f(k) \cdot |V(G)|^3$.

By guessing the image of every vertex of H, we get:

Corollary [Robertson and Seymour]

Given graphs H and G, it can be tested in time $f(k) \cdot |V(G)|^{O(V(H))}$ if H is a topological subgraph of G.

Algorithm for minor testing

A vertex $v \in V(G)$ is irrelevant if its removal does not change if $H \leq G$.

Ingredients of minor testing by [Robertson and Seymour]

1. Solve the problem on bounded-treewidth graphs.
2. If treewidth is large, either find an irrelevant vertex or the model of a large clique minor.
3. If we have a large clique minor, then either we are done (if the clique minor is "close" to the roots), or a vertex of the clique minor is irrelevant.

By iteratively removing irrelevant vertices, eventually we arrive to a graph of bounded treewidth.

Algorithm for minor testing

A vertex $v \in V(G)$ is irrelevant if its removal does not change if $H \leq G$.

Ingredients of minor testing by [Robertson and Seymour]

1. Solve the problem on bounded-treewidth graphs.
   By now, standard (e.g., Courcelle's Theorem).
2. If treewidth is large, either find an irrelevant vertex or the model of a large clique minor.
   Really difficult part (even after the significant simplifications of [Kawarabayashi and Wollan STOC 2010]).
3. If we have a large clique minor, then either we are done (if the clique minor is "close" to the roots), or a vertex of the clique minor is irrelevant.
   Idea is to use the clique model as a "crossbar switch."

By iteratively removing irrelevant vertices, eventually we arrive to a graph of bounded treewidth.

Algorithm for topological subgraphs

1. Solve the problem on bounded-treewidth graphs.
   No problem!
2. If treewidth is large, either find an irrelevant vertex or the model of a large clique minor.
   Painful, but the techniques of Kawarabayashi-Wollan go though.
3. If we have a large clique minor, then either we are done (if the clique minor is "close" to the roots), or a vertex of the clique minor is irrelevant.
   Approach completely fails: a large clique minor does not help in finding a topological subgraph if the degrees are not good.

Note: we solve a more general rooted version of topological subgraph testing.

Ideas

New ideas:

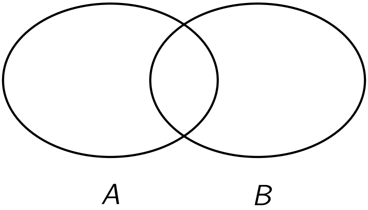
- Idea #1: Recursion and replacement on small separators.
- Idea #2: Reduction to bounded-degree graphs (high degree vertices + clique minor: topological clique).
- Idea #3: Solution for the bounded-degree case (distant vertices do not interfere).

Additionally, we are using a tool of Robertson and Seymour:

- Using a clique minor as a "crossbar switch."

Idea #1: Recursion and replacement

Suppose we have found a “small” separator such that both sides are “large.” We recursively “understand” the properties of one side, and replace it with a smaller “equivalent” graph.



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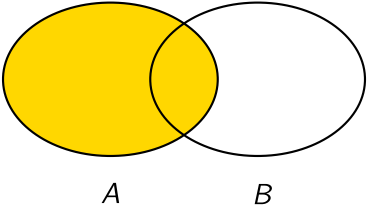

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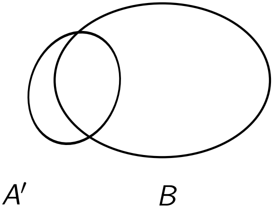

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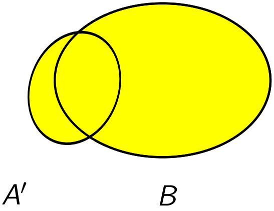

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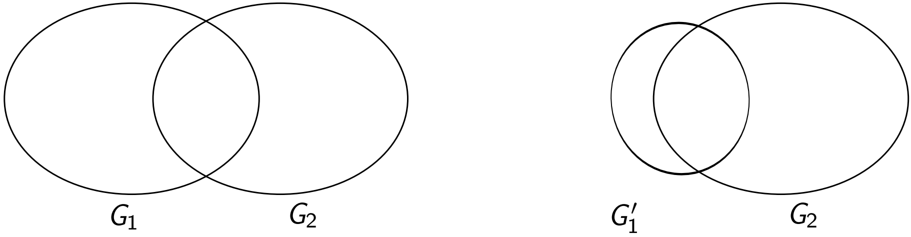

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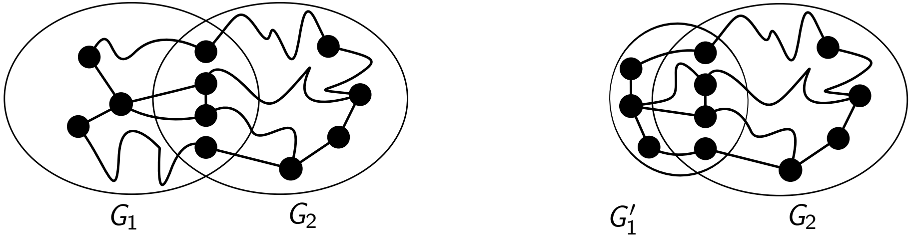

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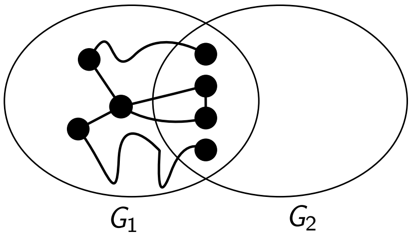

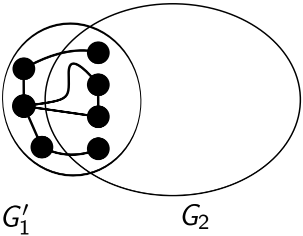

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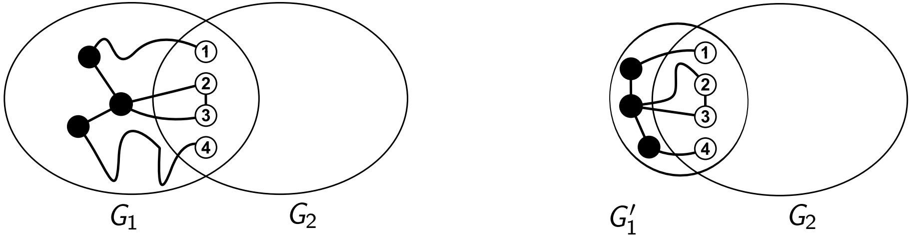

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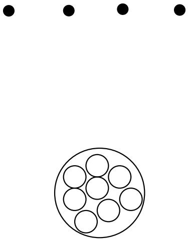
Note: we need rooted topological subgraphs to express this, therefore we solve the more general Rooted Topological Subgraph problem.

Idea #2: Reduction to bounded degree

Suppose that there is a set S of $|V(H)|$ vertices with huge degree.



Two possibilities:

(1) There are many disjoint paths from S to the clique minor
$\Rightarrow$ Using the clique minor as a crossbar, we can complete the paths into a topological subgraph.

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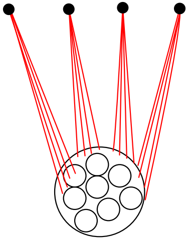

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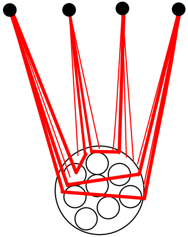

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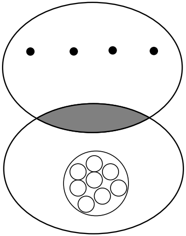

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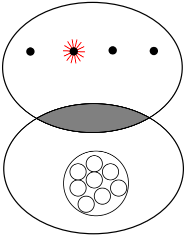

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Idea #3: The bounded-degree case

Suppose we are looking for a 4-regular graph H as topological subgraph.

- If there are only few vertices of degree ≥ 4
  $\Rightarrow$ We can guess the images of the vertices and use the disjoint paths algorithm.
- If there are many vertices if degree ≥ 4, then we can select a set S of $|V(H)|$ vertices of degree ≥ 4 that are very very far from each other (because the graph has bounded degree).

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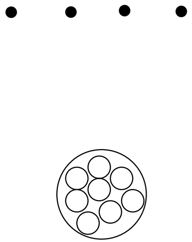

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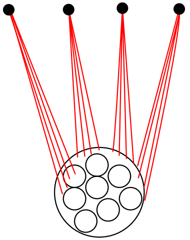

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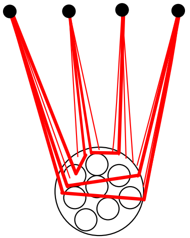

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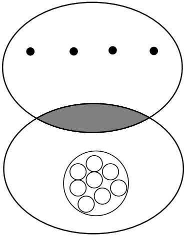

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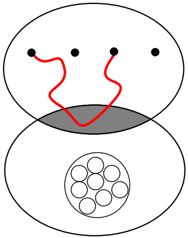

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A variant of topological subgraphs:

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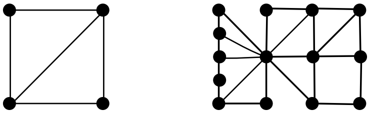
Immersion: The edges of H correspond to edge-disjoint paths between the images of the vertices in G.



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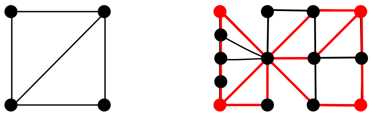

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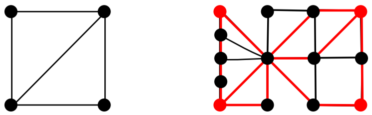

Theorem

Given graphs H and G, it can be tested in time $f(|V(H)|) \cdot |V(G)|^3$ if H has an immersion in G (for some function f).

An elementary reduction from immersion to topological subgraph testing.

Conclusions

- Main result: topological subgraph testing is FPT.
- Immersion testing follows as a corollary.
- Main new part: what to do with a large clique minor?
- Very roughly: large clique minor + vertices of the correct degree = topological subgraph.
- Recursion, high-degree vertices.