Finding topological subgraphs is fixed-parameter tractable

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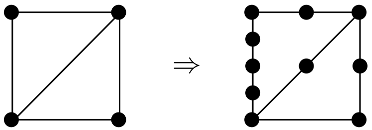
Topological subgraphs

Definition

Subdivision of a graph: replacing each edge by a path of length 1 or more.

Graph H is a topological subgraph of G (or topological minor of G, or $H \leq_T G$) if a subdivision of H is a subgraph of G.



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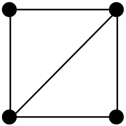

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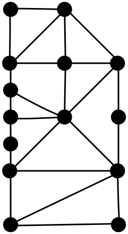

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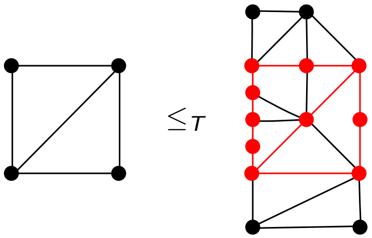

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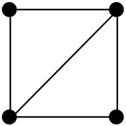

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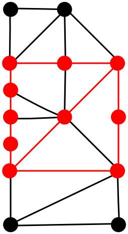
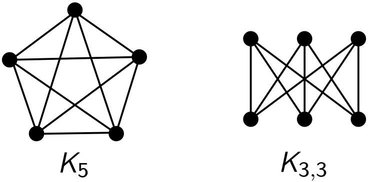


Equivalently, H is a topological subgraph of G if H can be obtained from G by removing vertices, removing edges, and dissolving degree two vertices.

Some combinatorial results

Theorem [Kuratowski 1930]

A graph G is planar if and only if $K_5 \not\leq_T G$ and $K_{3,3} \not\leq_T G$.



Theorem [Mader 1972]

For every graph H there is a constant c_H such that $H \leq_T G$ for every graph G with average degree at least c_H.

Algorithms

Theorem [Robertson and Seymour]

Given graphs H and G, it can be tested in time $|V(G)|^{O(V(H))}$ if $H \leq_T G$.

Main result

Given graphs H and G, it can be tested in time $f(|V(H)|) \cdot |V(G)|^3$ if $H \leq_T G$ (for some computable function f).

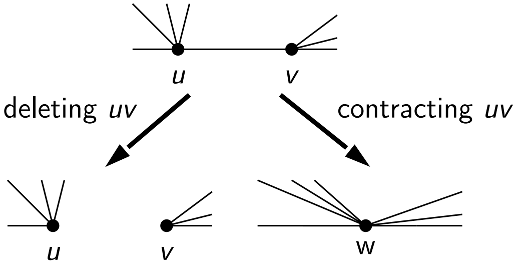
$\Rightarrow$ Topological subgraph testing is fixed-parameter tractable.

Answers an open question of [Downey and Fellows 1992].

Minors

Definition

Graph H is a minor G ($H \leq G$) if H can be obtained from G by deleting edges, deleting vertices, and contracting edges.



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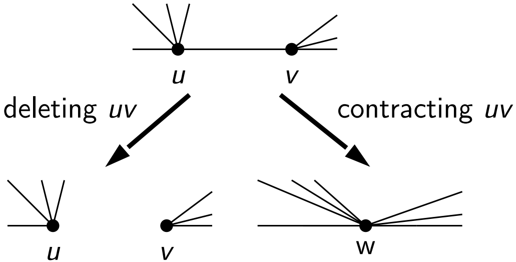


Note: $H \leq_T G \Rightarrow H \leq G$, but the converse is not necessarily true.

Theorem: [Wagner 1937]

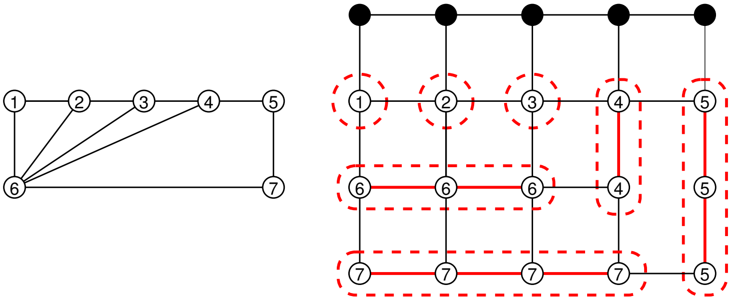
A graph G is planar if and only if $K_5 \not\leq G$ and $K_{3,3} \not\leq G$.

Minors

Equivalent definition

Graph H is a minor of G if there is a mapping ϕ (the minor model) that maps each vertex of H to a connected subset of G such that

- $\phi(u)$ and $\phi(v)$ are disjoint if $u \neq v$, and
- if $uv \in E(G)$, then there is an edge between $\phi(u)$ and $\phi(v)$.



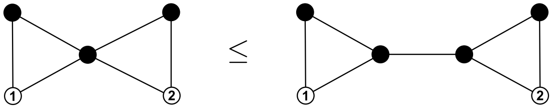
Algorithm for minor testing

Theorem [Robertson and Seymour]

Given graphs H and G, it can be tested in time $f(|V(H)|) \cdot |V(G)|^3$ if $H \leq G$ (for some computable function f).

In fact, they solve a more general rooted problem:

- H has a special set $R(H)$ of vertices (the roots),
- for every $v \in R(H)$, a vertex $\rho(v) \in V(G)$ is specified, and
- the model ϕ should satisfy $\rho(v) \in \phi(v)$.



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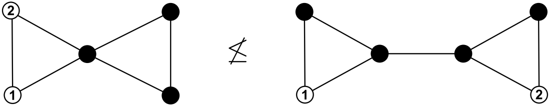


Algorithm for minor testing

Special case of rooted minor testing: k-Disjoint Paths problem (connect $(s_1, t_1), \dots, (s_k, t_k)$ with vertex-disjoint paths).

Corollary [Robertson and Seymour]

k-Disjoint Paths is FPT.

By guessing the image of every vertex of H, we get:

Corollary [Robertson and Seymour]

Given graphs H and G, it can be tested in time $|V(G)|^{O(V(H))}$ if H is a topological subgraph of G.

Algorithm for minor testing

A vertex $v \in V(G)$ is irrelevant if its removal does not change if $H \leq G$.

Ingredients of minor testing by [Robertson and Seymour]

1. Solve the problem on bounded-treewidth graphs.
2. If treewidth is large, either find an irrelevant vertex or the model of a large clique minor.
3. If we have a large clique minor, then either we are done (if the clique minor is "close" to the roots), or a vertex of the clique minor is irrelevant.

By iteratively removing irrelevant vertices, eventually we arrive to a graph of bounded treewidth.

Algorithm for minor testing

A vertex $v \in V(G)$ is irrelevant if its removal does not change if $H \leq G$.

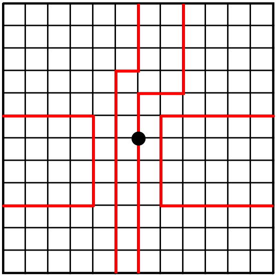
Ingredients of minor testing by [Robertson and Seymour]

1. Solve the problem on bounded-treewidth graphs.
   By now, standard (e.g., Courcelle's Theorem).
2. If treewidth is large, either find an irrelevant vertex or the model of a large clique minor.
   Really difficult part (even after the significant simplifications of [Kawarabayashi and Wollan 2010]).
3. If we have a large clique minor, then either we are done (if the clique minor is "close" to the roots), or a vertex of the clique minor is irrelevant.
   Idea is to use the clique model as a "crossbar."

By iteratively removing irrelevant vertices, eventually we arrive to a graph of bounded treewidth.

Sketch of Step 2 (very simplified!)

- The Graph Minor Theorem says that if G excludes a K_ℓ minor for some ℓ, then G is almost like a graph embeddable on some surface.
  $\Rightarrow$ Assume now that G is planar.
- The Excluded Grid Theorem says that if G has large treewidth, then G has a large grid/wall minor.
  $\Rightarrow$ Assume that G has a large grid far away from all the roots.
- The middle vertex of the grid is irrelevant: we can surely reroute any solution using it.



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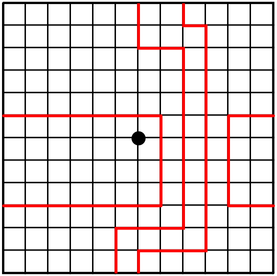


Algorithm for topological subgraphs

1. Solve the problem on bounded-treewidth graphs.
   No problem!
2. If treewidth is large, either find an irrelevant vertex or the model of a large clique minor.
   Painful, but the techniques of Kawarabayashi-Wollan go though.
3. If we have a large clique minor, then either we are done (if the clique minor is "close" to the roots), or a vertex of the clique minor is irrelevant.
   Approach completely fails: a large clique minor does not help in finding a topological subgraph if the degrees are not good.

Ideas

New ideas:

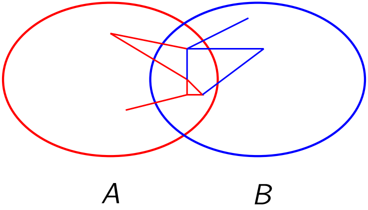
- Idea #1: Recursion and replacement on small separators.
- Idea #2: Reduction to bounded-degree graphs
  (high degree vertices + clique minor: topological clique).
- Idea #3: Solution for the bounded-degree case
  (distant vertices do not interfere).

Additionally, we are using a tool of Robertson and Seymour:

- Using a clique minor as a "crossbar."

Separations

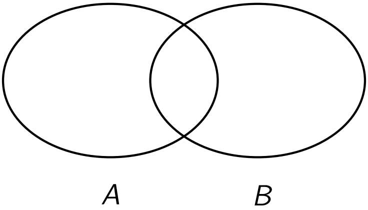
- A separation of a graph G is a pair (A, B) of subgraphs such that $V(G) = V(A) \cup V(B)$, $E(G) = E(A) \cup E(B)$, and $E(A) \cap E(B) = \emptyset$.
- The *order* of the separation (A, B) is $|V(A) \cap V(B)|$.
- The set $V(A) \cap V(B)$ is the separator.



Recursion

Idea #1: Recursion and replacement on small separators.

Suppose we have found a separation of "small" order such that both sides are "large." We recursively "understand" the properties of one side, and replace it with a smaller "equivalent" graph.



What do "small", "large", "understand," and "equivalent" mean exactly?

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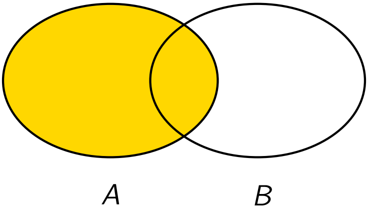

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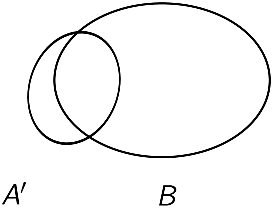

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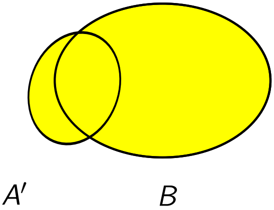

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Formal definitions

A rooted graph G has a set $R(G) \subseteq V(G)$ of roots and an injective mapping $\rho_G : R(G) \to \mathbb{N}$ of root number.

H is a topological minor of rooted graph G if there is a mapping ψ (a *model* of H in G) that assigns to each $v \in V(H)$ a vertex $\psi(v) \in V(G)$ and to each $e \in E(H)$ a path $\psi(e)$ in G such that

1. The vertices $\psi(v)$ ($v \in V(H)$) are distinct.
2. If $u, v \in V(H)$ are the endpoints of $e \in E(H)$, then path $\psi(e)$ connects $\psi(u)$ and $\psi(v)$.
3. The paths $\psi(e)$ ($e \in E(H)$) are pairwise internally vertex disjoint, i.e., the internal vertices of $\psi(e)$ do not appear as an (internal or end) vertex of $\psi(e')$ for any $e' \neq e$.
4. For every $v \in R(H)$, $\rho_G(\psi(v)) = \rho_H(v)$.

Folios

- The folio of rooted graph G is the set of all topological minors of G.
- The δ-folio of G contains every topological minor H of G with $|E(H)| + \text{number-of-isolated-vertices}(H) \leq \delta$.
- Observation: The number of distinct graphs (up to isomorphism) in the δ-folio of G can be bounded by a function of δ and $|R(G)|$.
- Extended δ-folio: for every set X of edges on $R(G)$, it contains the δ-folio of $G + X$ (so the extended δ-folio is a tuple of $2^{\binom{|R(G)|}{2}}$ folios).

Main result (more general version)

The extended δ-folio of G can be computed in time $f(\delta, |R(G)|) \cdot |V(G)|^3$.

Algorithms

FindFolio(G, δ)

Returns the extended δ-folio of G.

FindIrrelevantOrSeparation(G, δ)

Returns either

- the extended δ-folio of G, or
- a vertex v irrelevant to the extended δ-folio, or
- a separation (G_1, G_2) of "small" order with both sides "large".

FindIrrelevantOrClique(G, δ)

Returns either

- the extended δ-folio of G, or
- a vertex v irrelevant to the extended δ-folio, or
- a model of a "large" clique minor.

Algorithms

FindFolio(G, δ)

$\Uparrow$

Recursion and replacement.

$\Uparrow$

FindIrrelevantOrSeparation(G, δ)

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Using the clique as a crossbar, reducing the degree

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Graph structure theory along the lines of [Kawarabayashi-Wollan 2010].

Algorithms

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Using the clique as a crossbar, reducing the degree

$\Leftarrow$ FindFolio$(G, \delta - 1)$

$\Uparrow$

FindIrrelevantOrClique(G, δ)

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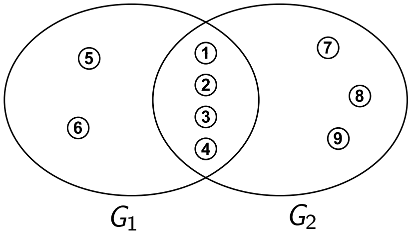
Graph structure theory along the lines of [Kawarabayashi-Wollan 2010].

Folios and replacement

Lemma: Let (G_1, G_2) be a separation of G such that

- $V(G_1) \cap V(G_2) \subseteq R(G)$,
- G_1' is a graph having the same root numbers as G_1, and
- G_1 and G_1' have the same extended δ-folio.

If we replace G_1 with G_1' in the separation (G_1, G_2), then the new graph has the same extended δ-folio as G.



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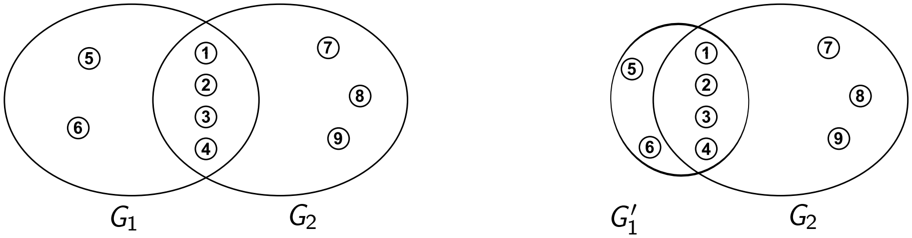

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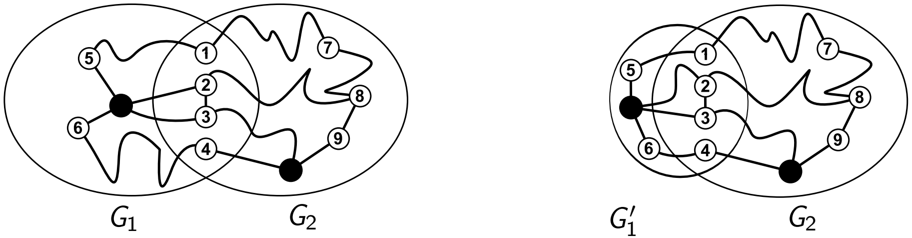

FindFolio(G, δ)

Notes:

- Small separator: $\leq \delta^2$
- We work with graphs having at most $2\delta^2$ roots.
- A graph with at most $2\delta^2$ roots is large if there is a smaller graph with the same extended δ-folio.

FindFolio(G, δ)

Notes:

- Small separator: $\leq \delta^2$
- We work with graphs having at most $2\delta^2$ roots.
- A graph with at most $2\delta^2$ roots is large if there is a smaller graph with the same extended δ-folio.

Algorithm FindFolio(G, δ):

- Call FindIrrelevantOrSeparation(G, δ)
  - If it returns the extended δ-folio: return it.
  - If it returns an irrelevant vertex v: return FindFolio($G \setminus v, \delta$).
  - If it returns a separation (G_1, G_2) of G having order $\leq \delta^2$ and with both sides large:
    1. Assume $|R(G_1)| \leq |R(G_2)|$.
    2. Make $S := V(G_1) \cap V(G_2)$ roots in $G_1 \Rightarrow G_1^+$ (note $|R(G_1^+)| \leq 2\delta^2$).
    3. FindFolio(G_1^+, δ).
    4. Let G_1' be the smallest graph having the same extended δ-folio as G_1^+.
    5. Replace G_1 with G_1' in $(G_1, G_2) \Rightarrow G'$.
    6. return FindFolio(G', δ).

FindIrrelevantOrSeparation(G, δ)

First we use FindIrrelevantOrClique(G, δ) to find a large clique minor.

The idea is that the clique minor makes realizing a topological subgraph easy, if we have vertices whose degrees are suitable.

Two cases:

1. Case 1: Many ($\geq 2\delta$) vertices with large degree.
2. Case 2: Few vertices vertices with large degree.

Clique minor as a crossbar

Definition

We say that $Z \subseteq V(G)$ is well-attached to a k-clique minor model ϕ, if there is no separation (G_1, G_2) of order $< |Z|$ with $Z \subseteq V(G_1)$ and $\phi(v) \cap V(G_1) = \emptyset$ for some vertex v of the k-clique.

Lemma [Robertson-Seymour, GM13]

Let Z be a set that is well-attached to a k-clique minor with $k \geq \frac{3}{2}|Z|$. Then for every partition $(Z_1, \ldots, Z_n)$ of Z, there are pairwise disjoint connected sets $T_1, \ldots, T_n$ with $T_i \cap Z = Z_i$.

Clique minor as a crossbar — weighted version

Definition

Let $Z \subseteq V(G)$ be a set and $w : V(G) \to \mathbb{Z}^+$ be a function such that $w(v) = 1$ for every $v \notin Z$. We say that $Z \subseteq V(G)$ is well-attached to a k-clique minor model ϕ, if there is no separation (G_1, G_2) with $w(V(G_1) \cap V(G_2)) < w(Z)$, $Z \subseteq V(G_1)$, and $\phi(v) \cap V(G_1) = \emptyset$ for some vertex v of the k-clique.

Lemma

Let Z be a set that is well-attached to a k-clique minor with $k \geq \frac{3}{2}w(Z)$. Then for every H and injective mapping $\psi : V(H) \to Z$ with $w(\psi(v)) \geq d(v)$ for every $v \in V(H)$, mapping ψ can be extended to a topological minor model.

Clique minor as a crossbar — weighted version

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d-attached: well-attached for $w(v) = d$ for $v \in Z$.

Corollary

If Z is a set of δ vertices having degree $\geq \delta$ such that Z is δ-attached to a k-clique minor with $k \geq \frac{3}{2}w(Z)$, then every graph with δ vertices is topological minor of G.

Case 1: Many high degree vertices

Idea #2: Reduction to bounded-degree graphs
(high degree vertices + clique minor: topological clique).

- Simpler case: assume for now that G has no roots.
- Let U be a set of 2δ vertices having "large" degree.
- If U is δ-attached to the clique model: the δ-folio of G contains every graph with at most δ edges and at most 2δ vertices!
- If there is a small separation (G_1, G_2) with $U \subseteq V(G_1)$ and $\phi(v) \cap V(G_1) = \emptyset$:
  - $V(G_1)$ is large, since it contains a high-degree vertex.
  - $V(G_2)$ is large, since it (mostly) contains the large clique minor.
  - We can return (G_1, G_2) as a good separation!

Case 2: Few high degree vertices

Idea #3: Solution for the bounded-degree case
(distant vertices do not interfere).

Assumptions:

- No roots and no vertices with large degree in G.
- H is (say) 9-regular and it has a model ψ where the branch vertices are at large distance from each other.

Case 2: Few high degree vertices

Idea #3: Solution for the bounded-degree case (distant vertices do not interfere).

Assumptions:

- No roots and no vertices with large degree in G.
- H is (say) 9-regular and it has a model ψ where the branch vertices are at large distance from each other.

Claim

Every branch vertex is 9-attached to the clique (or we find a separation).

- Suppose that there is a separation (G_1, G_2) of order < 9.
- G_1 contains at least two branch vertices $\Rightarrow$ G_1 is large.
- G_2 contains the clique minor $\Rightarrow$ G_2 is large.
- We can return the separation (G_1, G_2)!

Case 2: Few high degree vertices

Assumption: no high-degree vertices and no roots in G.

Claim

If there is a set Z of $|V(H)|$ 9-attached vertices that are at large distance from each other, then H has a model in G.

We prove that the set $Z = \{z_1, \ldots, z_{|V(H)|}\}$ itself is 9-attached.

- Suppose that there is a separation (G_1, G_2) of order $< 9|Z|$.
- Let S_i be the set of vertices in $V(G_1) \cap V(G_2)$ reachable from z_i.
- As z_i is 9-attached, $|S_i| \geq 9 \Rightarrow$ some S_i and S_j have to intersect.
- As the distance of z_i and z_j is large, G_1 is large $\Rightarrow$ we can return the separation (G_1, G_2)!

So we essentially need to find an independent set in a **bounded-degree** graph (easy).

Summary of ideas

New ideas:

- Idea #1: Recursion and replacement on small separators.
- Idea #2: Reduction to bounded-degree graphs
  (high degree vertices + clique minor: topological clique).
- Idea #3: Solution for the bounded-degree case
  (distant vertices do not interfere).

Additionally, we are using a tool of Robertson and Seymour:

- Using a clique minor as a "crossbar."

Immersion

Definition

An immersion of a graph H into graph G is a mapping ψ that assigns to each $v \in V(H)$ a vertex $\psi(v) \in V(G)$ and to each $e \in E(G)$ a path $\psi(e)$ in G such that

1. The vertices $\psi(v)$ ($v \in V(H)$) are distinct.
2. If $u, v \in V(H)$ are the endpoints of $e \in E(H)$, then path $\psi(e)$ connects $\psi(u)$ and $\psi(v)$.
3. The paths $\psi(e)$ ($e \in E(H)$) are pairwise **edge disjoint.**

Theorem

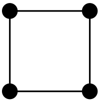
Given graphs H and G, it can be tested in time $f(|V(H)|) \cdot |V(G)|^3$ if H has an immersion in G (for some computable function f).

Similar result for **strong immersion:** $\psi(e)$ cannot go through any $\psi(v)$.

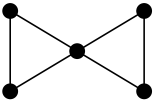
Immersion

Theorem: Given graphs H and G, it can be tested in time $f(|V(H)|) \cdot |V(G)|^3$ if H has an immersion in G.

- G' : subdivide edges of H and make $|E(H)|$ copies of each vertex.
- If H has an immersion in G, then H is a topological minor of G'.
- Converse is not true: a topological minor model in G' can use copies of the same vertex as branch vertices.



H



G

Immersion

Theorem: Given graphs H and G, it can be tested in time $f(|V(H)|) \cdot |V(G)|^3$ if H has an immersion in G.

- G' : subdivide edges of H and make $|E(H)|$ copies of each vertex.
- If H has an immersion in G, then H is a topological minor of G'.
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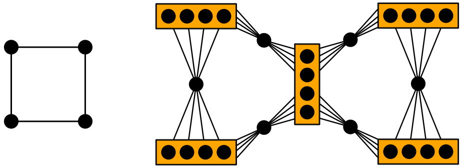

H G'

Immersion

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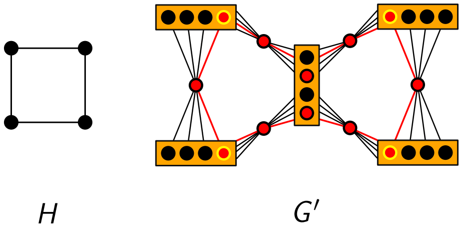

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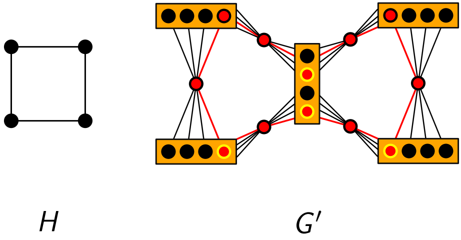

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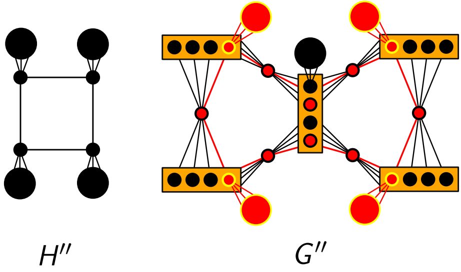
- If H has an immersion in G, then H is a topological minor of G'.
- Converse is not true: a topological minor model in G' can use copies of the same vertex as branch vertices.
- Fix:
  - If G has a large topological clique minor, then we are done.
  - Otherwise, decorate the vertices in H and G' with cliques.



Conclusions

- Main result: topological subgraph testing is FPT.
- Immersion testing follows as a corollary.
- Main new part: what to do with a large clique minor?
- Very roughly: large clique minor + vertices of the correct degree = topological minor.
- Recursion, high-degree vertices.