Complexity results for Minimum Sum Edge Coloring*

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Abstract

In the Minimum Sum Edge Coloring problem we have to assign positive integers to the edges of a graph such that adjacent edges receive different integers and the sum of the assigned numbers is minimal. We show that the problem is (a) NP-hard for planar bipartite graphs with maximum degree 3, (b) NP-hard for 3-regular planar graphs, (c) NP-hard for partial 2-trees, and (d) APX-hard for bipartite graphs.

1 Introduction

A *vertex coloring* of a graph is an assignment of colors to the vertices of a graph such that if two vertices are adjacent, then they are assigned different colors. In this paper, we assume that the colors are the positive intergers; a vertex k-coloring is a coloring where the color of each vertex is taken from the set $\{1, 2, \ldots, k\}$. Given a vertex coloring of a graph G, the *sum* of the coloring is the sum of the colors assigned to the vertices. The *chromatic sum* $\Sigma(G)$ of G is the smallest sum that can be achieved by any proper coloring of G. In the Minimum Sum Coloring problem we have to find a coloring of G with sum $\Sigma(G)$.

Minimum Sum Coloring was introduced independently by Kubicka [15] and Supowit [25]. Besides its combinatorial interest, the problem is motivated by applications in scheduling [2, 3, 11] and VLSI design [22, 26]. In [16] it is shown that the problem is NP-hard in general, but polynomial-time solvable for trees. The dynamic programming algorithm for trees can be extended to partial k-trees [14]. For further complexity results and approximation algorithms, see [2, 3, 9, 24].

One can analogously define the edge coloring version of Minimum Sum Coloring. Formally, we will investigate the following optimization problem:

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Table 1: Results for MINIMUM SUM EDGE COLORING.

Class	Algorithm	Hardness
Trees	P [10, 24, 28]	–
Bipartite graphs	1.414-approx [8]	APX-hard (Theorem 4.2)
Planar graphs	PTAS [19]	NP-hard (Theorem 3.3)
Partial k-trees	PTAS [19]	NP-hard (Theorem 5.6)
General graphs	2-approx [2]	APX-hard (Theorem 4.2)

MINIMUM SUM EDGE COLORING
Input: A graph $G(V, E)$.
Find: An edge coloring $\psi : E \to \mathbb{N}$ such that if e_1 and e_2 have a common vertex, then $\psi(e_1) \neq \psi(e_2)$.
Goal: Minimize $\Sigma'_\psi(E) = \sum_{e \in E} \psi(e)$, the *sum* of the coloring.

In this paper we prove complexity results for MINIMUM SUM EDGE COLORING restricted to certain classes of graphs. These results nicely complement the approximation algorithms published in the literature, as they show that the constant-factor approximation algorithms of [11, 2] cannot be improved to a polynomial-time approximation scheme (PTAS), and the approximation schemes of [19] cannot be replaced by a polynomial-time exact algorithm.

Table 1 summarizes the algorithmic and complexity results known for MINIMUM SUM EDGE COLORING. The problem is NP-hard in general (even for bipartite graphs [10]) and trees are the only class of graphs where MINIMUM SUM EDGE COLORING is known to be polynomial-time solvable [10, 24, 28]. Therefore, most of the algorithmic results presented in the literature are approximation algorithms.

For general graphs, a 2-approximation algorithm for MINIMUM SUM EDGE COLORING is presented in [2]. For bipartite graphs better approximation ration is possible: a 1.796-approximation algorithm follows from [11], and a 1.414-approximation algorithm is given in [8]. It is proved in Section 4 that the problem is APX-hard for bipartite graphs, hence these constant-factor approximations cannot be improved to a PTAS.

For partial k-trees (graphs of bounded tree width) and planar graphs, MINIMUM SUM EDGE COLORING admits a PTAS [19]. (In fact, the approximation scheme of [19] works also for the more general multicoloring version of the problem.) We show that a polynomial-time exact algorithm for these classes cannot be expected, as the problem is NP-hard for partial 2-trees (Section 5) and for planar graphs (Section 3).

As noted above, for trees MINIMUM SUM EDGE COLORING can be solved in polynomial time [10, 24, 28] by a dynamic programming algorithm that uses weighted bipartite matching as a subroutine. In most cases, when a problem can be solved in trees by dynamic programming, then this easily generalizes to partial k-trees, and a similar dynamic programming approach can solve the problem in partial k-trees. For example, that is the case with the vertex coloring version of MINIMUM SUM COLORING on trees and partial k-trees. Other examples include the MAXIMUM INDEPENDENT SET, VERTEX COLORING, and

Vertex Disjoint Paths (see [5, 6, 7] for more information on partial k-trees). Therefore, it is somewhat surprising that Minimum Sum Edge Coloring is NP-hard for partial 2-trees. There are only two other examples that we are aware of where the algorithm for trees does not generalize to partial 2-trees. The Edge Disjoint Paths problem is trivial for trees, but it becomes NP-hard for partial 2-trees [23]. Furthermore, the Edge Precoloring Extension problem is polynomial-time solvable for trees [18], but NP-hard for partial 2-trees [20].

2 Preliminaries

For the rest of the paper, we consider only edge colorings, hence even if it is not noted explicitly, "coloring" will mean "edge coloring." We introduce notation and new parameters that turn out to be useful in studying minimum sum edge colorings. Let ψ be an edge coloring of $G(V, E)$, and let E_v be the set of edges incident to vertex v. For every $v \in V$, let $\Sigma'_\psi(v) = \sum_{e \in E_v} \psi(e)$ be the *sum* of v, and for a subset $V' \subseteq V$, let $\Sigma'_\psi(V') = \sum_{v \in V'} \Sigma'_\psi(v)$. Clearly, $\Sigma'_\psi(V) = 2\Sigma'_\psi(G)$; therefore, minimizing $\Sigma'_\psi(V)$ is equivalent to minimizing $\Sigma'_\psi(G)$.

The degree of vertex v is denoted by $d(v) := |E_v|$. For every vertex v, let $\ell(v) := \sum_{i=1}^{d(v)} i = d(v)(d(v)+1)/2$, and for a set of vertices $V' \subseteq V$, let $\ell(V') := \sum_{v \in V'} \ell(v)$. Since $\Sigma'_\psi(v)$ is the sum of $d(v)$ distinct positive integers, $\Sigma'_\psi(v) \geq \ell(v)$ in every proper coloring ψ. Let $\epsilon_\psi(v) = \Sigma'_\psi(v) - \ell(v) \geq 0$ be the *error* of vertex v in coloring ψ. For $V' \subseteq V$ we define $\epsilon_\psi(V') = \sum_{v \in V'} \epsilon_\psi(v)$, and call $\epsilon_\psi(V)$ the *error of coloring* ψ. The error is always non-negative: $\Sigma'_\psi(V) \geq \ell(V)$, hence $\epsilon_\psi(V) = \Sigma'_\psi(V) - \ell(V) \geq 0$. Notice that $\epsilon_\psi(V)$ has the same parity for every coloring ψ. Minimizing the error of the coloring is clearly equivalent to minimizing the sum of the coloring. In particular, if ψ is a *zero error* coloring, that is, $\epsilon_\psi(V) = 0$, then ψ is a minimum sum coloring of G. In a zero error coloring, the edges incident to vertex v are colored with the colors $1, 2, \ldots, d(v)$.

However, in general, G does not necessarily have a zero error coloring. Deciding whether G has a zero error coloring is a special case of Minimum Sum Edge Coloring. It might be worth pointing out that finding a zero error coloring is very different from finding a minimum sum coloring: zero error is a local constraint on the coloring (every vertex has to have zero error), while minimizing the sum is a global constraint.

Parallel edges are not allowed for the graphs considered in this paper. However, for convenience we extend the problem by introducing *half-loops*. A half-loop is a loop that contributes only 1 to the degree of its end vertex. Every vertex has at most one half-loop. If a graph is allowed to have half-loops, then it will be called a *quasigraph* (the terminology half-loop and quasigraph is borrowed from [17]). In a quasigraph, the sum of an edge coloring is defined to be the sum of the color of the edges plus *half* the sum of the color of the half-loops; therefore, the sum of a quasigraph is not necessarily an integer. The sum $\Sigma'_\psi(v)$ is defined to be the integer $\sum_{e \in E_v} \psi(e)$, as before, thus a half-loop contributes to the sum of exactly one vertex. Thus it remains true that the error of a coloring is always integer and the sum of the vertices is twice the sum of the edges.

The following observation shows that allowing half-loops does not make the problem more difficult, thus any hardness result for quasigraphs immediately implies hardness for ordinary graphs as well. This observation was used in

[21] to obtain complexity results for the related problem Chromatic Edge Strength. We reproduce the proof here for completeness.

Proposition 2.1. *Given a quasigraph G, one can create in polynomial time a graph G' such that $\Sigma'(G') = 2\Sigma'(G)$.*

Proof. To obtain G', take two disjoint copies G_1, G_2 of G and remove every half-loop. If there was a half-loop at v in G, then add an edge v_1v_2 to G', where v_1 and v_2 are the vertices corresponding to v in G_1 and G_2, respectively. In graph G', give to every edge the color of the corresponding edge in G. If the sum of the coloring in G was S, then we obtain a coloring in G' with sum $2S$: two edges of G' correspond to every edge of G, but only one edge corresponds to every half-loop of G.

On the other hand, one can show that if G' has a k-coloring with sum S, then G has a k-coloring with sum at most $S/2$. The edges of G' can be partitioned into three sets E_1, E_2, E': set E_i contains the edges induced by G_i $(i = 1, 2)$, and E' contains the edges corresponding to the half-loops. If ψ is an edge coloring of G' with sum S, then $S = \Sigma'_\psi(E_1)+\Sigma'_\psi(E_2)+\Sigma'_\psi(E')$. Without loss of generality, it can be assumed that $\Sigma'_\psi(E_1) \le \Sigma'_\psi(E_2)$, hence $\Sigma'_\psi(E_1) + \Sigma'_\psi(E')/2 \le S/2$. The k-coloring of G_1 induced by ψ has sum $\Sigma'_\psi(E_1) + \Sigma'_\psi(E')/2 \le S/2$, since the edges in E' correspond to half-loops. □

Therefore, minimizing the sum of the coloring on G' is the same problem as minimizing the sum on G. Notice that if G is bipartite, then G' is bipartite as well. On the other hand, the transformation does not preserve planarity in general. Therefore, quasigraphs will be used only when proving hardness results for bipartite graphs (Section 4), but not in the case of planar graphs (Section 3).

3 Planar graphs

In this section we show that Minimum Sum Edge Coloring is NP-hard for planar bipartite graphs of maximum degree 3, and for planar 3-regular graphs. The proof is by reduction from Edge Precoloring Extension.

In Precoloring Extension a graph G is given with some of the vertices having preassigned colors, and it has to be decided whether this precoloring can be extended to a proper vertex k-coloring of the whole graph. One can analogously define the problem Edge Precoloring Extension. It is shown in [20] that Edge Precoloring Extension is NP-complete for 3-regular planar bipartite graphs. For more background on Precoloring Extension and Edge Precoloring Extension, the reader is referred to [27, 4, 12, 13].

In the following theorem, we reduce the NP-complete Edge Precoloring Extension (a problem with local constraints) to deciding whether a graph has a zero error coloring. This proves that Minimum Sum Edge Coloring is NP-hard.

Theorem 3.1. *It is* NP*-hard to decide if a planar bipartite graph with degree at most 3 has a zero error coloring.*

Proof. Using simple local replacements, we reduce Edge Precoloring Extension to the problem of finding a zero error coloring, which is a special case of Minimum Sum Edge Coloring. Given a 3-regular graph G with some

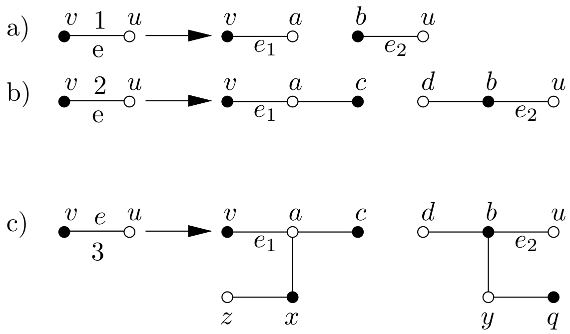

Figure 1: Each precolored edge is replaced by the corresponding subgraph on the right.

of the edges having preassigned colors, construct a graph G' by replacing the precolored edges with the subgraphs shown in Figure 1. If we replace the edge $e = uv$ with such a subgraph, then the two new edges incident to v and u will be called e_1 and e_2. If G is planar and bipartite, then clearly G' is planar and bipartite as well.

We show that G' has a zero error coloring if and only if G has a precoloring extension with 3 colors. Assume that ψ is a zero error coloring. We show that for every precolored edge e, the edges e_1 and e_2 receive the color of e. If e is precolored to 1 (see case a) in Figure 1), then $d(a) = d(b) = 1$, thus e_1 and e_2 receive color 1 in every zero error coloring. If e has color 2, then edges ac and bd must have color 1, thus edges e_1, e_2 have color 2 in every zero error coloring. Finally, if e has color 3, then ac and bd have color 1, edges ax and by have color 2, hence e_1 and e_2 have color 3. Therefore, ψ extends the precoloring of G.

The converse is also easy to see: given a precoloring extension of G, for each edge e in G we assign the color of e to edges e_1 and e_2 in G', and extend this coloring the straightforward way. It can be verified that this is a zero error coloring of G', there is no vertex v that is incident to an edge with color greater than $d(v)$ (here we use that G is 3-regular). □

As finding a zero error coloring is a special case of Minimum Sum Edge Coloring, we have

Corollary 3.2. Minimum Sum Edge Coloring *is* NP*-hard for planar bipartite graphs having degrees at most 3.* □

It is tempting to try to strengthen Corollary 3.2 by replacing "degree at most 3" with "3-regular." However, Minimum Sum Edge Coloring becomes polynomial-time solvable for bipartite, regular graphs. In fact, every such graph has a zero error coloring: by the line coloring theorem of Kőnig, every bipartite graph G has a $\Delta(G)$-edge-coloring, which has zero error if G is regular. However, if we add the requirement of 3-regularity, but drop the requirement that the graph is bipartite, then the problem remains NP-complete.

Theorem 3.3. Minimum Sum Edge Coloring *is* NP*-complete for planar* 3*-regular graphs.*

Proof. The reduction is from zero error coloring of planar graphs with degree at most 3 (Theorem 3.1). We attach certain gadgets to the graph G to make it a 3-regular graph G'. The gadgets are attached in such a way that G has a zero error coloring if and only if G' has a coloring with error K, where K is an integer determined during the reduction.

Figure 2 shows three gadgets R_1, R_2, R_3, each gadget has a pendant edge e. We show that gadget R_i has the following property: if its edges are colored in such a way that the total error on the internal vertices is as small as possible, then the pendant edge receives color i. The figure shows such a coloring for each gadget, the circled vertices are the vertices where there are errors in the coloring.

Gadget R_1 (see Figure 2) has a pendant edge e, 5 internal vertices (denoted by S), and 7 edges connecting the internal vertices. Since each color can be used at most twice on these 7 edges, they have sum at least $2\cdot 1+2\cdot 2+2\cdot 3+1\cdot 4 = 16$ in every coloring. Therefore, if a coloring assigns color i to edge e, then the vertices in S have sum at least $32 + i$ and error at least $32 + i - \ell(S) = 2 + i$. Thus the error of S is at least 3 and it can be 3 only if the pendant edge e is colored with color 1.

In gadget R_3 (second graph on Figure 2), two copies of gadget R_1 are attached to vertex v. The error on the internal vertices is at least 6 in every coloring: there are at least 3 errors in each of S_1 and S_2. However, the error is strictly greater than this: at least one of e_1 and e_2 is colored with a color greater than 1, hence either S_1 or S_2 has error at least 4. Moreover, if the error of the internal vertices in R_3 is 7, then one of e_1 and e_2 is colored with color 1, the other edge is colored with color 2; therefore, edge e has to be colored with color 3.

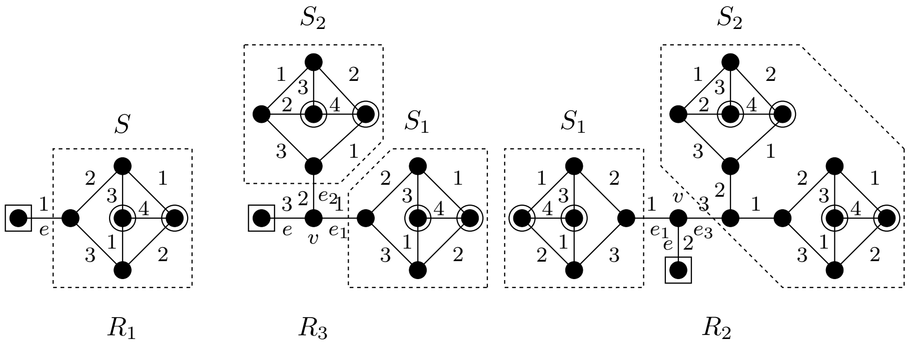

Figure 2: The gadgets R_1, R_2, R_3. The coloring given on the figure has as few errors on the internal vertices as possible. The circles show the errors on the internal vertices in this coloring.

Gadget R_2 (third graph on Figure 2) contains a gadget R_1 and R_3 attached to vertex v. It has error at least $3 + 7 = 10$, since the internal vertices of these gadgets have at least that much error in every coloring. Furthermore, if the error on the internal vertices of R_2 is exactly 10, then this is only possible if the error in S_1 is 3 and the error in S_2 is 7. This implies that the edge e_1 has color 1 and edge e_2 has color 3; therefore, edge e has color 2.

Given a planar graph G with degree at most 3, we attach a gadget R_2 and a gadget R_3 to every vertex of degree 1. Furthermore, we attach a gadget R_3 to every degree 2 vertex. Clearly, the resulting graph G' is planar and 3-regular. Let n be the number of R_3 gadgets attached, and let m be the number of R_2 gadgets. We claim that G has a zero error coloring if and only if G' has a coloring with error at most $K = 7n + 10m$.

Assume first that G has zero error. This coloring can be extended in such a way that the error on every attached R_3 (resp., R_2) gadget is 7 (resp., 10), and the edge that connects an R_2 (resp., R_3) gadget to G has color 2 (resp., 3). If v is a vertex of G (not an internal vertex of a gadget), then the three colors 1, 2, and 3 appear at v. Therefore, the error of the coloring is the total error of the gadgets, that is, $K = 7n + 10m$.

Assume now that G' has a coloring with error at most K. As we have seen, every gadget R_3 has error at least 7 in every coloring, and every gadget R_2 has error at least 10; therefore, if the coloring has error $7n + 10m$, then every R_3 gadget has error exactly 7, and every R_2 gadget has error exactly 10. This means that every edge connecting an R_2 (resp., R_3) gadget to G has color 2 (resp., 3). Since G is a subgraph of G', the coloring of G' induces a coloring of G. We show that this coloring is a zero error coloring of G. If v is a degree 1 vertex of G, then two additional edges connect v to an R_2 and an R_3 gadget in G', and these two edges have colors 2 and 3. The error of v is zero in the coloring; therefore, the edge incident to v in G receives color 1. Similarly, if v has degree 2 in G, then an additional edge with color 3 is connected to v in G, and it follows that the two edges incident to v in G have the colors 1 and 2, as required. $\square$

4 Approximability

A *polynomial-time approximation scheme (PTAS)* is an approximation algorithm that has an input parameter ϵ, and for every $\epsilon > 0$ it produces a solution with cost at most $(1 + \epsilon)$ times the optimum. The running time has to be polynomial in the size of the input for every fixed value of ϵ, i.e., it is of the form $n^{f(\epsilon)}$. If a problem admits a PTAS, then this means that there is no "best" approximation algorithm: an approximation ratio arbitrarily close to 1 can be achieved. On the other hand, by proving that a problem is APX-hard we can show that the problem does not admit a PTAS (unless P = NP), that is, there is a $c > 1$ such that there is no polynomial-time approximation algorithm with approximation ratio better than c. Here we prove that Minimum Sum Edge Coloring is APX-hard, even for bipartite graphs. Therefore, the approximation schemes for partial k-trees and planar graphs presented in [19] cannot be generalized to arbitrary graphs.

Theorem 4.1. Minimum Sum Coloring *is* APX-*hard for graphs with maximum degree* 3.

Proof. The theorem is proved by an L-reduction from Minimum Vertex Cover for 3-regular graphs, which is shown to be APX-hard in [1]. For every graph $G(V, E)$ with minimum vertex cover size $\tau(G)$, a graph G'' is constructed that has edge chromatic sum $C = c_1|V| + c_2|E| + \tau(G)$, where c_1 and c_2 are constants to be determined later. To see that this is an L-reduction, notice that $|E| = \frac{3}{2}|V|$ and $\tau(G) \geq |V|/4$ follows from the fact that G is 3-regular. Therefore, $C \leq 4c_1\tau(G) + 6c_2\tau(G) + \tau(G) = c_3\tau(G)$, as required. Furthermore, we show that given an edge coloring of G'' with sum at most $c_1|V| + c_2|E| + t$, one can find a vertex cover of size t. This proves the correctness of the L-reduction.

The graph G'' is constructed in two steps: first we create a quasigraph G', then apply the transformation of Proposition 2.1 to obtain the graph G''. The graph G' consists of vertex gadgets and edge gadgets. The vertex gadget shown in Figure 3 has 3 pendant edges e_1, e_2, e_3, and satisfies the following two properties:

- If a coloring has zero error on the internal vertices of the variable gadget, then it colors all three pendant edges with color 1.
- There is a coloring that colors all three pendant edges with color 2 and has only 1 error on the internal vertices.

Figure 3 shows two possible colorings of the gadget, the two numbers on each edge show the color of the edge in the two colorings. The first coloring is the unique coloring with zero error on the internal vertices. To see this, notice first that an edge incident to a degree 1 internal vertex has to be colored with color 1. Furthermore, if an edge of a degree 2 vertex is colored with color 1, then the other edge has to be colored with color 2. Applying these and similar implications repeatedly, we get the first coloring of Figure 3. In particular, edges e_1, e_2, e_3 have color 1, proving the first property. The second coloring has one error (at v), and colors e_1, e_2, e_3 with color 2, proving the second property.

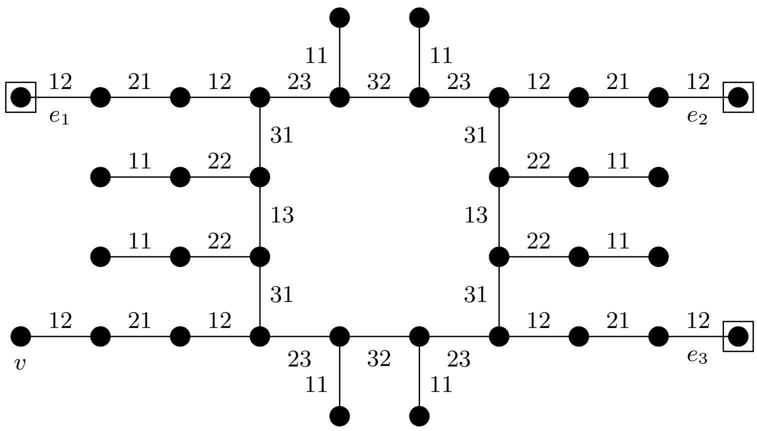

Figure 3: The vertex gadget.

The edge gadget shown in Figure 4 has two pendant edges f and g. If a coloring has zero error on the internal vertices of the gadget, then clearly f and g have color 1 or 2. There are 4 different ways of coloring f and g with colors 1 or 2. In 3 out of 4 of these combinations, when at least one of f and g is colored with color 2, the coloring can be extended to the whole gadget with zero error (Figure 4 shows these 3 colorings). On the other hand, if both f and g have color 1, then there is at least one error on the internal vertices of the gadget. The reader can verify this by following the implications of coloring f and g with color 1, and requiring that every internal vertex has zero error.

The quasigraph $G'(V', E')$ is constructed as follows. A vertex gadget S_v corresponds to every vertex v of G, and an edge gadget S_e corresponds to every

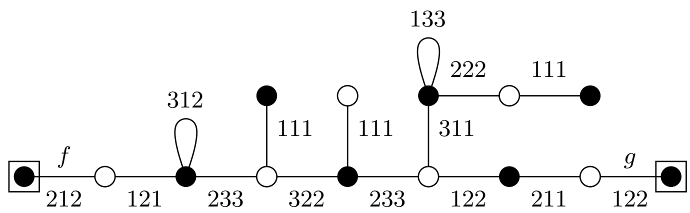

Figure 4: The edge gadget.

edge e of G. Direct the edges of G arbitrarily. If the i-th edge incident to $v \in V$ ($i = 1, 2, 3$) is the head of some edge $e \in E$, then identify edge e_i of S_v with edge f of S_e. If the i-th edge incident to $v \in V$ is the tail of some edge $e \in E$, then identify edge e_i of S_v with edge g of S_e. Thus every vertex of G' is an internal vertex of a vertex gadget S_v or an edge gadget S_e. Denote by V_v the internal vertices of gadget S_v and by V_e the internal vertices of S_e; clearly these sets form a partition of V'.

We claim that G' has a coloring with error t if and only if G has a vertex cover of size t. Assume first that $D \subseteq V$ is a vertex cover of G. If $v \in D$, then color gadget S_v such that every pendant edge has color 2 (and there is one error on the internal vertices), otherwise color S_v in such a way that every pendant edge has color 1, and there is no error on the internal vertices. Now consider a gadget S_e for some $e \in E$. The two pendant edges f and g are already colored with colors 1 or 2. However, at least one of these two edges is colored with 2, since at least one end vertex of e is in D. Therefore, using one of the three colorings shown in Figure 4, we can extend the coloring to every edge of S_e with zero error on the internal vertices of the gadget. This means that errors appear only on the internal vertices of S_v for $v \in D$, and the total error is $|D|$.

On the other hand, consider a coloring of G' with error t. Let $\widehat{V} \subseteq V$ be the set of those $v \in V$ for which V_v is colored with error. Similarly, let $\widehat{E} \subseteq E$ be the set of those $e \in E$ for which V_e is colored with error. Clearly, the coloring has error at least $|\widehat{V}| + |\widehat{E}| \leq t$. Let $\overline{V}$ be a set of $|\widehat{E}|$ vertices in G that cover every edge in $\widehat{E}$. The set of vertices $\widehat{V} \cup \overline{V}$ has size at most $|\widehat{V}| + |\widehat{E}| \leq t$. We show that this set is a vertex cover of G. It is clear that every edge $e \in \widehat{E}$ is covered, since there is a $v \in \overline{V}$ covering e. Now consider an edge $e \notin \widehat{E}$, this means that V_e is colored with zero error, thus, as we have observed, at least one pendant edge of S_e is colored with color 2. If this edge is the pendant edge of the vertex gadget S_v, then there is at least one error in V_v and v is in $\widehat{V}$. If the pendant edge of S_e and S_v is identified in the construction, this means that e is incident to v, thus $v \in \widehat{V}$ covers e.

We have proved that the error of a minimum sum edge coloring of G' is at least $\tau(G)$. Furthermore, $\Sigma'(G') = (c_1/2)|V| + (c_2/2)|E| + \tau(G)/2$ for some constants c_1 and c_2. To see this, notice that the lower bound $\ell(V_v)$ is the same for every $v \in V$ (denote it by c_1), and $\ell(V_e)$ is the same for every $e \in E$ (denote it by c_2). Therefore, the sum of the vertices in the optimum coloring is $\ell(V') + \tau(G) = c_1|V| + c_2|E| + \tau(G)$. The edge chromatic sum is the half of this value, $(c_1/2)|V| + (c_2/2)|E| + \tau(G)/2$. Now construct graph G'' from G' as in Proposition 2.1. We have that $\Sigma'(G'') = 2\Sigma'(G') = c_1|V| + c_2|E| + \tau(G)$. Furthermore, a coloring of G'' with sum $c_1|V| + c_2|E| + t$ gives a coloring of

G' with sum $(c_1/2)|V| + (c_2/2)|E| + t/2$, that is a coloring with error t. It was shown above that given a coloring of G' with error t, one can find a vertex cover of G with size at most t. This completes the proof of the L-reduction. □

Theorem 4.1 can be strengthened: the problem remains APX-hard for bipartite graphs. The graph constructed in the proof of Theorem 4.1 is not bipartite, since the vertex gadget in Figure 3 is not bipartite. However, the vertex gadget can be replaced by the slightly more complex quasigraph shown in Figure 5, which is bipartite and has the same properties. That is, if a coloring has zero error on the internal vertices, then the pendant edges have color 1, and there is a coloring that has error 1 on the internal vertices, and assigns color 2 to the pendant edges. The vertex and edge gadgets are bipartite, and they are connected in a way that ensures that the resulting graph G' is bipartite as well.

Theorem 4.2. *Minimum sum edge coloring is* APX*-hard for bipartite graphs with maximum degree* 3. □

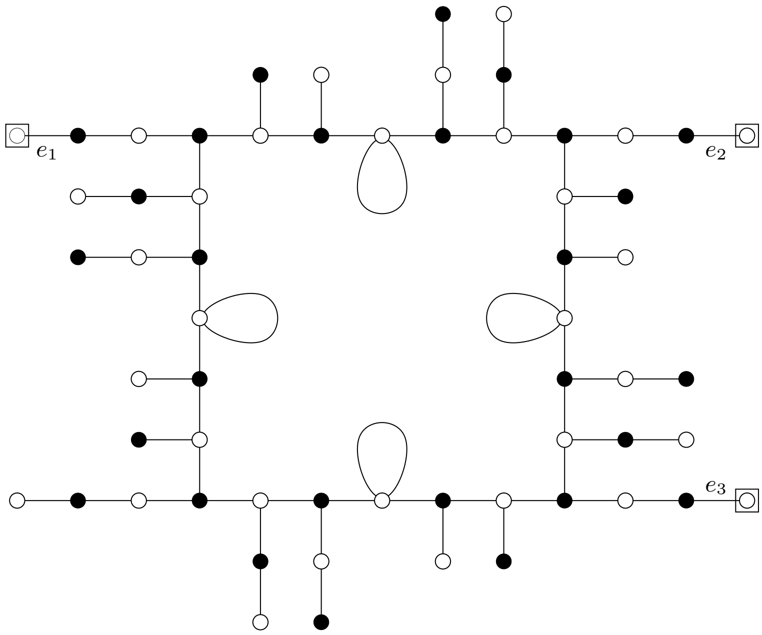

Figure 5: The bipartite quasigraph version of the vertex gadget.

5 Partial k-trees

In this section, we show that Minimum Sum Edge Coloring is NP-hard for partial 2-trees. A *k-tree* is a graph defined by the following three rules:

1. A clique of size $k+1$ is a k-tree.
2. If G is a k-tree, and K is a clique of size k in G, then the graph G' that is obtained by adding a new vertex v and connecting v to every vertex of K is also a k-tree.

3. Every k-tree can be obtained using 1 and 2.

Another way to define k-trees is to say that a graph is a k-tree if and only if it is a chordal graph with clique number $k+1$. A graph is a *partial k-tree* if it is a subgraph of a k-tree. The notion of tree width gives an alternate characterization of partial k-trees: a graph is partial k-tree if and only if it has tree width at most k. For more information on the algorithmic and combinatorial significance of partial k-trees and tree width, the reader is referred to [7, 6].

Before presenting the proof of NP-completeness, we introduce some gadgets used in the reduction. These gadgets are trees with a single pendant edge, and have the following general property: if a coloring is "cheap," meaning that it has as small error on the internal vertices as possible, then the color of the pendant edge has to be one of the special allowed colors of the gadget. For the gadget F_n, this means that in every such cheap coloring, the pendant edge has color n. In the gadget L_n, the color of the pendant edge has to be either $n-1$ or $n+1$ in such a coloring. In the gadget A_n, the color of the pendant edge has to be an odd number not greater than n.

The reduction is from 3-SAT; therefore, we need satisfaction testing gadgets and variable setting gadgets. All these gadget are connected to a central vertex v. The satisfaction testing gadget has the property that in every cheap coloring the pendant edge (the edge that connects the gadget to v) has one of the three preassigned colors. The variable setting gadget W_n is different from the other gadgets. First, it is not a tree, but a partial 2-tree. Moreover, there are *two* edges connecting it to the central vertex v. The crucial property of this gadget is that in every cheap coloring, these two edges either use the colors $n+1$, $n+3$, or they use the colors $n+5$, $n+7$.

In the following lemmas, we formally define the properties of the gadgets, describe how they are constructed, and prove the required properties.

Lemma 5.1. *For every $n \geq 2$, there is a tree F_n and an integer f_n, such that*

1. *F_n has one pendant edge e,*
2. *the internal vertices of F_n have error at least f_n in every coloring,*
3. *if a coloring has error f_n on the internal vertices of F_n, then this coloring assigns color n to the pendant edge e, and*
4. *F_n can be constructed in time polynomial in n.*

Proof. The tree F_n is a star with a central vertex v, and n leaves $v_1, v_2, \ldots, v_n$. The pendant edge e is the edge $v_n v$, thus the internal vertices are $v, v_1, v_2, \ldots, v_{n-1}$. Let $f_n := (n-1)(n-2)/2$. The $n-1$ edges $v_1 v, \ldots, v_{n-1} v$ have different colors, hence the sum of the vertices $v_1, \ldots, v_{n-1}$ is at least $\sum_{i=1}^{n-1} i = n(n-1)/2$. Therefore, the error on these vertices is at least $n(n-1)/2-(n-1) = f_n$. There is equality if and only if the sum of these vertices is exactly $n(n-1)/2$ and there is no error on v. This implies that edge $v_n v$ has color n, as required. $\square$

Lemma 5.2. *For every even $n \geq 1$, there is a tree L_n and an integer k_n, such that*

1. *L_n has one pendant edge e,*

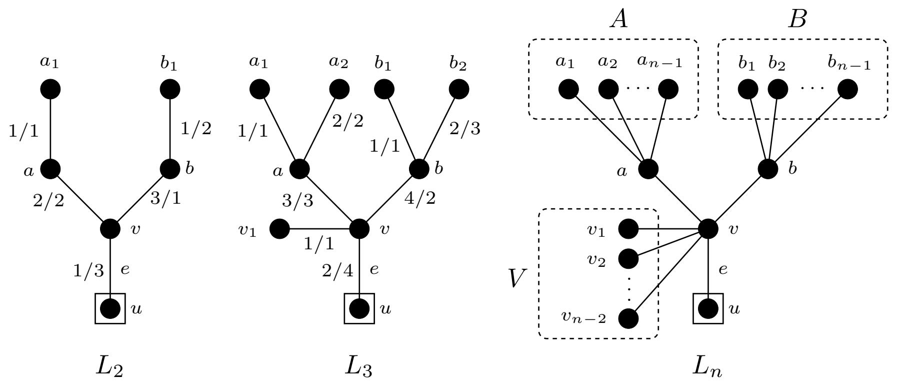

Figure 6: The gadget L_n.

2. *the internal vertices of L_n have error at least k_n in every coloring,*

3. *if a coloring has error k_n on the internal vertices of L_n, then this coloring assigns either color $n-1$ or $n+1$ to the pendant edge e,*

4. *there are colorings ψ_{n-1} and ψ_{n+1} of L_n with $\psi_{n-1}(e) = n-1$, $\psi_{n+1}(e) = n+1$, such that they have error k_n on the internal vertices, and*

5. *L_n can be constructed in time polynomial in n.*

Proof. The tree L_n is constructed as follows (see Figure 6). The pendant edge e connects external vertex u and internal vertex v. A set V of $n-2$ vertices $v_1, v_2, \ldots, v_{n-2}$ are connected to v. There are two additional neighbors of v: vertices a and b. Besides v, vertex a has $n-1$ neighbors $a_1, a_2, \ldots, a_{n-1}$, let A be the set containing these $n-1$ vertices. Similarly, vertex b has $n-1$ additional neighbors $B = \{b_1, b_2, \ldots, b_{n-1}\}$.

Since the edges v_1v, v_2v, $\ldots$, $v_{n-2}v$ have different colors in every coloring of L_n, the sum of V is at least $\sum_{i=1}^{n-2} i = (n-2)(n-1)/2$ in every coloring. Therefore, there is error at least $(n-2)(n-1)/2 - \ell(V) = (n-2)(n-1)/2 - (n-2) = (n-2)(n-3)/2$ on V in every coloring. This minimum is reached if and only if the edges $v_1v, \ldots, v_{n-2}v$ have the colors $1, \ldots, n-2$ (in some order). Similarly, there is error at least $(n-1)n/2 - (n-1) = (n-1)(n-2)/2$ on both A and B. Therefore, there is error at least $(n-2)(n-3)/2 + 2 \cdot (n-1)(n-2)/2$ on the internal vertices in every coloring. However, the error is always strictly greater than that. If the error is exactly $(n-1)(n-2)/2$ on both A and B, and there is zero error on a and b, then edges va and vb both have to receive color n. Thus we can conclude that there is error at least $k_n := (n-2)(n-3)/2 + 2 \cdot (n-1)(n-2)/2 + 1$ in every coloring.

The coloring ψ_{n-1} is defined as

- $\psi_{n-1}(e) = n-1$,
- $\psi_{n-1}(va) = n$,
- $\psi_{n-1}(vb) = n+1$,
- $\psi_{n-1}(v_iv) = i$ for $1 \le i \le n-2$,

- $\psi_{n-1}(a_i a) = i$ for $1 \le i \le n-1$, and
- $\psi_{n-1}(b_i b) = i$ for $1 \le i \le n-1$.

It can be verified that $\epsilon_{\psi_{n-1}}(V) = (n-2)(n-3)/2$, $\epsilon_{\psi_{n-1}}(A) = \epsilon_{\psi_{n-1}}(B) = (n-1)(n-2)/2$, $\epsilon_{\psi_{n-1}}(a) = \epsilon_{\psi_{n-1}}(v) = 0$, and $\epsilon_{\psi_{n-1}}(b) = 1$; therefore, the error of ψ_{n-1} on the internal vertices of L_n is exactly k_n. Coloring ψ_{n+1} is the same as coloring ψ_{n-1}, except that

- $\psi_{n+1}(e) = n+1$,
- $\psi_{n+1}(vb) = n-1$, and
- $\psi_{n+1}(b_{n-1}b) = n$.

This change decreases the error on b to zero, and increases the error on b_{n-1} to 1. Therefore, ψ_{n+1} also has error k_n on the internal vertices, and this proves Property 4.

To show that Property 3 holds, assume that coloring ψ has error k_n on the internal vertices of L_n. As we have observed, $e_\psi(A \cup \{a\}) = (n-1)(n-2)/2$ implies $\psi(va) = n$. Similarly, $e_\psi(B \cup \{b\}) = (n-1)(n-2)/2$ implies $\psi(vb) = n$; therefore, at least one of $A \cup \{a\}$ and $B \cup \{b\}$ have error strictly greater than $(n-1)(n-2)/2$. Assume, without loss of generality, that $e_\psi(A \cup \{a\}) > (n-1)(n-2)/2$. In this case, the error of ψ can be k_n only if $e_\psi(B \cup \{b\}) = (n-1)(n-2)/2$, $e_\psi(V) = (n-2)(n-3)/2$, thus v has zero error. Therefore, color n is used by edge vb, and the colors $1, 2, \ldots, n-2$ are used by the edges $v_1v, v_2v, \ldots, v_{n-2}v$ (not necessarily in this order). Since there is zero error at v, and v has degree $n+1$, edge e has a color not greater than $n+1$. This can be only $n-1$ or $n+1$, since the other colors are already used by edges incident to v. $\square$

Lemma 5.3. *For every odd $n \ge 1$, there is a tree A_n and an integer a_n such that*

1. *A_n has one pendant edge e,*
2. *the internal vertices of A_n have error at least a_n in every coloring,*
3. *if a coloring ψ has error a_n on the internal vertices of A_n, then $\psi(e)$ is odd and $\psi(e) \le n$,*
4. *for every odd c not greater than n, there is a coloring ψ_c of A_n such that $\psi_c(e) = c$ and it has error a_n on the internal vertices,*
5. *A_n can be constructed in time polynomial in n.*

Proof. The pendant edge e of A_n connects external vertex u and internal vertex v. Attach the pendant edges of the $(n-1)/2$ trees $F_2, F_4, \ldots, F_{n-1}$ (Lemma 5.1) to vertex v, let the pendant edges of these trees be $v_2v, v_4v, \ldots, v_{n-1}v$, respectively (see Figure 7). Similarly, attach the pendant edges of the $(n-1)/2$ trees $L_2, L_4, \ldots, L_{n-1}$ (Lemma 5.2) to v, let the pendant edges of these trees be $w_2v, w_4v, \ldots, w_{n-1}v$, respectively. Therefore, the degree of v in A_n is n.

Let $a_n = (f_2 + f_4 + \cdots + f_{n-1}) + (k_2 + k_4 + \cdots + k_{n-1})$. Since A_n contains a copy of the trees $F_2, F_4, \ldots, F_{n-1}$, and a copy of the trees $L_2, L_4, \ldots, L_{n-1}$, it

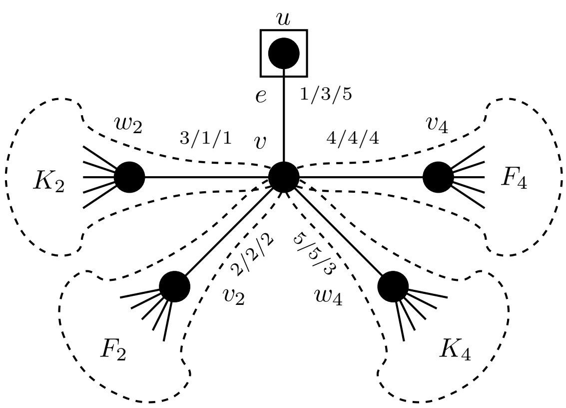

Figure 7: The gadget A_5.

is clear that every coloring of A_n has at least a_n errors on the internal vertices. Moreover, if a coloring ψ has error a_n on the internal vertices, then $\psi(v_i v) = i$ for $i = 2, 4, \ldots, n-1$, and the error of v is zero. This implies that $\psi(e) \leq n$ and not even, as required.

The coloring ψ_c required by Property 4 is the following. For every $i = 2, 4, \ldots, n-1$, coloring ψ_c colors the edges of the tree F_i in such a way that the pendant edge $v_i v$ receives color i, and there is error f_i on the internal vertices of F_i; by Lemma 5.1, such a coloring exists. For every $i = 2, 4, \ldots, c-1$, the tree L_i is colored such that the pendant edge $w_i v$ has color $i-1$, and the error on the internal vertices of L_i is k_i. Similarly, for $i = c+1, \ldots, n-1$, the tree L_i is colored such that the pendant edge $w_i v$ has color $i+1$, and there is error k_i on the internal vertices of L_i. Coloring ψ_c assigns color c to edge e, thus every color 1, 2, …, n appears on exactly one edge incident to v. Therefore, v has zero error, and the error on the internal vertices of A_n is a_n. ☐

Lemma 5.4 (Satisfaction testing gadget). *For odd integers $x_1 < x_2 < x_3$, there is a tree S_{x_1,x_2,x_3} and an integer s_{x_1,x_2,x_3} such that*

1. *S_{x_1,x_2,x_3} has one pendant edge e,*
2. *the internal vertices of S_{x_1,x_2,x_3} have error at least s_{x_1,x_2,x_3} in every coloring,*
3. *if a coloring ψ has error s_{x_1,x_2,x_3} on the internal vertices of S_{x_1,x_2,x_3}, then $\psi(e) \in \{x_1, x_2, x_3\}$*
4. *for $i = 1, 2, 3$, there is a coloring ψ_i of S_{x_1,x_2,x_3} such that $\psi_i(e) = x_i$ and it has error s_{x_1,x_2,x_3} on the internal vertices,*
5. *S_{x_1,x_2,x_3} can be constructed in time polynomial in x_3.*

Proof. The pendant edge e of S_{x_1,x_2,x_3} connects external vertex u and internal vertex v. Attach to vertex v the pendant edges of

- $x_1 - 1$ trees F_1, F_2, …, F_{x_1-1} (Lemma 5.1),

- $x_2 - x_1 - 1$ trees $F_{x_1+1}, \ldots, F_{x_2-1}$,
- $x_3 - x_2 - 1$ trees $F_{x_2+1}, \ldots, F_{x_3-1}$, and
- 2 copies of the tree A_{x_3} (Lemma 5.3).

Vertex v has degree x_3 in S_{x_1,x_2,x_3}. Set $s_{x_1,x_2,x_3} := f_1 + f_2 + \cdots + f_{x_1-1} + f_{x_1+1} + \cdots + f_{x_2-1} + f_{x_2+1} + \cdots + f_{x_3-1} + 2a_{x_3}$. Because of the way S_{x_1,x_2,x_3} is constructed, it is clear that every coloring of S_{x_1,x_2,x_3} has error at least s_{x_1,x_2,x_3} on the internal vertices. If ψ has error exactly s_{x_1,x_2,x_3} on the internal vertices, then v has zero error and $\psi(e) \le d(v) = x_3$. Furthermore, it also follows that the colors $1, \ldots, x_1-1, x_1+1, \ldots, x_2-1, x_2+1, \ldots, x_3-1$ are used at v by the pendant edges of the attached trees $F_1, \ldots, F_{x_1-1}, F_{x_1+1}, \ldots, F_{x_2-1}, F_{x_2+1}, \ldots, F_{x_3-1}$, respectively. Therefore, edge e has one of the remaining colors x_1, x_2, x_3, proving Property 3.

The colorings ψ_1, ψ_2, ψ_3 required by Property 4 color the $(x_1 - 1) + (x_2 - x_1 - 1) + (x_3 - x_2 - 1)$ trees of type F_i in the same way: all three colorings color these trees such that there is error $f_1 + f_2 + \cdots + f_{x_1-1} + f_{x_1+1} + \cdots + f_{x_2-1} + f_{x_2+1} + \cdots + f_{x_3-1}$ on the internal vertices of the trees, and their pendant edges use the colors $1, \ldots, x_1 - 1, x_1 + 1, \ldots, x_2 - 1, x_2 + 1, \ldots, x_3 - 1$ at v, respectively. Coloring ψ_i assigns color x_i to the pendant edge e, hence two colors not greater than x_3 remains unused at v: only the colors $\{x_1, x_2, x_3\} \setminus x_i$ are not yet assigned. These two colors are odd and not greater than x_3, thus by Property 4 of Lemma 5.3, we can color the two copies of A_{x_3} attached to v such that their pendant edges have these two colors, and the additional error that we introduce is $2a_{x_3}$. Since there is zero error on v, the error of this coloring is exactly s_{x_1,x_2,x_3} on the internal vertices of S_{x_1,x_2,x_3}, as required by Property 4. ☐

Lemma 5.5 (Variable setting gadget). *For every $n \ge 0$, there is a partial 2-tree W_n and an integer w_n such that*

1. *W_n has an external vertex v, and two edges e_1 and e_2 incident to v,*
2. *every coloring of W_n has error at least w_n on the internal vertices of W_n,*
3. *if a coloring ψ of W_n has error w_n on the internal vertices, then either*
   - *$\psi(e_1) = n + 1$, $\psi(e_2) = n + 3$ or*
   - *$\psi(e_1) = n + 5$, $\psi(e_2) = n + 7$ holds,*
4. *there are colorings ψ_1 and ψ_2 of W_n with error w_n on the internal vertices such that*
   - *$\psi_1(e_1) = n + 1$, $\psi_1(e_2) = n + 3$,*
   - *$\psi_2(e_1) = n + 5$, $\psi_2(e_2) = n + 7$, and*
5. *W_n can be constructed in time polynomial in n.*

Proof. The graph W_n is constructed as follows (see Figure 8 for the case $n = 0$). The external vertex v is connected to vertex v_1 by edge e_1, and to v_2 by e_2. Vertices v_1 and v_2 are connected by an edge e. We attach several trees to vertices v_1 and v_2:

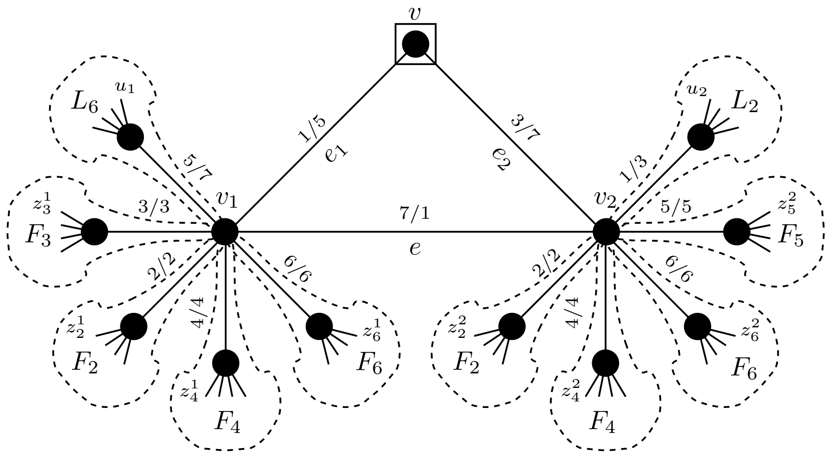

Figure 8: The variable setting gadget W_0.

- Attach n trees $F_1, F_2, \ldots, F_n$ to v_1, let the pendant edges of these trees be $z_1^1 v_1, z_2^1 v_1, \ldots, z_n^1 v_1$, respectively.
- Similarly, attach a copy of these n trees to v_2, let the pendant edges be $z_1^2 v_2, z_2^2 v_2, \ldots, z_n^2 v_2$.
- Attach to v_1 the trees $F_{n+2}, F_{n+3}, F_{n+4}, F_{n+6}$ with pendant edges $z_{n+2}^1 v_1$, $z_{n+3}^1 v_1$, $z_{n+4}^1 v_1$, $z_{n+6}^1 v_1$, respectively.
- Attach to v_1 a tree L_{n+6} with pendant edge $u_1 v_1$.
- Attach to v_2 the trees $F_{n+2}, F_{n+4}, F_{n+5}, F_{n+6}$ with pendant edges $z_{n+2}^2 v_2$, $z_{n+4}^2 v_2$, $z_{n+5}^2 v_2$, $z_{n+6}^2 v_2$, respectively.
- Attach to v_2 a tree L_{n+2} with pendant edge $u_2 v_2$.

Notice that both v_1 and v_2 have degree $n+7$. The graph W_n is a partial 2-tree: it is chordal, and it has clique number 3.

Set $w_n := 2(f_1 + f_2 + \cdots + f_n) + (f_{n+2} + f_{n+3} + f_{n+4} + f_{n+6} + k_{n+6}) + (f_{n+2} + f_{n+4} + f_{n+5} + f_{n+6} + k_{n+2})$. It is clear that every coloring of W_n has error at least w_n on the internal vertices: the combined error in the attached trees is always at least w_n. Moreover, if the error of coloring ψ is w_n on the internal vertices, then there has to be zero error on v_1 and v_2. Furthermore, from Lemma 5.1 and Lemma 5.2, in this case we also have that

- $\psi(z_i^1 v_1) = \psi(z_i^2 v_2) = i$ for $i = 1, 2, \ldots, n$,
- $\psi(z_i^1 v_1) = \psi(z_i^2 v_2) = i$ for $i = n+2, n+4, n+6$,
- $\psi(z_{n+3}^1 v_1) = n+3$,
- $\psi(z_{n+5}^2 v_2) = n+5$,
- $\psi(u_1 v_1)$ is either $n+5$ or $n+7$, and
- $\psi(u_2 v_2)$ is either $n+1$ or $n+3$.

Since the degree of v_1 is $n+7$ and there is zero error on v_1, it follows that $\psi(e) \le n+7$. Moreover, $\psi(e)$ is either $n+1$ or $n+7$: as shown above, every other color not greater than $n+7$ is already used on at least one of v_1 or v_2. Assume first that $\psi(e) = n+1$. In this case u_2v_2 cannot have color $n+1$; therefore, $\psi(u_2v_2) = n+3$ follows. Now the only unused color not greater than $n+7$ at v_2 is $n+7$, hence $\psi(e_2) = n+7$. There remains two unused colors at v_1: color $n+5$ and color $n+7$. However, edge e_1 cannot have color $n+7$, since edge e_2 already has this color. Thus we have $\psi(e_1) = n+5$ and $\psi(e_2) = n+7$, as required by Property 4. Similarly, assume that $\psi(e) = n+7$, it follows that $\psi(u_1v_1) = n+5$. The only unused color not greater than $n+7$ at v_1 is $n+1$, hence edge e_1 has to receive this color. Colors $n+3$ and $n+1$ are the only remaining colors at v_2; therefore, e_2 has color $n+3$, since $n+1$ is already used by e_1. Thus we have $\psi(e_1) = n+1$ and $\psi(e_2) = n+3$, as required.

The two colorings ψ_1 and ψ_2 required by Property 4 are given as follows (see Figure 8 for the case $n = 0$). Consider the (partial) coloring ψ with

- $\psi(z_i^1 v_1) = \psi(z_i^2 v_2) = i$ for $i = 1, 2, \dots, n$,
- $\psi(z_i^1 v_1) = \psi(z_i^2 v_2) = i$ for $i = n+2, n+4, n+6$,
- $\psi(z_{n+3}^1 v_1) = n+3$ and
- $\psi(z_{n+5}^2 v_2) = n+5$.

Both ψ_1 and ψ_2 assign the same colors as ψ, but we also have

- $\psi_1(e_1) = n+1$, $\psi_1(e_2) = n+3$, $\psi_1(e) = n+7$,
- $\psi_1(u_1v_1) = n+5$,
- $\psi_1(u_2v_2) = n+1$.
- $\psi_2(e_1) = n+5$, $\psi_2(e_2) = n+7$, $\psi_2(e) = n+1$,
- $\psi_2(u_1v_1) = n+7$,
- $\psi_2(u_2v_2) = n+3$.

In these colorings vertices v_1 and v_2 have zero error. Furthermore, these colorings can be extended to the attached trees with error w_n: the colors assigned to the pendant edges of the attached trees are compatible with the "best" coloring of the attached trees (see Property 4 of Lemma 5.2 and Property 3 of Lemma 5.1). This gives Property 4 of the lemma being proved. □

Theorem 5.6. Minimum Sum Edge Coloring *is* NP*-hard for partial* 2*-trees.*

Proof. The proof is by reduction from 3-SAT: given a 3-CNF formula φ, we construct a partial 2-tree G and determine an integer K such that $\Sigma'(G) \le K$ if and only if φ is satisfiable.

We assume that every variable occurs exactly twice positively and exactly twice negated in ϕ. This can be achieved as follows. It is well-known that 3-SAT remains NP-complete if every variable occurs exactly twice positively, exactly once negated, and every clause contains two or three literals. Let us assume that the number of variables is even, if not, then duplicate every variable and every clause. Let x_1, x_2, $\dots$, x_n be the variables of ϕ. We add $n/2$ new variables

y_1, y_2, …, $y_{\frac{n}{2}}$ and n new clauses $(\bar{x}_1 \vee y_1 \vee \bar{y}_1)$, $(\bar{x}_2 \vee y_1 \vee \bar{y}_1)$, $(\bar{x}_3 \vee y_2 \vee \bar{y}_2)$, $(\bar{x}_4 \vee y_2 \vee \bar{y}_2)$, …, $(\bar{x}_{n-1} \vee y_{\frac{n}{2}} \vee \bar{y}_{\frac{n}{2}})$, $(\bar{x}_n \vee y_{\frac{n}{2}} \vee \bar{y}_{\frac{n}{2}})$ to the formula. Now every variable occurs exactly twice positively and twice negated. These new clauses are satisfied in every variable assignment, hence the new formula is satisfiable if and only if the original is satisfiable. Furthermore, if there is a clause $(x \vee y)$ containing only two literals, then add a new variable z, and replace this clause with $(x \vee z \vee z) \wedge (\bar{z} \vee \bar{z} \vee y)$. It is easy to see that this transformation does not change the satisfiability of the formula.

Let x_0, x_1, …, x_{n-1} be the n variables of φ. The number of clauses is therefore $m = 4n/3$. For every literal of φ, there is a corresponding color, as follows:

- color $8i + 1$ corresponds to the first positive occurrence of x_i,
- color $8i + 3$ corresponds to the second positive occurrence of x_i,
- color $8i + 5$ corresponds to the first negated occurrence of x_i, and
- color $8i + 7$ corresponds to the second negated occurrence of x_i.

Notice that these numbers are odd, and every odd number not greater than $8n$ corresponds to a literal.

Take a vertex v, we will attach several gadgets to v to obtain the graph G. Attach $4n$ trees F_2, F_4, …, F_{8n} to v, let the pendant edges of the attached trees be u_2v, u_4v, …, $u_{8n}v$, respectively. Attach n variable setting gadgets W_0, W_8, W_{16}, …, $W_{8(n-1)}$ to v, let the two edges of W_{8i} incident to v be called $w_{i,1}v$ and $w_{i,2}v$. For every clause C_j of φ, we attach a satisfaction testing gadget to v in the following way: if colors $c_{j,1} < c_{j,2} < c_{j,3}$ correspond to the three literals in clause C_j, then attach a tree S_{c_1,c_2,c_3} to v, and let s_iv be its pendant edge. Finally, attach $m/2$ copies of the tree A_{8n-1} to v, let the pendant edges of these trees be t_1v, t_2v, …, $t_{\frac{m}{2}}v$. This completes the description of the graph G. Since every gadget is a partial 2-tree (or even a tree), the graph G is a partial 2-tree as well: joining graphs at a single vertex does not increase the tree width of the graphs.

Let $K^{(1)} := f_2 + f_4 + \cdots + f_{8n}$. In every coloring of G the error is at least $K^{(1)}$ on the internal vertices of the $4n$ trees F_i attached to v. Let $K^{(2)} := w_0 + w_8 + \cdots + w_{8(n-1)}$. In every coloring the error is at least $K^{(2)}$ on the internal vertices of the n variable setting gadgets. Let $K^{(3)} := m/2 \cdot a_{8n-1}$. In every coloring the error is at least $K^{(3)}$ on the internal vertices of the $m/2$ copies of A_{8n-1}. Let $K^{(4)} := \sum_{j=1}^{m} s_{c_{j,1},c_{j,2},c_{j,3}}$ where $c_{j,k}$ is the color corresponding to the k-th literal in clause C_j. In every coloring of G, the error on the internal vertices of the m satisfaction testing gadget is at least $K^{(4)}$. Finally, set $K := K^{(1)} + K^{(2)} + K^{(3)} + K^{(4)}$. It is clear that every coloring of G has error at least K. We claim that G has a coloring with error exactly K if and only if φ is satisfiable.

Assume first that coloring ψ has error K. This is possible only if ψ has zero error on v, and the error is exactly K on the internal vertices of the attached gadgets. By Lemmas 5.1, 5.3, 5.4, and 5.5, this implies that

- $\psi(u_iv) = i$ for $i = 2, 4, \ldots, 8n$,
- for every $i = 0, 1, \ldots, n-1$, either

  - $\psi(w_{i,1}v) = 8i + 1$ and $\psi(w_{i,2}v) = 8i + 3$, or
  - $\psi(w_{i,1}v) = 8i + 5$ and $\psi(w_{i,2}v) = 8i + 7$,

- $\psi(s_i v) \in \{c_{j,1}, c_{j,2}, c_{j,3}\}$ for every $j = 1, \ldots, m$, and
- $\psi(t_i v) \leq 8n - 1$ and odd for every $i = 1, 2, \ldots, m/2$.

Consider the following variable assignment: set variable x_i to true if $\psi(w_{i,1}v) = 8i+5$, $\psi(w_{i,2}v) = 8i+7$, and set x_i to false if $\psi(w_{i,1}v) = 8i+1$ and $\psi(w_{i,2}v) = 8i+3$. We show that this is a satisfying assignment of φ, i.e., every clause C_j is satisfied. Assume that $\psi(s_j v) = c_{j,k}$ for some $k = 1, 2, 3$, and let the k-th literal in clause C_j be an occurrence of the variable x_i. In this case, the k-th literal of clause C_j is true in the constructed variable assignment: otherwise color $c_{j,w}$ would appear also on edge $w_{i,1}v$ or $w_{i,2}v$. Therefore, every clause contains at least one true literal, and the formula is satisfied by the variable assignment.

Now assume that φ has a satisfying variable assignment. Consider the following (partial) coloring ψ:

- $\psi(u_i v) = i$ for $i = 2, 4, \ldots, 8n$,
- for every $i = 0, 1, \ldots, n-1$,
  - if variable x_i is true, then $\psi(w_{i,1}v) = 8i + 5$ and $\psi(w_{i,2}v) = 8i + 7$,
  - if variable x_i false, then $\psi(w_{i,1}v) = 8i + 1$ and $\psi(w_{i,2}v) = 8i + 3$,

It is clear from the construction that for every $j = 1, 2, \ldots, m$, one of the colors $c_{j,1}$, $c_{j,2}$, $c_{j,3}$ is not already assigned: otherwise this would imply that clause C_j contains only false literals in the satisfying variable assignment, a contradiction. Therefore, we can set $\psi(s_j v)$ to one of these three colors. So far coloring ψ assigns $4n$ even and $2n + m$ odd colors to the edges incident to v, thus there remains exactly $m/2$ odd colors not greater than $8n$. Assign these colors to the edges $t_1 v$, $t_2 v$, $\ldots$, $t_{\frac{m}{2}} v$ in some order. Now every color not greater than $8n$ is used exactly once at v, hence there is zero error on vertex v in ψ. It is straightforward to verify that this coloring can be extended to the whole graph G such that the resulting coloring has error exactly K: in every gadget, the edges incident to v are colored in such a way that makes this extension possible. $\square$

References

[1] P. Alimonti and V. Kann. Some APX-completeness results for cubic graphs. *Theoret. Comput. Sci.*, 237(1-2):123–134, 2000.

[2] A. Bar-Noy, M. Bellare, M. M. Halldórsson, H. Shachnai, and T. Tamir. On chromatic sums and distributed resource allocation. *Inform. and Comput.*, 140(2):183–202, 1998.

[3] A. Bar-Noy and G. Kortsarz. Minimum color sum of bipartite graphs. *J. Algorithms*, 28(2):339–365, 1998.

[4] M. Biró, M. Hujter, and Zs. Tuza. Precoloring extension. I. Interval graphs. *Discrete Math.*, 100(1-3):267–279, 1992.

[5] H. L. Bodlaender. A tourist guide through treewidth. *Acta Cybernet.*, 11(1-2):1–21, 1993.

[6] H. L. Bodlaender. Treewidth: algorithmic techniques and results. In *Mathematical Foundations of Computer Science 1997 (Bratislava)*, pages 19–36. Springer, Berlin, 1997.

[7] H. L. Bodlaender. A partial k-arboretum of graphs with bounded treewidth. *Theoret. Comput. Sci.*, 209(1-2):1–45, 1998.

[8] R. Gandhi and J. Mestre. Combinatorial algorithms for data migration, To appear in *Algorithmica.*

[9] K. Giaro, R. Janczewski, M. Kubale, and M. Małafiejski. A 27/26-approximation algorithm for the chromatic sum coloring of bipartite graphs. In *APPROX 2002*, volume 2462 of *Lecture Notes in Comput. Sci.*, pages 135–145. Springer, Berlin, 2002.

[10] K. Giaro and M. Kubale. Edge-chromatic sum of trees and bounded cyclicity graphs. *Inform. Process. Lett.*, 75(1-2):65–69, 2000.

[11] M. M. Halldórsson, G. Kortsarz, and H. Shachnai. Sum coloring interval and k-claw free graphs with application to scheduling dependent jobs. *Algorithmica*, 37(3):187–209, 2003.

[12] M. Hujter and Zs. Tuza. Precoloring extension. II. Graph classes related to bipartite graphs. *Acta Mathematica Universitatis Comenianae*, 62(1):1–11, 1993.

[13] M. Hujter and Zs. Tuza. Precoloring extension. III. Classes of perfect graphs. *Combin. Probab. Comput.*, 5(1):35–56, 1996.

[14] K. Jansen. The optimum cost chromatic partition problem. In *Algorithms and complexity (Rome, 1997)*, volume 1203 of *Lecture Notes in Comput. Sci.*, pages 25–36. Springer, Berlin, 1997.

[15] E. Kubicka. *The Chromatic Sum of a Graph*. PhD thesis, Western Michigan University, 1989.

[16] E. Kubicka and A. J. Schwenk. An introduction to chromatic sums. In *Proceedings of the ACM Computer Science Conf.*, pages 15–21. Springer, Berlin, 1989.

[17] L. Lovász. The membership problem in jump systems. *J. Combin. Theory Ser. B*, 70(1):45–66, 1997.

[18] O. Marcotte and P. D. Seymour. Extending an edge-coloring. *J. Graph Theory*, 14(5):565–573, 1990.

[19] D. Marx. Minimum sum multicoloring on the edges of planar graphs and partial k-trees. In *2nd International Workshop on Approximation and Online Algorithms (WAOA) 2004 (Bergen)*, volume 3351 of *Lecture Notes in Computer Science*, pages 9–22. Springer, Berlin, 2005.

[20] D. Marx. NP-completeness of list coloring and precoloring extension on the edges of planar graphs. *J. Graph Theory*, 49(4):313–324, 2005.

[21] D. Marx. The complexity of chromatic strength and chromatic edge strength. *Comput. Complex.*, 14(4):308–340, 2006.

[22] S. Nicoloso, M. Sarrafzadeh, and X. Song. On the sum coloring problem on interval graphs. *Algorithmica*, 23(2):109–126, 1999.

[23] T. Nishizeki, J. Vygen, and X. Zhou. The edge-disjoint paths problem is NP-complete for series-parallel graphs. *Discrete Appl. Math.*, 115(1-3):177–186, 2001.

[24] M. R. Salavatipour. On sum coloring of graphs. *Discrete Appl. Math.*, 127(3):477–488, 2003.

[25] K. J. Supowit. Finding a maximum planar subset of nets in a channel. *IEEE Trans. Comput. Aided Design*, 6(1):93–94, 1987.

[26] T. Szkaliczki. Routing with minimum wire length in the dogleg-free Manhattan model is NP-complete. *SIAM J. Comput.*, 29(1):274–287, 1999.

[27] Zs. Tuza. Graph colorings with local constraints—a survey. *Discuss. Math. Graph Theory*, 17(2):161–228, 1997.

[28] X. Zhou and T. Nishizeki. Algorithm for the cost edge-coloring of trees. *J. Comb. Optim.*, 8(1):97–108, 2004.