An exact characterization of tractable demand patterns for maximum disjoint path problems

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Disjoint paths

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Input: graph G, two sets of vertices S and T, integer k.
Task: find k pairwise vertex-disjoint $S - T$ paths.

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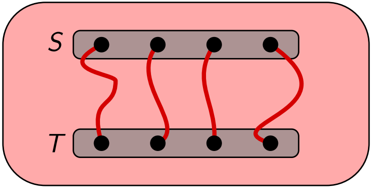

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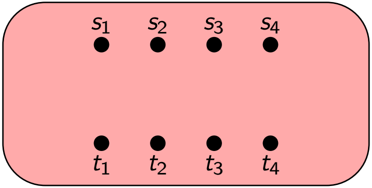
Disjoint paths – specified endpoints

k-Disjoint Paths
Input: graph G and pairs of vertices $(s_1, t_1), \ldots, (s_k, t_k)$.
Task: find pairwise vertex-disjoint paths $P_1, \ldots, P_k$ such that P_i connects s_i and t_i.



NP-hard, but FPT parameterized by k:

Theorem [Robertson and Seymour]
The k-Disjoint Paths problem can be solved in time $f(k)n^3$.

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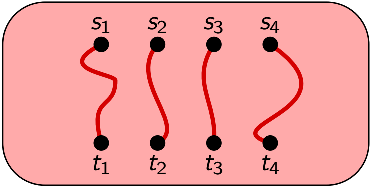

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Maximization version

We consider now a maximization version of the problem.

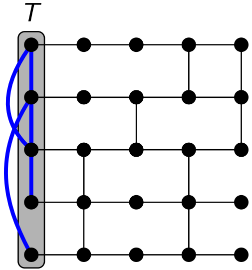
MAXIMUM DISJOINT PATHS
Input: graph G, pairs of vertices $(s_1, t_1), \ldots, (s_m, t_m)$, integer k.
Task: find k pairwise vertex-disjoint paths, each of them connecting some pair (s_i, t_i).

Can be solved in time $n^{O(k)}$, but W[1]-hard in general.

Maximization version

A different formulation:

Maximum Disjoint Paths
Input: supply graph G, set $T \subseteq V(G)$ of terminals and a demand graph H on T.
Task: find k pairwise vertex-disjoint paths such that the two end-points of each path are adjacent in H.



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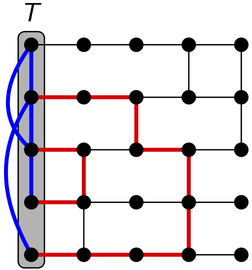

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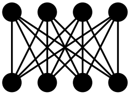
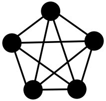
Maximum Disjoint $\mathcal{H}$-Paths

Maximum Disjoint $\mathcal{H}$-Paths:
special case when H restricted to be a member of $\mathcal{H}$.



bicliques:
in P



cliques:
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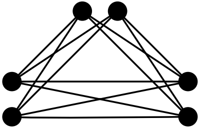
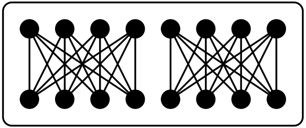
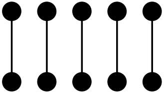


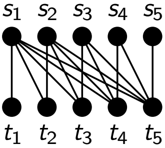
complete multipartite graphs:
in P



two disjoint bicliques:
FPT



matchings:
W[1]-hard



skew bicliques:
W[1]-hard

Maximum Disjoint $\mathcal{H}$-Paths

Questions:

- Algorithmic: FPT vs. W[1]-hard.
  - complete multipartite graphs: FPT.
  - union of two bicliques: FPT.
  - what else is FPT?

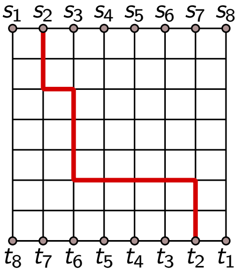
Maximum Disjoint $\mathcal{H}$-Paths

Questions:

- Algorithmic: FPT vs. W[1]-hard.
  - complete multipartite graphs: FPT.
  - union of two bicliques: FPT.
  - what else is FPT?
- Combinatorial (Erdős-Pósa): is there a function f such that there is either a set of k vertex-disjoint good paths of a set of $f(k)$ vertices covering every good path?
  - bicliques: tight Erdős-Pósa property with $f(k) = k - 1$ (Menger's Theorem)
  - cliques: Erdős-Pósa property with $f(k) = 2k - 2$
  - but false in general.

Erdős-Pósa property

Erdős-Pósa property does not hold in general:



Maximum number of disjoint valid paths is 1, but we need n vertices to cover every valid path.

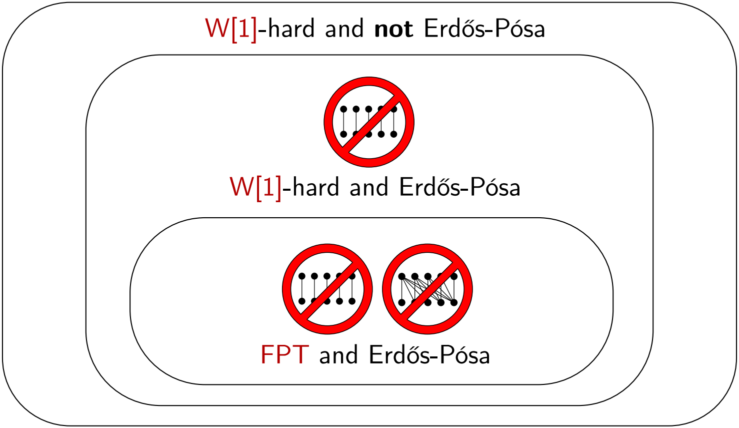
Main result

Theorem

Let $\mathcal{H}$ be a hereditary class of graphs.

1. If $\mathcal{H}$ does not contain every matching and every skew biclique, then Maximum Disjoint $\mathcal{H}$-Paths is FPT and has the Erdős-Pósa Property.
2. If $\mathcal{H}$ does not contain every matching, but contains every skew biclique, then Maximum Disjoint $\mathcal{H}$-Paths is W[1]-hard, but has the Erdős-Pósa Property.
3. If $\mathcal{H}$ contains every matching, then Maximum Disjoint $\mathcal{H}$-Paths is W[1]-hard, and does not have the Erdős-Pósa Property.

Main result

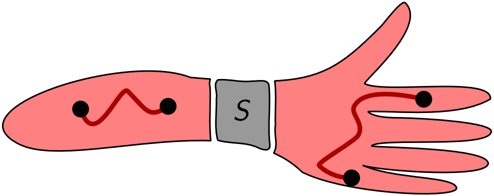

W[1]-hard and **not** Erdős-Pósa
W[1]-hard and Erdős-Pósa
FPT and Erdős-Pósa

Erdős-Pósa property

Theorem

If $\mathcal{H}$ is a hereditary class, then Maximum Disjoint $\mathcal{H}$-Paths has the Erdős-Pósa Property if and only $\mathcal{H}$ contains every matching.

A standard first step:



If there is small set S separating two valid paths, then we can do recursion.

Erdős-Pósa property

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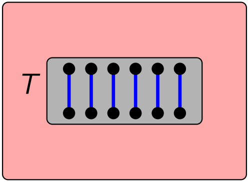
If $\mathcal{H}$ is a hereditary class, then MAXIMUM DISJOINT $\mathcal{H}$-PATHS has the Erdős-Pósa Property if and only $\mathcal{H}$ contains every matching.

We arrive to a large set T of terminals such that



- T has a perfect matching in the demand graph and
- T is highly connected in the supply graph: for any $X, Y \subseteq T$ with $|X| = |Y|$, there exists $|X|$ disjoint $X - Y$ paths.

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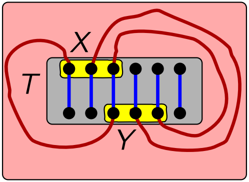

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What is this good for?

A combinatorial lemma

Observation

If a graph H on n vertices has a perfect matching, then either

- H contains an **induced** matching of size $\Omega(\log n)$ or
- H has two sets X, Y of size $\Omega(\log n)$ that are completely connected to each other.

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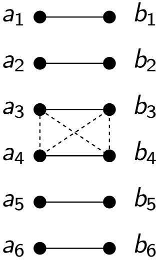
- For every $i < j$, there are 2^4 possibilities for the 4 edges between $\{a_i, b_i\}$ and $\{a_j, b_j\}$.
- If there is a large matching, then there is a large matching that is homogeneous with respect to these 16 possibilities.



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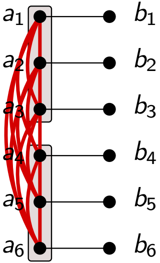
- In each of the 16 cases, we find a matching, clique, or biclique as induced subgraph.



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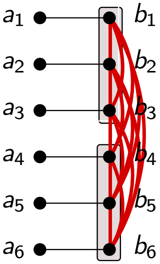

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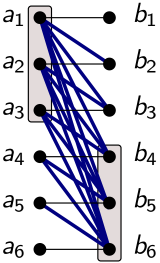

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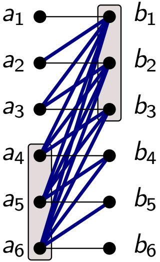

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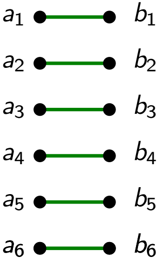

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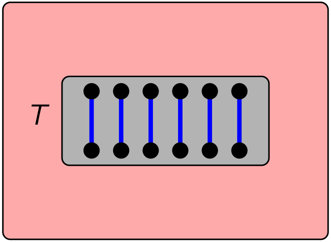

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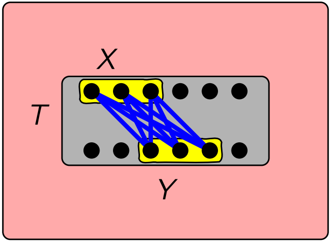

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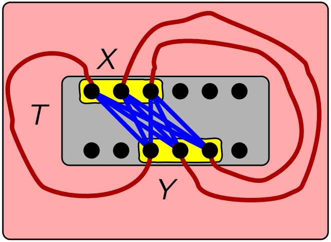

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$$\Downarrow$$

Connectedness condition implies that there are many disjoint $X - Y$ paths, which are valid paths.

Approximation

The Erdős-Pósa result can be stated algorithmically, giving an approximation algorithm as a byproduct:

Theorem

Let $\mathcal{H}$ be a hereditary class of graph not containing every matching. Given an instance of MAXIMUM DISJOINT $\mathcal{H}$-PATHS, in time $2^{2^{O(k)}} \cdot n^{O(1)}$ we can either

- find k disjoint valid paths or
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A standard consequence:

Theorem

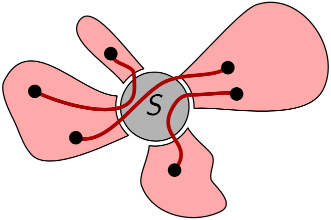
If $\mathcal{H}$ is a hereditary class of graphs not containing every matching, then there is a polynomial-time algorithm for MAXIMUM DISJOINT $\mathcal{H}$-PATHS that finds a solution of size $O(\log\log OPT)$.

From approximation to exact

Theorem

If $\mathcal{H}$ is a hereditary class of graphs that does not contain every matching and every skew biclique, then Maximum Disjoint $\mathcal{H}$-Paths is FPT.

We use the approximation algorithm to find a small set S covering every valid path.



Goal: reduce the number of terminals to $f(k)$.

Then brute force + the Robertson-Seymour algorithm gives an FPT algorithm.

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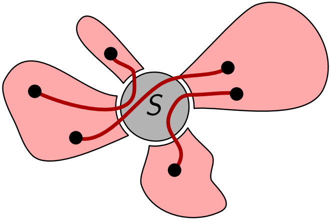

Main argument: we can mark $f(k)$ terminals in each component of $G - S$ and show that every solution can be modified to use only marked terminals.

Representative sets

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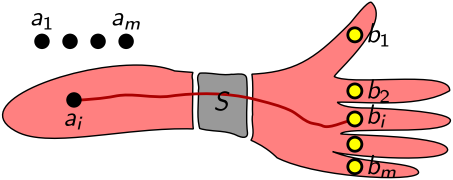
Bad news: seems impossible to do in general without looking at the other components.

Example: Suppose that the demand graph contains only the edges $a_i b_i$.



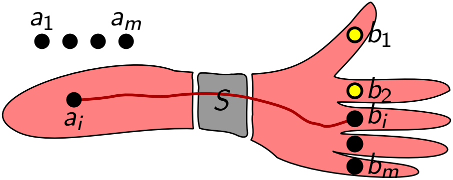
We cannot decide which a_i to mark without knowing which b_i is on the other side.

Representative sets

Main argument: we can mark $f(k)$ terminals in each component of $G - S$ and show that every solution can be modified to use only marked terminals.

Good news: much easier if we exclude induced matchings and induced skew bicliques from the demand graph.

Example: Suppose that the demand graph contains only the edges $a_i b_j$ with $i \neq j$.



It is sufficient to mark, say, a_1 and a_2: no matter which b_j is reachable, one of them is compatible.

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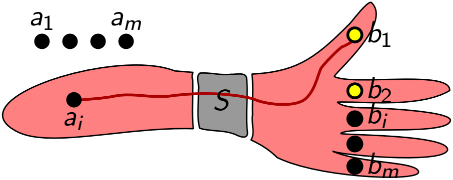

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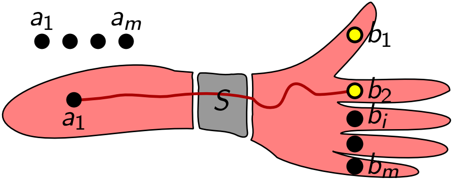

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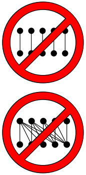

If we exclude induced matchings and induced skew bicliques, then we can compute a **representative set** of $f(k)$ partial solutions for each component such that every solution can be modified to use only these partial solutions.

$\Downarrow$

We can mark $f(k)$ terminals in each component of $G - S$.

Conceptually similar to other FPT applications of representative sets, but here works only if there are no induced matchings and induced skew bicliques (again some Ramsey statement behind this).

Summary

- Complete characterization of classes $\mathcal{H}$ for which Maximum Disjoint $\mathcal{H}$-Paths is FPT or has the Erdős-Pósa properties.
- Interesting collection of technical tools: Ramsey's Theorem, tangles, important separators, representative sets, ...
- **Open:** FPT-approximation for Maximum Disjoint Paths for arbitrary patterns?