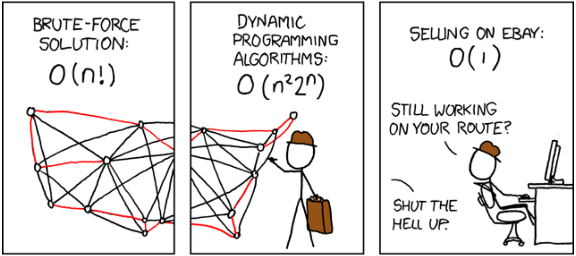
A subexponential parameterized algorithm for Subset TSP on planar graphs

Philip N. Klein   Dániel Marx



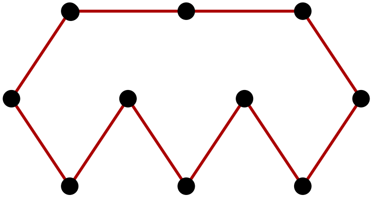
http://xkcd.com/399/

SODA 2014
January 7, 2014
Portland, OR

TSP

TSP

Input: A set T of cities and a distance function d on T
Output: A tour on T with minimum total distance



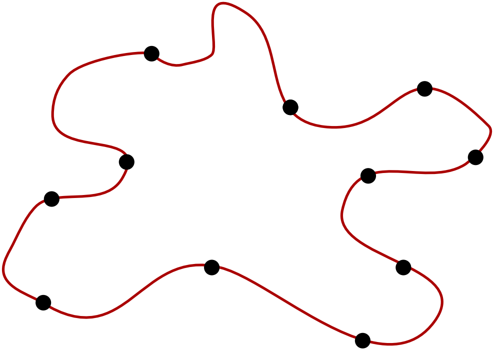
Theorem [Held and Karp 1962]

TSP with n cities can be solved in time $2^n \cdot n^2 \cdot \log D$, where D is the maximum (integer) distance.

Dynamic programming:
Let $x(v, T')$ be the minimum length of path from v_{start} to v visiting all the cities $T' \subseteq T$.

c-change TSP

- c-change operation: removing c steps of the tour and connecting the resulting c paths in some other way.
- A solution is c-OPT if no c-change can improve it.
- We can find a c-OPT solution in $n^{O(c)} \cdot D$ time, where D is the maximum (integer) distance.



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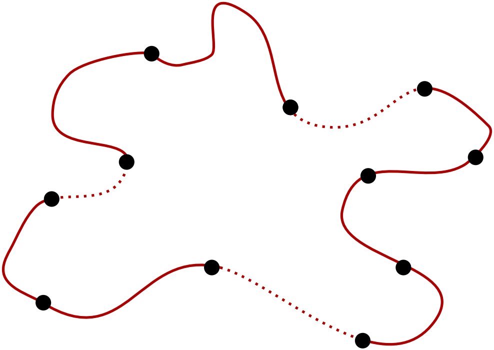

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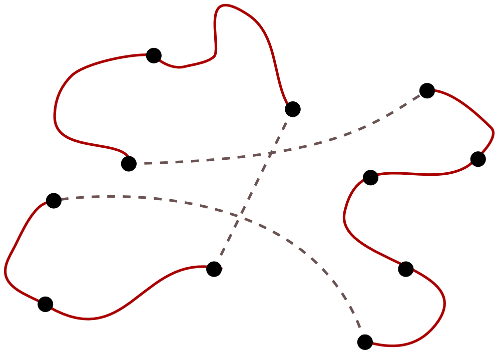
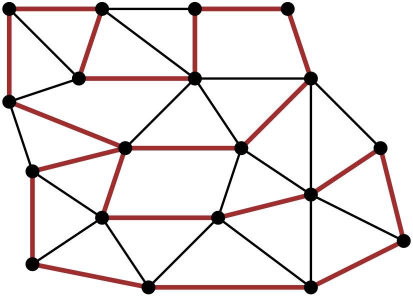


TSP on planar graphs

Assume that the cities correspond to the set of all vertices of a (weighted) planar graph and distance is measured in this (weighted) planar graph.



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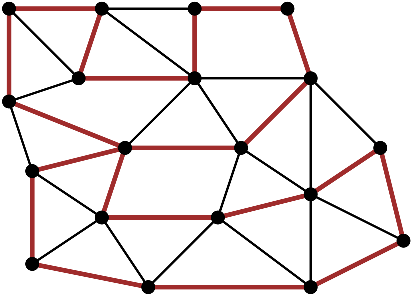
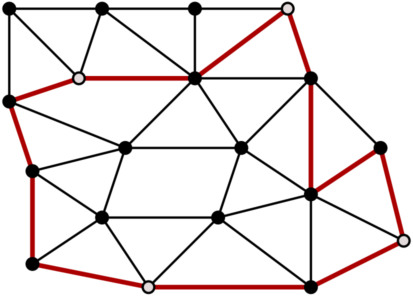


- Can be solved in time $n^{O(\sqrt{n})}$.
- Admits a PTAS.

Subset TSP on planar graphs

Assume that the cities correspond to a subset T of vertices of a planar graph and distance is measured in this planar graph.



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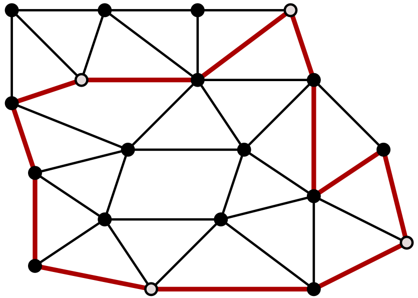

- Can be solved in time $n^{O(\sqrt{n})}$.
- Can be solved in time $2^k \cdot n^{O(1)}$.
- **Question:** Can we restrict the exponential dependence to k **and** exploit planarity?

Subset TSP on planar graphs

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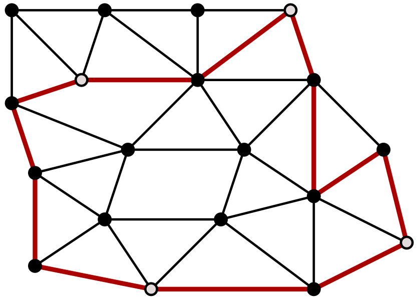

Theorem

Subset TSP for k cities in a unit-weight planar graph can be solved in time $2^{O(\sqrt{k}\log k)} \cdot n^{O(1)}$.

Subset TSP on planar graphs

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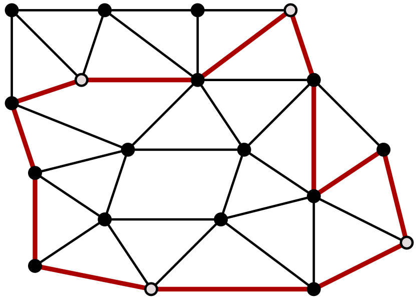

Theorem

Subset TSP for k cities in a weighted planar graph can be solved in time $(2^{O(\sqrt{k}\log k)} + W) \cdot n^{O(1)}$ if the weights are integers not more than W.

Partial solutions

General idea: build larger and larger partial solutions.

Held-Karp algorithm: the partial solutions are $v_{\text{start}} - v$ paths visiting a subset T' of cities.

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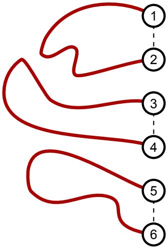


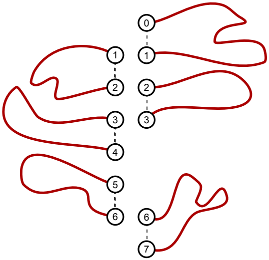
Generalization: a partial solution is a set of at most d pairwise disjoint paths with specified cities as endpoints.

The **type** of a partial solution can be described by

- the set of endpoints of the paths,
- a matching between the endpoints, and
- the subset T' of visited cities.

Merging partial solutions

Two compatible partial solutions can be merged in an obvious way:



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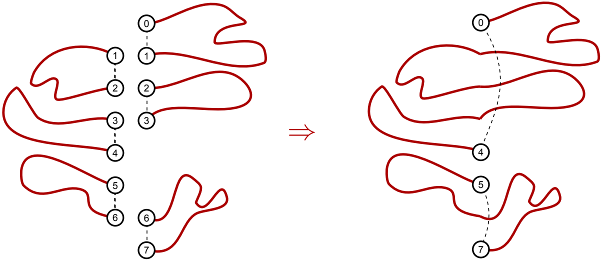

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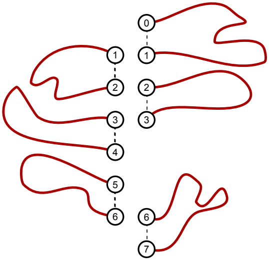

Algorithm

- Start with an initial set of trivial partial solutions.
- Combine two partial solutions as long as possible.
- Keep at most one partial solution from each type: the best one encountered so far.
- Return the best partial solution that consists of a single path (cycle) visiting all vertices.

Running time

Algorithm

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With careful implementation, the running time is dominated by the number of types, whose number has two factors:

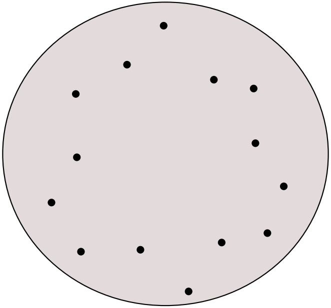
- endpoints described by at most d pairs of vertices
  $\Rightarrow k^{2d}$ possibilities,
- describing the subset T' of visited cities
  $\Rightarrow 2^k$ possibilities.

We can increase d up to $O(\sqrt{k})$, but we need to reduce somehow the number of possible subsets of cities!

Restricting the subset of cities

We restrict attention to a collection $\mathcal{T}$ of subsets of cities and consider only partial solutions that visit a subset in $\mathcal{T}$.

We need: a collection $\mathcal{T}$ of size $k^{O(\sqrt{k})}$ that guarantees finding an optimum solution.



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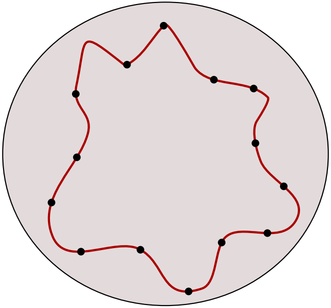

Definition of $\mathcal{T}$:

- Find a 4-OPT tour.

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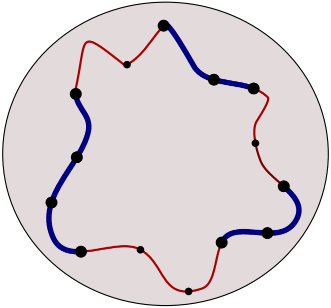


Definition of $\mathcal{T}$:

- Find a 4-OPT tour.
- A subset is in $\mathcal{T}$ if and only if it induces $O(\sqrt{k})$ consecutive intervals on the 4-OPT tour.

Main result

Definition of $\mathcal{T}$:

- Find a 4-OPT tour.
- A subset is in $\mathcal{T}$ if and only if it induces $O(\sqrt{k})$ consecutive intervals on the 4-OPT tour.

Theorem

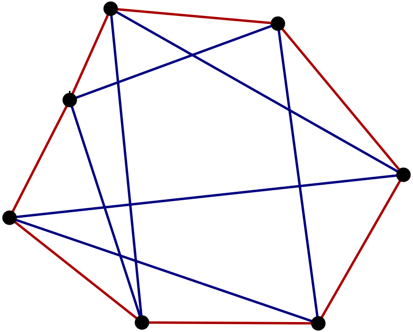
After setting $\mathcal{T}$ as above and $d = O(\sqrt{k})$, the Algorithm finds an optimum solution for Subset TSP on planar graphs.

Corollary

Subset TSP for k cities in a planar graph can be solved in time $(2^{O(\sqrt{k}\log k)} + W) \cdot n^{O(1)}$ if the weights are integers at most W.

The treewidth bound

Consider the union of an optimum solution and a 4-OPT solution as a graph on k vertices:



Lemma

For every 4-OPT solution, there is an optimum solution such that their union has treewidth $O(\sqrt{k})$.

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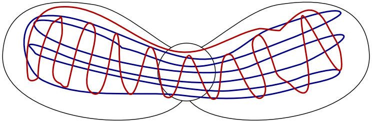
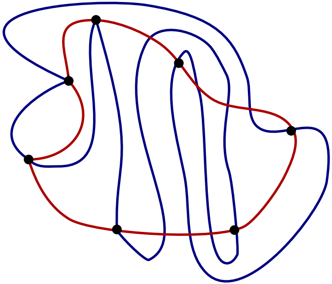


- The union has separators of size $O(\sqrt{k})$.
- In each component, the set of cities visited by the optimum solution is nice: it is the same as what $O(\sqrt{k})$ segments of the 4-OPT tour visited.
- We can use this tree decomposition to prove that the Algorithm finds an optimum solution.

Proof of the treewidth bound

Consider the closed walk corresponding to the 4-OPT solution and pick an optimum solution and a closed walk representing that.

The **union** is a planar graph (we ignore degree-2 vertices now):



Select the optimum solution and the closed walk such that the two tours cross each other the minimum number of times.

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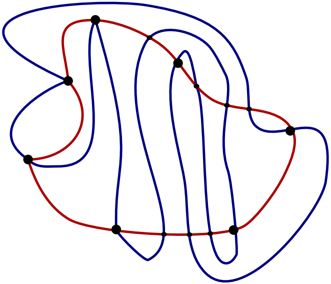

We give an $O(\sqrt{k})$ bound on the treewidth of this planar graph

$\Downarrow$

A $O(\sqrt{k})$ bound follows for the k-vertex graph, as it is a minor of this graph after duplicating the vertices.

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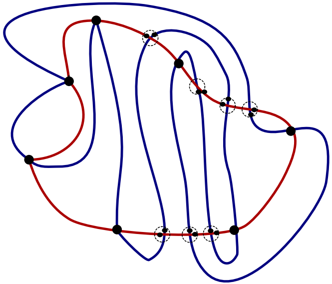

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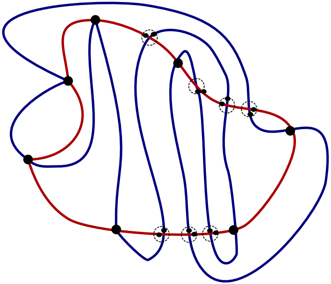


We prove that every 3-connected component of the planar graph has $O(k)$ vertices of degree > 2

$\Downarrow$

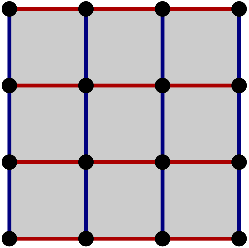
$O(\sqrt{k})$ treewidth bound on the 3-connected components

$\Downarrow$

same bound for the whole graph.

Grids

A **grid** is a 16-vertex subgraph of the union of the 4-OPT solution and the optimum solution:



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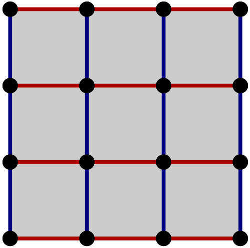


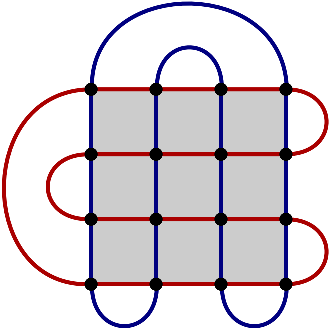
Lemma

If a 3-connected component of the union has size $\Omega(k)$, then there is a grid.

Proof idea: 4-regular and $O(k)$ faces have length < 4
$\Rightarrow$ Euler's formula implies that most of the faces have length 4
$\Rightarrow$ a 4-face surrounded by 4-faces should be a grid.

Grids

Suppose that the grid is used like this by two tours:



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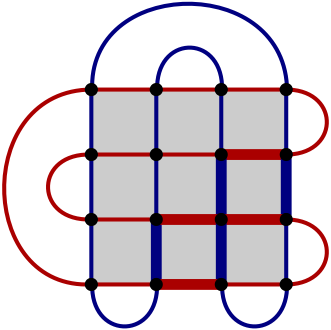

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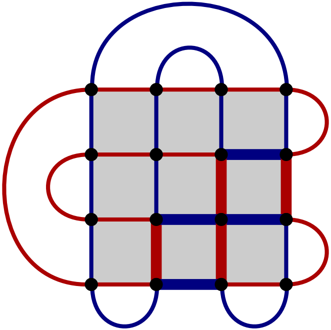
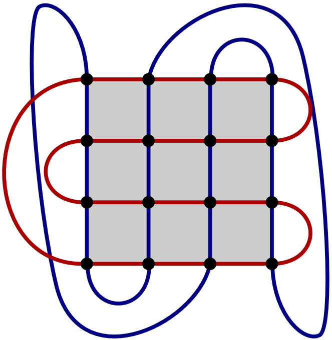


- Let us exchange these two sets of edges between the two tours.
- The 4-OPT tour cannot improve.
- The optimum tour cannot improve.
- We get another optimum tour that has fewer crossings with the 4-OPT tour.

Grids — other cases:

C type + S type:



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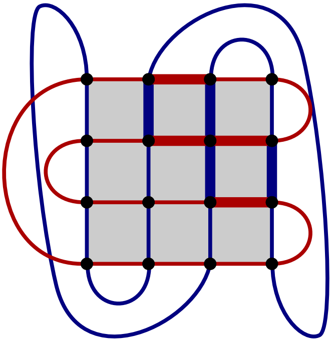

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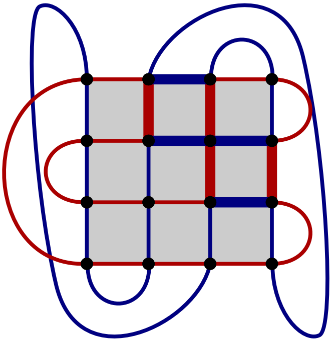

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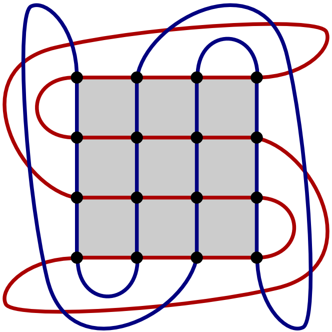

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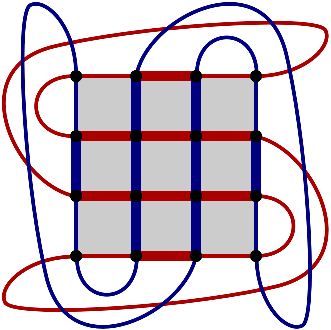

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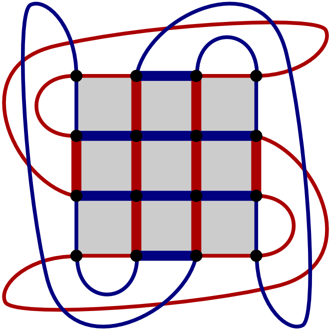
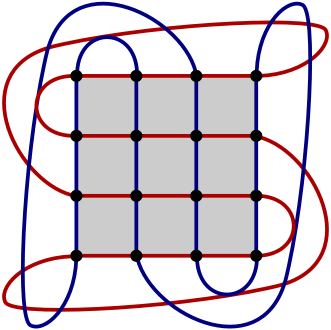


Grids — other cases:

S type + inverted S type:



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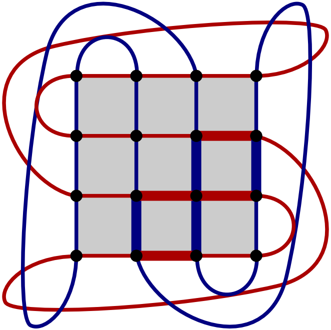

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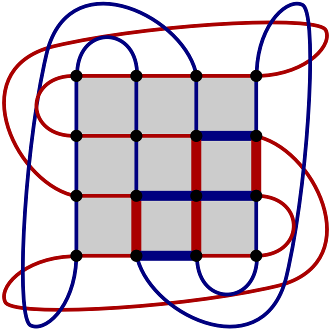

Overview

- Algorithm:
  - Find a 4-OPT tour.
  - Partial solutions: $O(\sqrt{k})$ disjoint paths, visiting $O(\sqrt{k})$ consecutive intervals on the 4-OPT tour.
  - Merge partial solutions until the optimum solution is found.
- Treewidth bound: the union of the 4-OPT tour and some optimum tour is a k-vertex graph with treewidth $O(\sqrt{k})$.
  - Study the union in the planar graph.
  - Every 3-connected component has $O(k)$ vertices of degree > 2, otherwise there is a grid and an exchange argument could be used.
  - Union in the planar graph has treewidth $O(\sqrt{k}) \Rightarrow$ the k-vertex graph has treewidth $O(\sqrt{k})$.