Finding small patterns in permutations in linear time

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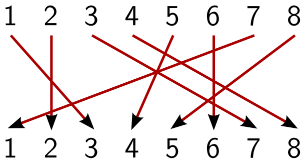
Permutations

Different interpretations:

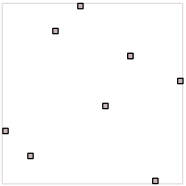
- Bijective mapping $\sigma : [n] \to [n]$:



- Ordering of $[n]$:

3 2 7 8 4 6 1 5

- n points in the plane "in general position":



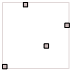
Subpermutations

There is a natural way of defining the meaning of σ being a subpermutation of π (or a "permutation pattern" in π).

Example:

σ

1 4 2 3



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π

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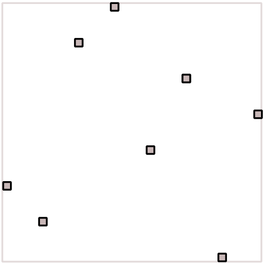

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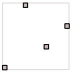

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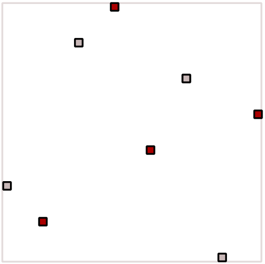

Excluding subpermutations

There are $n!$ permutations of length n, but their number is much smaller if we exclude a fixed permutation:

Theorem [Marcus and Tardos 2004]

For every fixed permutation σ, the number of permutations of length n avoiding σ is at most c^n for some constant c depending on σ.

Example:

The number of permutations of length n avoiding 231 is exactly the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n} < 4^n$.

Finding patterns in permutations

> PERMUTATION PATTERN
> **Input:** Two permutations σ and π.
> **Decide:** Is σ a subpattern of π?

- NP-hard in general [Bose, Buss, and Lubiw 1998].
- Can be solved in time $n^{\ell+O(1)}$ by brute force, where $\ell = |\sigma|$ and $n = |\pi|$.
- Can be solved in time $n^{0.47\ell+o(\ell)}$ [Ahal and Rabinovich 2008].

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Main result:

Theorem

PERMUTATION PATTERN can be solved in time $2^{O(\ell^2 \log \ell)} \cdot n$, where $\ell = |\sigma|$ and $n = |\pi|$.

Decompositions

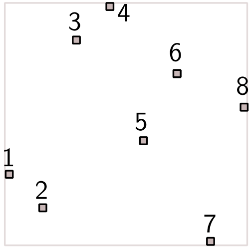
We define the notion of d-**wide decomposition** and solve the problem using the following win/win strategy:
($\ell = |\sigma|$, $n = |\pi|$)

1. There is an algorithm that either
   - finds σ in π or
   - finds a $2^{O(\ell \log \ell)}$-wide decomposition of π.
2. There is an algorithm that, given σ and a d-wide decomposition of π, decides if σ is a subpattern of π in time $(d\ell)^{O(\ell)} \cdot n$.

These two algorithms together give a $2^{O(\ell^2 \log \ell)} \cdot n$ time algorithm for Permutation Pattern.

Decompositions

Starting with a set of points (degenerate rectangles), the decomposition is a sequence of merges, where a merge consists of replacing two rectangles with their bounding box:



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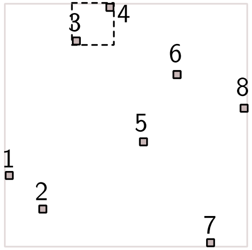

$(3, 4)$

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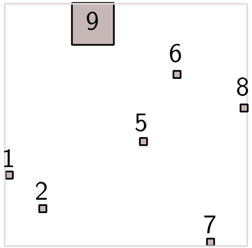

$(3, 4) \rightarrow 9$

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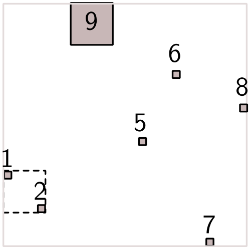

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$(1, 2)$

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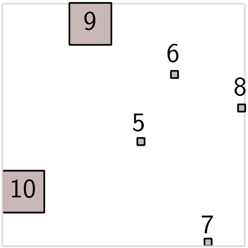

$(3,4) \to 9$
$(1,2) \to 10$

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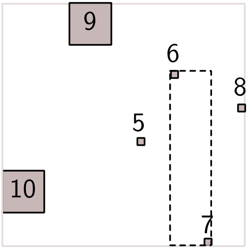

$(3,4) \to 9$
$(1,2) \to 10$
$(6,7)$

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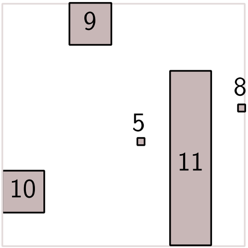

$(3, 4) \rightarrow 9$
$(1, 2) \rightarrow 10$
$(6, 7) \rightarrow 11$

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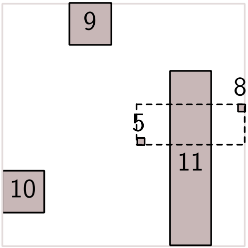

$(3, 4) \rightarrow 9$
$(1, 2) \rightarrow 10$
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$(5, 8)$

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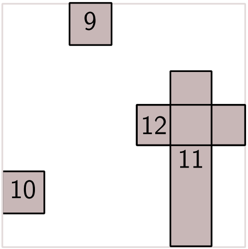

$(3,4) \to 9$
$(1,2) \to 10$
$(6,7) \to 11$
$(5,8) \to 12$

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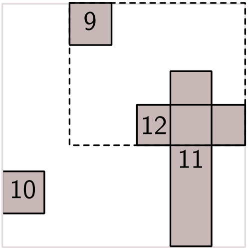

$(3,4) \to 9$ $\quad (9,12)$
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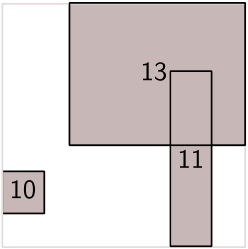

$(3,4) \rightarrow 9$ $\quad (9,12) \rightarrow 13$
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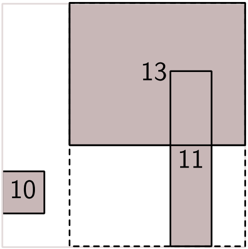

$(3,4) \to 9$
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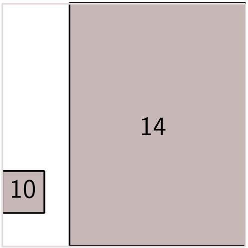

$(3,4) \rightarrow 9$ $\quad (9,12) \rightarrow 13$
$(1,2) \rightarrow 10$ $\quad (13,11) \rightarrow 14$
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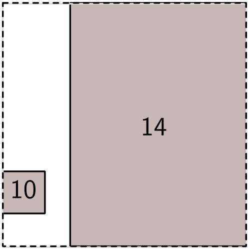

$(3,4) \to 9$

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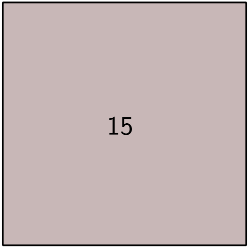

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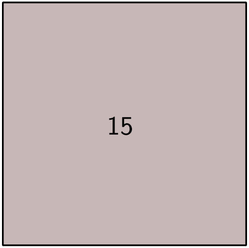
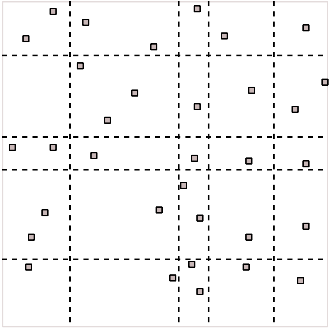


- R_1 sees R_2 horizontally (resp., vertically) if there is a horizontal (resp., vertical) line intersecting both.
- A rectangle family is d-**wide** if every rectangle sees less than d other rectangles horizontally and less than d other rectangles vertically.
- The decomposition is d-**wide** if it is d-wide in every step.

Grids

$r \times r$-grid: partitioning the rows and the columns into r classes such that every cell contains a point.



Observation: If a point set has an $r \times r$-grid, then it contains every permutation of length r.

Fact

If π has an $r \times r$-grid, then every decomposition of π is $\Omega(r)$-wide.

Finding decompositions

Large grids imply large width:

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If π contains an $r \times r$-grid, then every decomposition of π is $\Omega(r)$-wide.

Large width implies large grids:

Theorem

There is an $O(n)$ time algorithm that finds either an $r \times r$ grid in π or a $2^{O(r \log r)}$-wide decomposition of π.

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Theorem

There is an $O(n)$ time algorithm that finds either an $r \times r$ grid in π or a $2^{O(r \log r)}$-wide decomposition of π.

The algorithm relies on the following previous result:

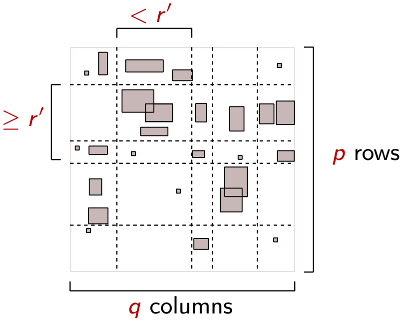
Theorem (essentially [Marcus and Tardos 2004])

If M is a point set in $[p] \times [q]$ with $|M| > r^4 \binom{r^2}{r}(p+q)$, then we can find an $r \times r$-grid in M.

Finding decompositions

We maintain a partition of rows and columns that is compatible with every current rectangle and satisfies that

- every row/column contains $< r'$ rectangles and
- every two adjacent rows/columns contain $\geq r'$ rectangles.



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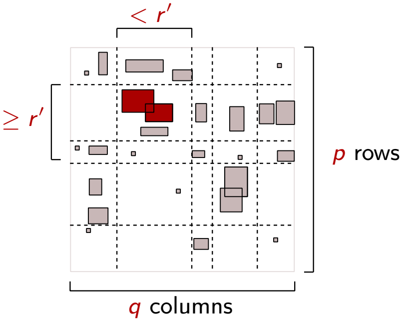

Case 1: If a cell contains two rectangles, merge them. If two adjacent rows/columns contain $< r'$ rectangles, then merge them.

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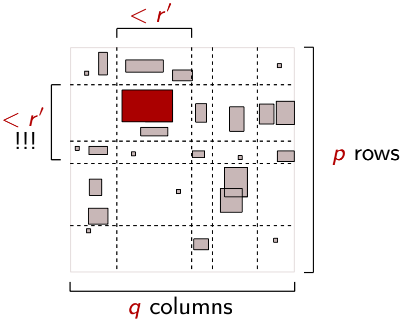

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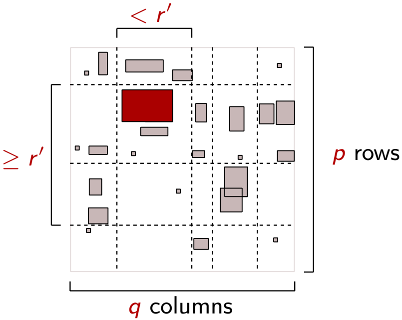

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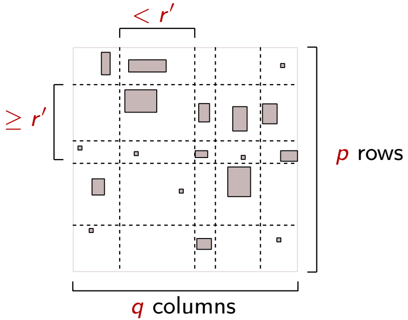


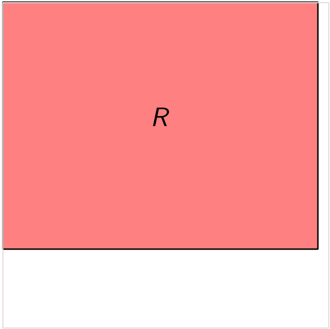
Case 2: Every cell contains at most one rectangle
$\Rightarrow$ There are $\Omega((p+q)r')$ nonempty cells
$\Rightarrow$ Result of [Marcus and Tardos 2004] implies that there is an $r \times r$-grid (if r' is sufficiently large).

Finding patterns

We would like to design a dynamic programming algorithm to solve PERMUTATION PATTERN using a given decomposition.

Problem:

The decomposition does not break the problem into independent subproblems.



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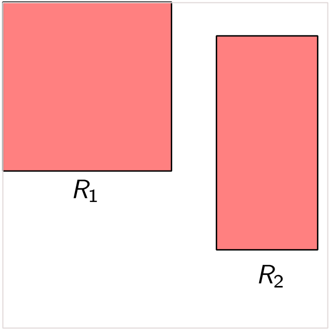

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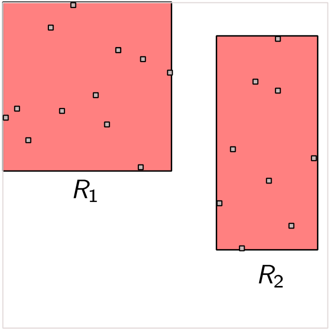

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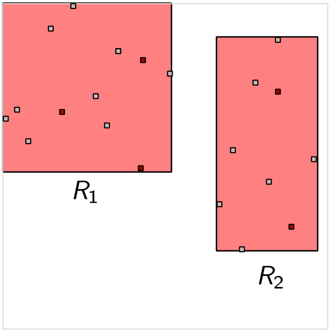

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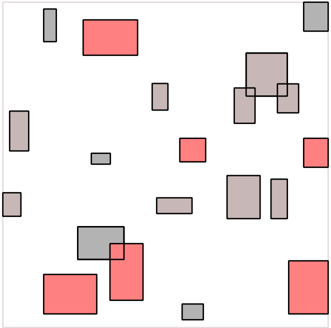
Connected components

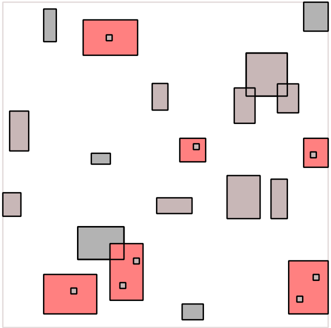
Visibility graph: two rectangles are adjacent if they see each other horizontally or vertically.



Observation: The degree of the visibility graph is less than $2d$ if the rectangle family is d-wide. Therefore, there are $d^{O(\ell)} \cdot n$ sets K of size $\leq \ell$ that are connected in the visibility graph.

Connected components

Visibility graph: two rectangles are adjacent if they see each other horizontally or vertically.

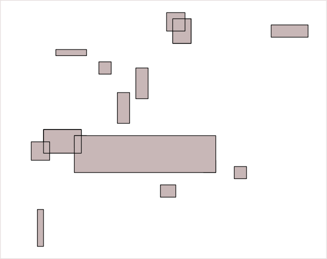


Subproblems defined by

- step i of the decomposition,
- a connected set K of size $\leq \ell$ in the visibility graph,
- a subpermutation σ' if σ, and
- a mapping of σ' into the rectangles of K.

Connected components

How to solve a subproblem at step i using the subproblems at step $i-1$?



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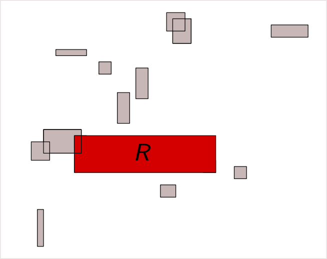

- If a rectangle R of K was created at step i by merging R_1 and R_2, then we have to distribute the points assigned to R in every possible way between R_1 and R_2.
- In step $i-1$, set K falls apart into some number of connected sets, but the interaction between them is completely understood.

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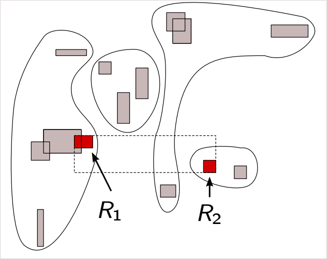

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Algorithm summary

Theorem

PERMUTATION PATTERN can be solved in time $2^{O(\ell^2 \log \ell)} \cdot n$, where $\ell = |\sigma|$ and $n = |\pi|$.

Win/win strategy:

1. There is an algorithm that either
   - finds σ in π or
   - finds a $2^{O(\ell \log \ell)}$-wide decomposition of π.
2. There is an algorithm that, given σ and a d-wide decomposition of π, decides if σ is a subpattern of π in time $(d\ell)^{O(\ell)} \cdot n$.

Hardness

PARTITIONED PERMUTATION PATTERN: for every $i \in \sigma$, a subset $S_i \subseteq \pi$ is given where it can be mapped.

Theorem

PARTITIONED PERMUTATION PATTERN is W[1]-hard parameterized by $|\sigma|$.

3-DIMENSIONAL PERMUTATION PATTERN: natural generalization to 3-dimensional points in general position.

Theorem

3-DIMENSIONAL PERMUTATION PATTERN is W[1]-hard parameterized by $|\sigma|$.

Conclusions

- Finding patterns in permutations is fixed-parameter tractable.
- Algorithm is based on a novel width measure for permutations.
- Win/win situation similar to certain algorithms based on minors and bidimensionality.
- But our decomposition is strictly speaking not a decomposition.