The Complexity Landscape of Fixed-Parameter Directed Steiner Network Problems

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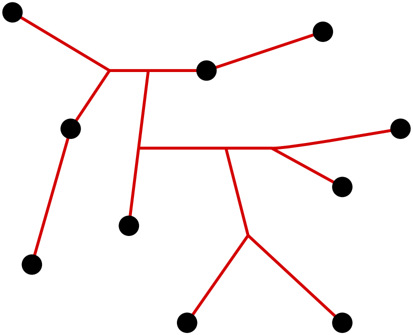
SIWAG 2016
Polignano a Mare
September 26, 2016

Steiner Tree

Steiner Tree

Given an edge-weighted graph G and set $T \subseteq V(G)$ of terminals, find a minimum-weight tree in G containing every vertex of T.



Steiner Tree

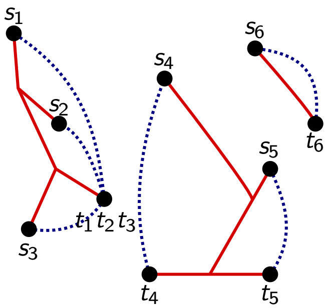
Some known results:

- NP-hard
- Easy 2-approximation: use a minimum spanning tree.
- 1.386-approximation [Byrka et al. 2013].
- $3^k \cdot n^{O(1)}$ time algorithm for k terminals using dynamic programming (i.e., fixed-parameter tractable parameterized by the number of terminals)
- Can be improved to $2^k \cdot n^{O(1)}$ time using fast subset convolution [Björklund et al. 2006].

Steiner Forest

Steiner Forest

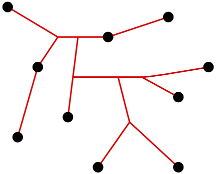
Given an edge-weighted graph G and a list $(s_1, t_1), \ldots, (s_k, t_k)$ of pairs of terminals, find a minimum-weight forest in G that connects s_i and t_i for every $1 \le i \le k$.



Fixed-parameter tractable parameterized by k: Guess a partition of the $2k$ terminals ($k^{O(k)} = 2^{O(k \log k)}$ possibilities) and solve a Steiner Tree for each class of the partition.

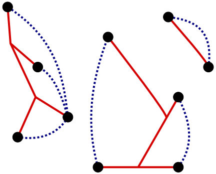
Variants of Steiner Tree

Steiner Tree



Connect all the terminals

Steiner Forest



Create connections
satisying every request

Variants of Steiner Tree

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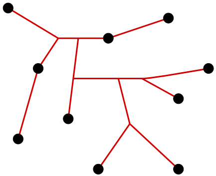

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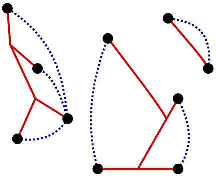

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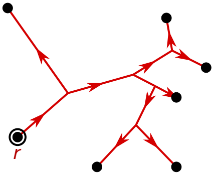
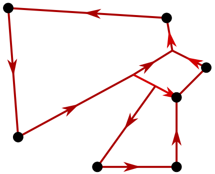
Steiner Tree



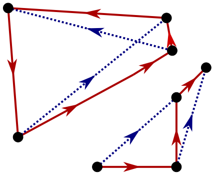
Make every terminal reachable from the root

Strongly Connected Steiner Subgraph (SCSS)



Make all the terminals reachable from each other

Directed Steiner Network (DSN)



Create connections satisying every request

Directed Steiner vs. SCSS

The DP for Steiner Tree generalizes to the directed version:

Directed Steiner Tree with k terminals can be solved in time $2^k \cdot n^{O(1)}$.

Directed Steiner vs. SCSS

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Directed Steiner Tree with k terminals can be solved in time $2^k \cdot n^{O(1)}$.

SCSS seems to be much harder:

Theorem [Feldman and Ruhl 2006]

Strongly Connected Steiner Subgraph with k terminals can be solved in time $n^{O(k)}$.

Theorem [Chitnis, Hajiaghayi, and M. 2014]

Assuming ETH, Strongly Connected Steiner Subgraph is W[1]-hard and has no $f(k)n^{o(k/\log k)}$ time algorithm for any function f.

Directed Steiner Network

Theorem [Feldman and Ruhl 2006]

Directed Steiner Network with k requests can be solved in time $n^{O(k)}$.

Corollary: Strongly Connected Steiner Subgraph with k terminals can be solved in time $n^{O(k)}$.

Proof is based on a "pebble game": $O(k)$ pebbles need to reach their destinations using certain allowed moves, tracing the solution.

Directed Steiner Network

A new combinatorial result:

Theorem [Feldmann and M. 2016]

[The underlying undireced graph of] every minimum cost solution of Directed Steiner Network with k requests has cutwidth and treewidth $O(k)$.

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A new algorithmic result:

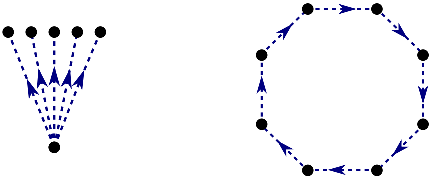
Theorem [Feldmann and M. 2016]

If a Directed Steiner Network instance with k requests has a minimum cost solution with treewidth w [of the underlying undirected graph], then it can be solved in time $f(k, w) \cdot n^{O(w)}$.

Corollary: A new proof that DSN and SCSS can be solved in time $f(k)n^{O(k)}$.

Special cases of Directed Steiner Network

Directed Steiner Tree and Strongly Connected Steiner Subgraph are both restrictions of Directed Steiner Network to certain type of patterns:



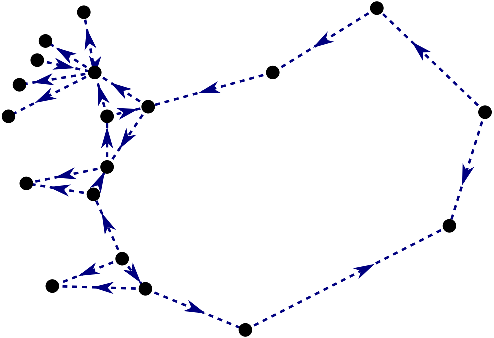
Directed Steiner Tree SCSS

Goal: characterize the patterns that give rise to FPT/W[1]-hard problems.

Patterns for Directed Steiner Network

Question:

What is the complexity of Directed Steiner Network for this pattern?



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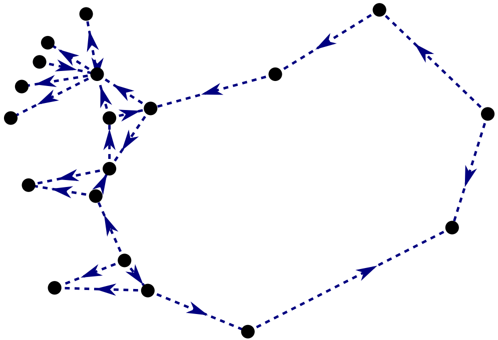


Answer:

Directed Steiner Network has an $n^{O(k)}$ algorithm for k requests, so it is polynomial-time solvable for every fixed pattern.

Patterns for Directed Steiner Network

Goal: For every class of $\mathcal{H}$ of directed patterns, characterize the complexity of Directed Steiner Network *when restricted to demand patterns from* $\mathcal{H}$.

Example:

- If $\mathcal{H}$ is the class of all directed in-stars (or out-stars), then $\mathcal{H}$-DSN is FPT.
- If $\mathcal{H}$ is the class of all directed cycles, then $\mathcal{H}$-DSN is W[1]-hard.

Patterns for DIRECTED STEINER NETWORK

Goal: For every class of $\mathcal{H}$ of directed patterns, characterize the complexity of DIRECTED STEINER NETWORK *when restricted to demand patterns from* $\mathcal{H}$.

Example:

- If $\mathcal{H}$ is the class of all directed in-stars (or out-stars), then $\mathcal{H}$-DSN is FPT.
- If $\mathcal{H}$ is the class of all directed cycles, then $\mathcal{H}$-DSN is W[1]-hard.

Main result:

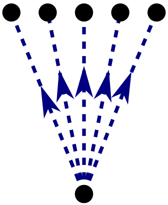
Theorem [Feldmann and M. 2016]

For any class $\mathcal{H}$ of directed patterns,

- if $\mathcal{H}$ has combinatorial property X, then $\mathcal{H}$-DSN and
- $\mathcal{H}$-DSN is W[1]-hard otherwise.

FPT special cases

What classes $\mathcal{H}$ give FPT cases of $\mathcal{H}$-DSN?



We know that out-stars are FPT.

FPT special cases

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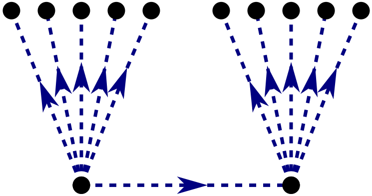

This is also FPT: minimal solutions have bounded treewidth.

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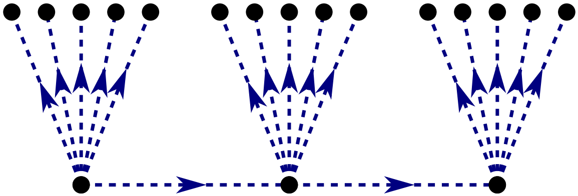

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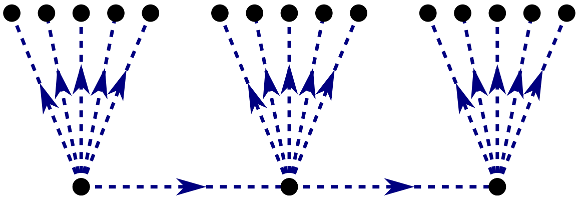

$\mathcal{C}_\lambda$: in- or out-caterpillar of length λ.

Lemma

If the pattern is in $\mathcal{C}_\lambda$, then every minimal solution has treewidth $O(\lambda^2)$.

FPT special cases

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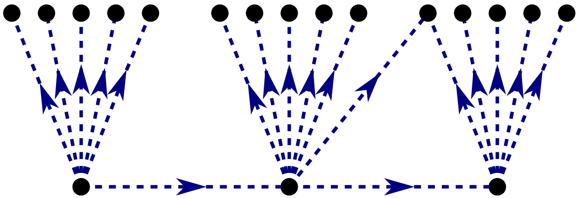
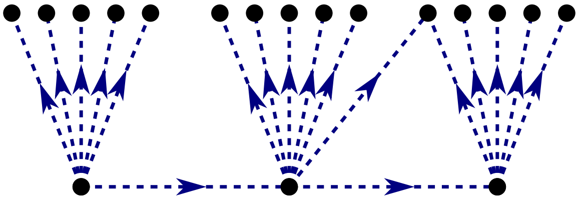


What about this pattern?

FPT special cases

What classes $\mathcal{H}$ give FPT cases of $\mathcal{H}$-DSN?

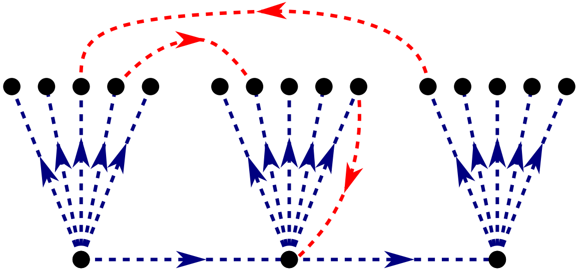


Lemma

If the pattern is **transitively equivalent** to a member of $\mathcal{C}_\lambda$, then every minimal solution has treewidth $O(\lambda^2)$.

FPT special cases

What classes $\mathcal{H}$ give FPT cases of $\mathcal{H}$-DSN?



$\mathcal{C}_{\lambda,\delta}$: in- or out-caterpillar of length λ with δ additional edges.

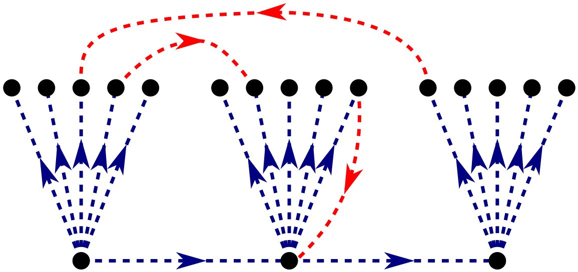
Lemma

If the pattern is **transitively equivalent** to a member of $\mathcal{C}_{\lambda,\delta}$, then every minimal solution has treewidth $O((1+\lambda)(\lambda+\delta))$.

FPT special cases

Theorem

If every $H \in \mathcal{H}$ is **transitively equivalent** to a member of $\mathcal{C}_{\lambda,\delta}$ for some constants $\lambda, \delta \geq 0$, then $\mathcal{H}$-DSN is FPT.



Does this cover all the FPT cases?

FPT special cases

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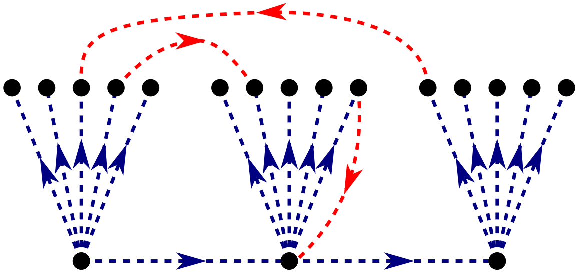
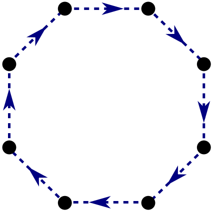


Does this cover all the FPT cases?

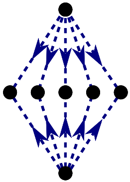
(Yes)

W[1]-hard special cases

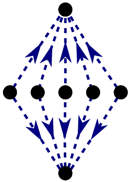
We show that the following classes $\mathcal{H}$ make $\mathcal{H}$-DSN W[1]-hard:



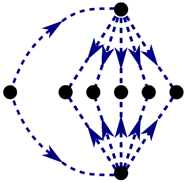
cycles (SCSS)



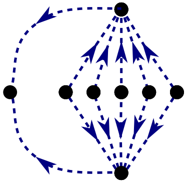
out-diamonds



in-diamonds



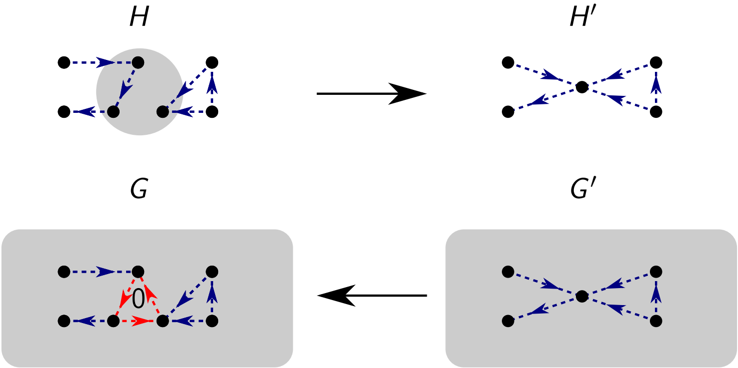
flawed out-diamonds



flawed in-diamonds

Identifying terminals

If H' is obtained from H by **identifying terminals**, then the problem cannot be harder for H' than for H:



$\Rightarrow$ We can assume that $\mathcal{H}$ is closed under identifying terminals.

Combinatorial classification

The following combinatorial result connects the algorithmic and the hardness results:

Theorem

Let $\mathcal{H}$ be a class of patterns closed under identifying terminals and transitive equivalence. Then exactly one of the following holds:

1. There are constants λ, δ such that every $H \in \mathcal{H}$ is transitively equivalent to a member of $\mathcal{C}_{\lambda,\delta}$
2. $\mathcal{H}$ contains either
   - all directed cycles,
   - all in-diamonds,
   - all out-diamonds,
   - all flawed in-diamonds, or
   - all flawed out-diamonds.

Classification result

Our main result:

Theorem [Feldmann and M. 2016]

Let $\mathcal{H}$ be a class of patterns.

1. If there are constants λ, δ such that every $H \in \mathcal{H}$ is transitively equivalent to a member of $\mathcal{C}_{\lambda,\delta}$, then $\mathcal{H}$-DSN is FPT,
2. and it is W[1]-hard otherwise.