Fine-Grained Complexity and Algorithm Design Boot Camp

Parameterized Reductions

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Lower bounds

So far we have seen positive results: basic algorithmic techniques for fixed-parameter tractability.

What kind of negative results we have?

- Can we show that a problem (e.g., Clique) is **not** FPT?

  $\Rightarrow$ Today

- Can we show that a problem (e.g., Vertex Cover) has **no** algorithm with running time, say, $2^{o(k)} \cdot n^{O(1)}$?

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This would require showing that $\mathrm{P} \neq \mathrm{NP}$: if $\mathrm{P} = \mathrm{NP}$, then, e.g., k-CLIQUE is polynomial-time solvable, hence FPT.

Can we give some evidence for negative results?

Goals of this talk

Two goals:

1. Explain the theory behind parameterized intractability.
2. Show examples of parameterized reductions.

Parameterized complexity

To build a complexity theory for parameterized problems, we need two concepts:

- An appropriate notion of reduction.
- An appropriate hypothesis.

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Example: Graph G has an independent set k if and only if it has a vertex cover of size $n - k$.

$\Rightarrow$ Transforming an Independent Set instance (G, k) into a Vertex Cover instance $(G, n - k)$ is a correct polynomial-time reduction.

However, Vertex Cover is FPT, but Independent Set is not known to be FPT.

Parameterized reduction

Definition

Parameterized reduction from problem P to problem Q: a function ϕ with the following properties:

- $\phi(x)$ is a yes-instance of $Q \iff x$ is a yes-instance of P.
- $\phi(x)$ can be computed in time $f(k) \cdot |x|^{O(1)}$, where k is the parameter of x,
- If k is the parameter of x and k' is the parameter of $\phi(x)$, then $k' \leq g(k)$ for some function g.

Fact: If there is a parameterized reduction from problem P to problem Q and Q is FPT, then P is also FPT.

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Fact: If there is a parameterized reduction from problem P to problem Q and Q is FPT, then P is also FPT.

Non-example: Transforming an INDEPENDENT SET instance (G, k) into a VERTEX COVER instance $(G, n-k)$ is **not** a parameterized reduction.

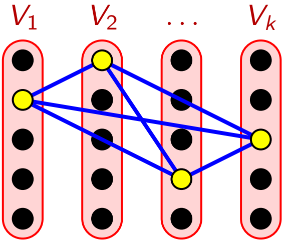
Example: Transforming an INDEPENDENT SET instance (G, k) into a CLIQUE instance $(\overline{G}, k)$ **is** a parameterized reduction.

Multicolored Clique

A useful variant of Clique:

> Multicolored Clique: The vertices of the input graph G are colored with k colors and we have to find a clique containing one vertex from each color.

(or Partitioned Clique)



Theorem

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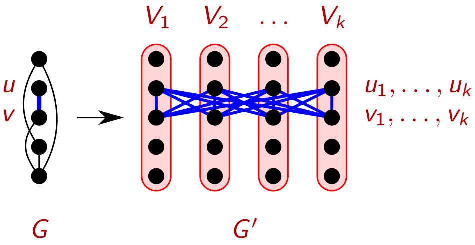
Create G' by replacing each vertex v with k vertices, one in each color class. If u and v are adjacent in the original graph, connect all copies of u with all copies of v.



k-clique in G $\iff$ multicolored k-clique in G'.

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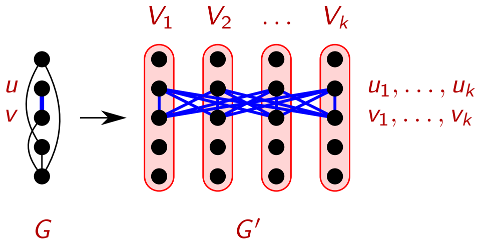


k-clique in G $\iff$ multicolored k-clique in G'.

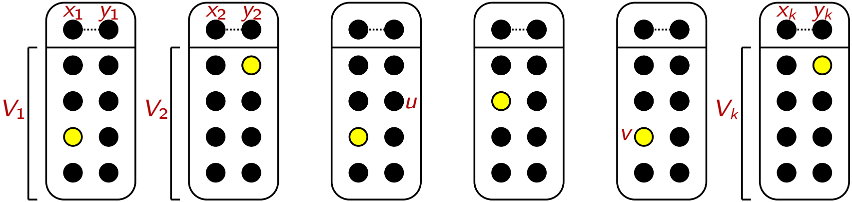
Similarly: reduction to Multicolored Independent Set.

Dominating Set

Theorem

There is a parameterized reduction from Multicolored Independent Set to Dominating Set.

Proof: Let G be a graph with color classes $V_1, \ldots, V_k$. We construct a graph H such that G has a multicolored k-clique iff H has a dominating set of size k.



- The dominating set has to contain one vertex from each of the k cliques $V_1, \ldots, V_k$ to dominate every x_i and y_i.

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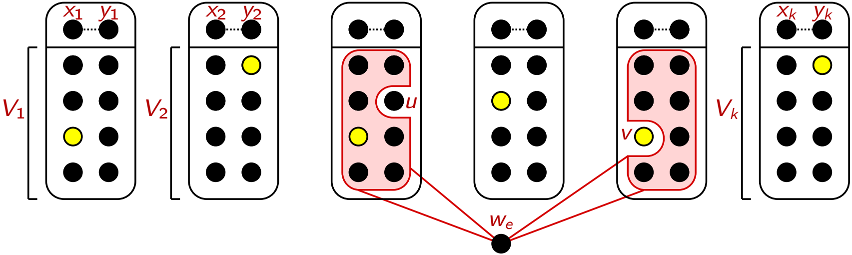


- The dominating set has to contain one vertex from each of the k cliques $V_1, \ldots, V_k$ to dominate every x_i and y_i.
- For every edge $e = uv$, an additional vertex w_e ensures that these selections describe an independent set.

Variants of Dominating Set

- Dominating Set: Given a graph, find k vertices that dominate every vertex.
- Red-Blue Dominating Set: Given a bipartite graph, find k vertices on the red side that dominate the blue side.
- Set Cover: Given a set system, find k sets whose union covers the universe.
- Hitting Set: Given a set system, find k elements that intersect every set in the system.

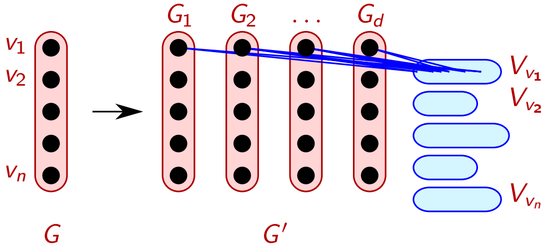
All of these problems are equivalent under parameterized reductions, hence at least as hard as Clique.

Regular graphs

Theorem

There is a parameterized reduction from CLIQUE to CLIQUE on regular graphs.

Proof: Given a graph G and an integer k, let d be the maximum degree of G. Take d copies of G and for every $v \in V(G)$, fully connect every copy of v with a set V_v of $d - d(v)$ vertices.



Observe the edges incident to V_v do not appear in any triangle, hence every k-clique of G' is a k-clique of G (assuming $k \geq 3$).

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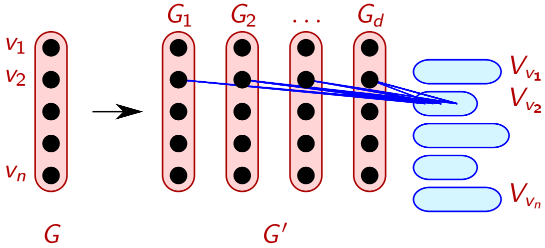

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Partial Vertex Cover

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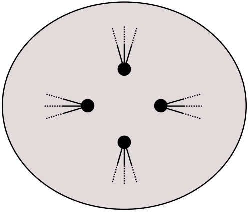
Proof: If G is d-regular, then k vertices can cover $s := kd$ edges if and only if there is a independent set of size k.



$d = 3$, $k = 4$, $s = 12$

Hard problems

Hundreds of parameterized problems are known to be at least as hard as Clique:

- Independent Set
- Set Cover
- Hitting Set
- Connected Dominating Set
- Independent Dominating Set
- Partial Vertex Cover parameterized by k
- Dominating Set in bipartite graphs
- ...

We believe that none of these problems are FPT.

Basic hypotheses

It seems that parameterized complexity theory cannot be built on assuming $\mathrm{P} \neq \mathrm{NP}$ – we have to assume something stronger.

Let us choose a basic hypothesis:

Engineers' Hypothesis

k-Clique cannot be solved in time $f(k) \cdot n^{O(1)}$.

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k-Step Halting Problem (is there a path of the given NTM that stops in k steps?) cannot be solved in time $f(k) \cdot n^{O(1)}$.

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Exponential Time Hypothesis (ETH)

n-variable 3SAT cannot be solved in time $2^{o(n)}$.

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INDEPENDENT SET ⇒ Turing machines

Theorem

There is a parameterized reduction from INDEPENDENT SET to the k-STEP HALTING PROBLEM.

Proof: Given a graph G and an integer k, we construct a Turing machine M and an integer $k' = O(k^2)$ such that M halts in k' steps if and only if G has an independent set of size k.

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The alphabet Σ of M is the set of vertices of G.

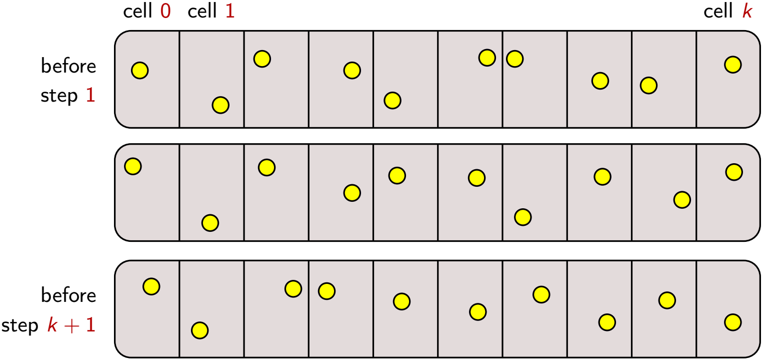
- In the first k steps, M nondeterministically writes k vertices to the first k cells.
- For every $1 \leq i \leq k$, M moves to the i-th cell, stores the vertex in the internal state, and goes through the tape to check that every other vertex is nonadjacent with the i-th vertex (otherwise M loops).
- M does k checks and each check can be done in $2k$ steps $\Rightarrow$ $k' = O(k^2)$.

Turing machines ⇒ INDEPENDENT SET

Theorem

There is a parameterized reduction from the k-STEP HALTING PROBLEM to INDEPENDENT SET.

Proof: Given a Turing machine M and an integer k, we construct a graph G that has an independent set of size $k' := (k+1)^2$ if and only if M halts in k steps.



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- G consists of $(k+1)^2$ cliques, thus a k'-independent set has to contain one vertex from each.
- The selected vertex from clique $K_{i,j}$ describes the situation before step i at cell j: what is written there, is the head there, and if so, what the state is, and what the next transition is.
- We add edges between the cliques to rule out inconsistencies: head is at more than one location at the same time, wrong character is written, head moves in the wrong direction etc.

Summary

- Independent Set and k-Step Halting Problem can be reduced to each other $\Rightarrow$ Engineers' Hypothesis and Theorists' Hypothesis are equivalent!
- Independent Set and k-Step Halting Problem can be reduced to Dominating Set.

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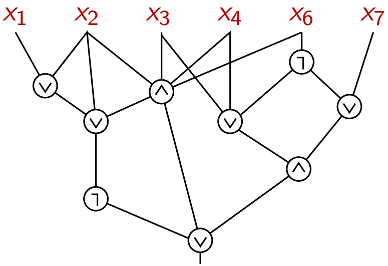
- Independent Set and k-Step Halting Problem can be reduced to each other $\Rightarrow$ Engineers' Hypothesis and Theorists' Hypothesis are equivalent!
- Independent Set and k-Step Halting Problem can be reduced to Dominating Set.
- Is there a parameterized reduction from Dominating Set to Independent Set?
- Probably not. Unlike in NP-completeness, where most problems are equivalent, here we have a hierarchy of hard problems.
  - Independent Set is W[1]-complete.
  - Dominating Set is W[2]-complete.
- Does not matter if we only care about whether a problem is FPT or not!

Boolean circuit

A **Boolean circuit** consists of input gates, negation gates, AND gates, OR gates, and a single output gate.



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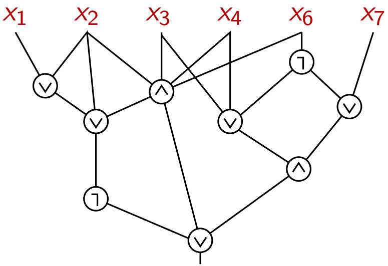


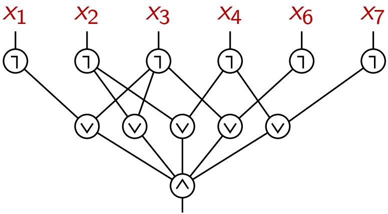
CIRCUIT SATISFIABILITY: Given a Boolean circuit C, decide if there is an assignment on the inputs of C making the output true.

Weight of an assignment: number of true values.

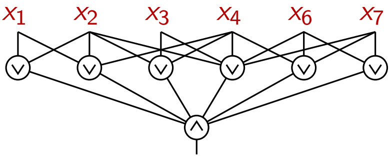
WEIGHTED CIRCUIT SATISFIABILITY: Given a Boolean circuit C and an integer k, decide if there is an assignment of weight k making the output true.

Weighted Circuit Satisfiability

Independent Set can be reduced to Weighted Circuit Satisfiability:



Dominating Set can be reduced to Weighted Circuit Satisfiability:



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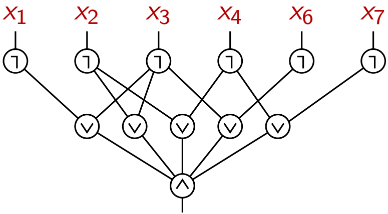

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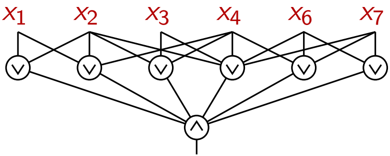


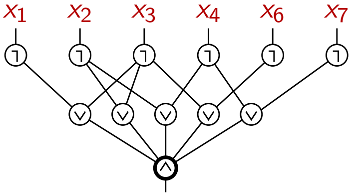
To express Dominating Set, we need more complicated circuits.

Depth and weft

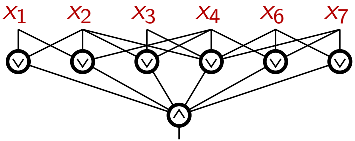
The **depth** of a circuit is the maximum length of a path from an input to the output.

A gate is **large** if it has more than 2 inputs. The **weft** of a circuit is the maximum number of large gates on a path from an input to the output.

INDEPENDENT SET: weft 1, depth 3



DOMINATING SET: weft 2, depth 2



The W-hierarchy

Let $C[t, d]$ be the set of all circuits having weft at most t and depth at most d.

Definition

A problem P is in the class W[t] if there is a constant d and a parameterized reduction from P to WEIGHTED CIRCUIT SATISFIABILITY of $C[t, d]$.

We have seen that INDEPENDENT SET is in W[1] and DOMINATING SET is in W[2].

Fact: INDEPENDENT SET is W[1]-complete.
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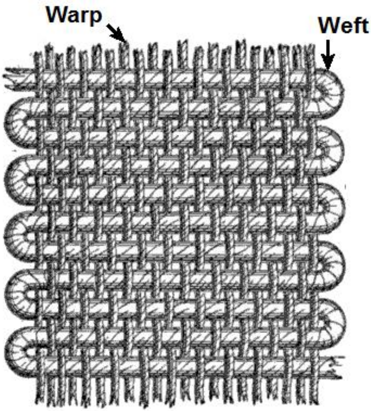
Fact: Independent Set is W[1]-complete.
Fact: Dominating Set is W[2]-complete.

If any W[1]-complete problem is FPT, then FPT = W[1] and **every** problem in W[1] is FPT.

If any W[2]-complete problem is in W[1], then W[1] = W[2].

$\Rightarrow$ If there is a parameterized reduction from Dominating Set to Independent Set, then W[1] = W[2].

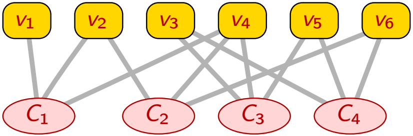
Weft



Weft is a term related to weaving cloth: it is the thread that runs from side to side in the fabric.

Parameterized reductions

Typical NP-hardness proofs: reduction from e.g., Clique or 3SAT, representing each vertex/edge/variable/clause with a gadget.



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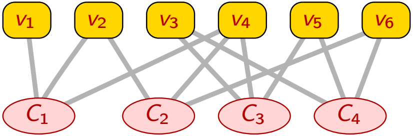

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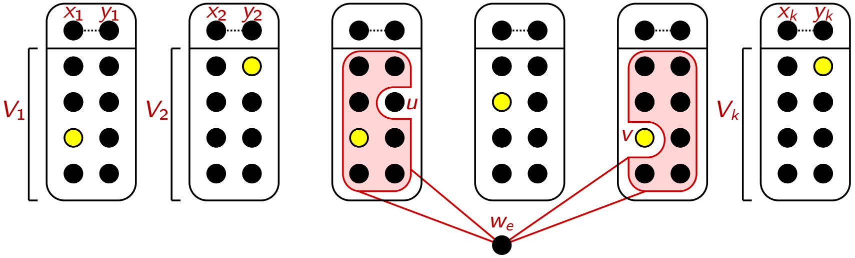
Types of parameterized reductions:

- Reductions keeping the structure of the graph.
  - CLIQUE $\Rightarrow$ INDEPENDENT SET
  - INDEPENDENT SET on regular graphs $\Rightarrow$ PARTIAL VERTEX COVER
- Reductions with vertex representations.
  - MULTICOLORED INDEPENDENT SET $\Rightarrow$ DOMINATING SET
- Reductions with vertex and edge representations.

Vertex representation

Recall: Reduction from MULTICOLORED INDEPENDENT SET to DOMINATING SET:



List Coloring

List Coloring is a generalization of ordinary vertex coloring: given a

- graph G,
- a set of colors C, and
- a list $L(v) \subseteq C$ for each vertex v,

the task is to find a coloring c where $c(v) \in L(v)$ for every v.

Theorem

Vertex Coloring is FPT parameterized by treewidth.

However, list coloring is more difficult:

Theorem

List Coloring is W[1]-hard parameterized by treewidth.

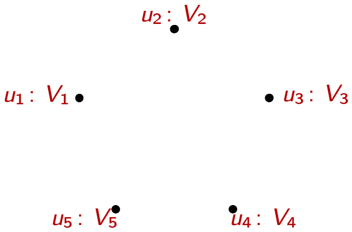
List Coloring

Theorem

List Coloring is W[1]-hard parameterized by treewidth.

Proof: By reduction from Multicolored Independent Set.

- Let G be a graph with color classes $V_1, \ldots, V_k$.
- Set C of colors: the set of vertices of G.
- The colors appearing on vertices $u_1, \ldots, u_k$ correspond to the k vertices of the clique, hence we set $L(u_i) = V_i$.



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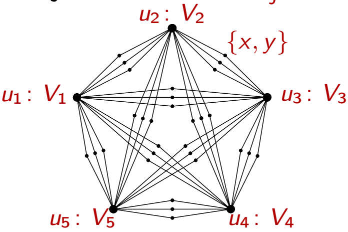
- Set C of colors: the set of vertices of G.
- The colors appearing on vertices $u_1, \ldots, u_k$ correspond to the k vertices of the clique, hence we set $L(u_i) = V_i$.
- If $x \in V_i$ and $y \in V_j$ are adjacent in G, then we need to ensure that $c(u_i) = x$ and $c(u_j) = y$ are not true at the same time $\Rightarrow$ we add a vertex adjacent to u_i and u_j whose list is $\{x, y\}$.



Vertex representation

Key idea

- Represent the k vertices of the solution with k gadgets.
- Connect the gadgets in a way that ensures that the represented values are **compatible.**

But sometimes it is very difficult to create connections that force two gadgets to be compatible...

Odd Set

Odd Set: Given a set system $\mathcal{F}$ over a universe U and an integer k, find a set S of at most k elements such that $|S \cap F|$ is odd for every $F \in \mathcal{F}$.

Theorem

Odd Set is W[1]-hard parameterized by k.

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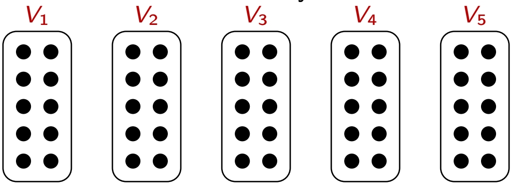
First try: Reduction from Multicolored Independent Set.
Let $U = V_1 \cup \dots V_k$ and introduce each set V_i into $\mathcal{F}$.
$\Rightarrow$ The solution has to contain exactly one element from each V_i.



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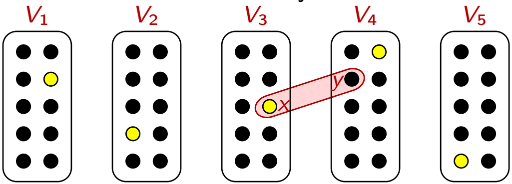

If $xy \in E(G)$, how can we express that $x \in V_i$ and $y \in V_j$ cannot be selected simultaneously? Seems difficult:

- introducing $\{x, y\}$ into $\mathcal{F}$ forces that **exactly one** of x and y appears in the solution,

ODD SET

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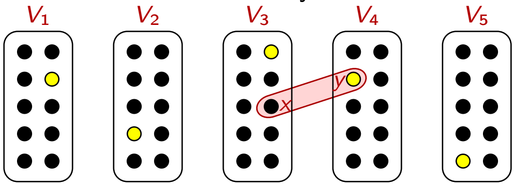

If $xy \in E(G)$, how can we express that $x \in V_i$ and $y \in V_j$ cannot be selected simultaneously? Seems difficult:

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ODD SET

Theorem

ODD SET is W[1]-hard parameterized by k.

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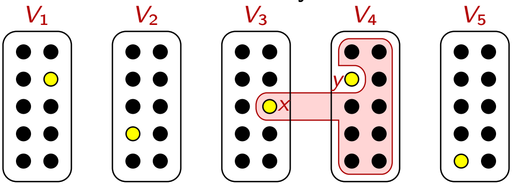

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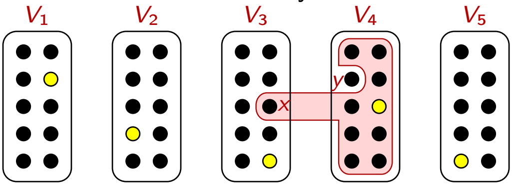

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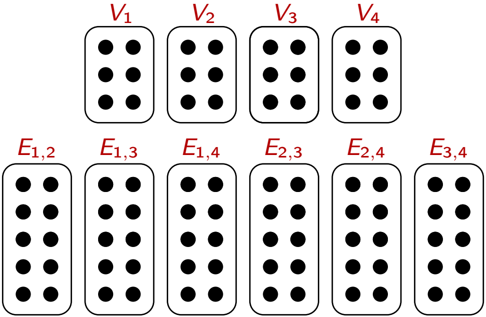
Reduction from Multicolored Clique.

- $U := \bigcup_{i=1}^{k} V_i \cup \bigcup_{1 \le i < j \le k} E_{i,j}$.
- $k' := k + \binom{k}{2}$.
- Let $\mathcal{F}$ contain V_i ($1 \le i \le k$) and $E_{i,j}$ ($1 \le i < j \le k$).



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  $(V_i \setminus \{v\}) \cup \{\text{every edge from } E_{i,x} \text{ with endpoint } v\}$
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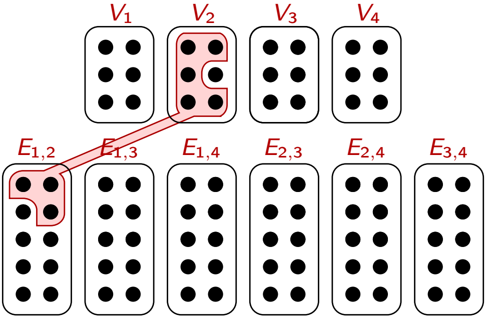

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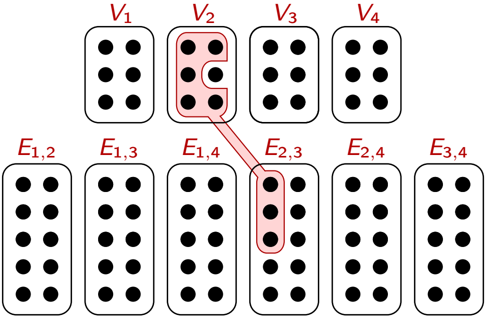

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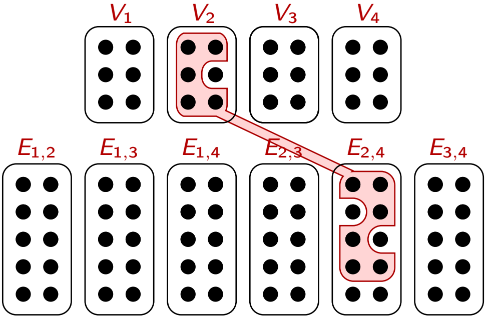

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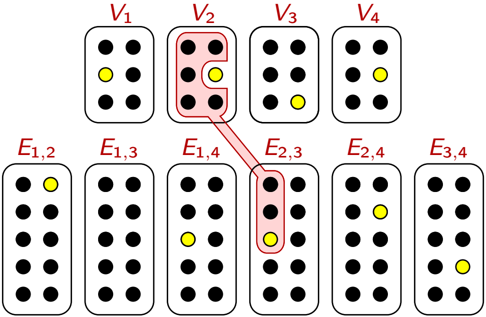

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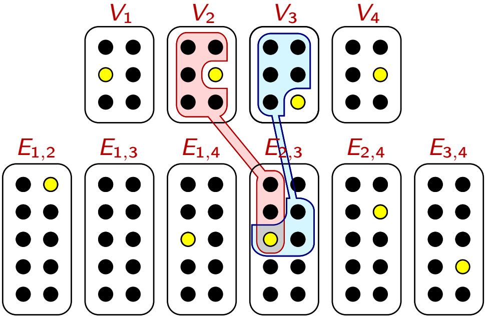
- $v \in V_i$ selected $\iff$ edges with endpoint v are selected from $E_{i,x}$ and $E_{x,i}$
- $v_i \in V_i$ selected, $v_j \in V_j$ selected $\iff$ edge $v_i v_j$ is selected in $E_{i,x}$



Vertex and edge representation

Key idea

- Represent the vertices of the clique by k gadgets.
- Represent the edges of the clique by $\binom{k}{2}$ gadgets.
- Connect edge gadget $E_{i,j}$ to vertex gadgets V_i and V_j such that if $E_{i,j}$ represents the edge between $x \in V_i$ and $y \in V_j$, then it **forces** V_i to x and V_j to y.

The connection between the edge gadget and a vertex gadget needs to express a simple projection relation: a selection of an edge forces a selection of a vertex.

Typically blows up the parameter to $O(k^2)$!

Variants of Odd Set

The following problems are W[1]-hard:

- Odd Set
- Exact Odd Set (find a set of size exactly k ...)
- Exact Even Set
- Unique Hitting Set
  (at most k elements that hit each set exactly once)
- Exact Unique Hitting Set
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Open question:

? Even Set: Given a set system $\mathcal{F}$ and an integer k, find a **nonempty** set S of at most k elements such $|F \cap S|$ is even for every $F \in \mathcal{F}$.

Summary

- By parameterized reductions, we can show that lots of parameterized problems are at least as hard as Clique, hence unlikely to be fixed-parameter tractable.
- Connection with Turing machines gives some supporting evidence for hardness (only of theoretical interest).
- The W-hierarchy classifies the problems according to hardness (only of theoretical interest).
- Important trick in W[1]-hardness proofs: vertex and edge representations.