Parameterized graph separation problems

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Terminal separation

MINIMUM TERMINAL SEPARATION

- **Given:** a graph G, an integer k, and a set T of ℓ vertices (the **terminals**)
- **Parameter:** k, ℓ
- **Find:** a set S of k vertices such that S separates every two vertices of T

Note: deleting a vertex in T separates it from every other vertex.

Polynomial time solvable for $\ell = 2$ (network flows).
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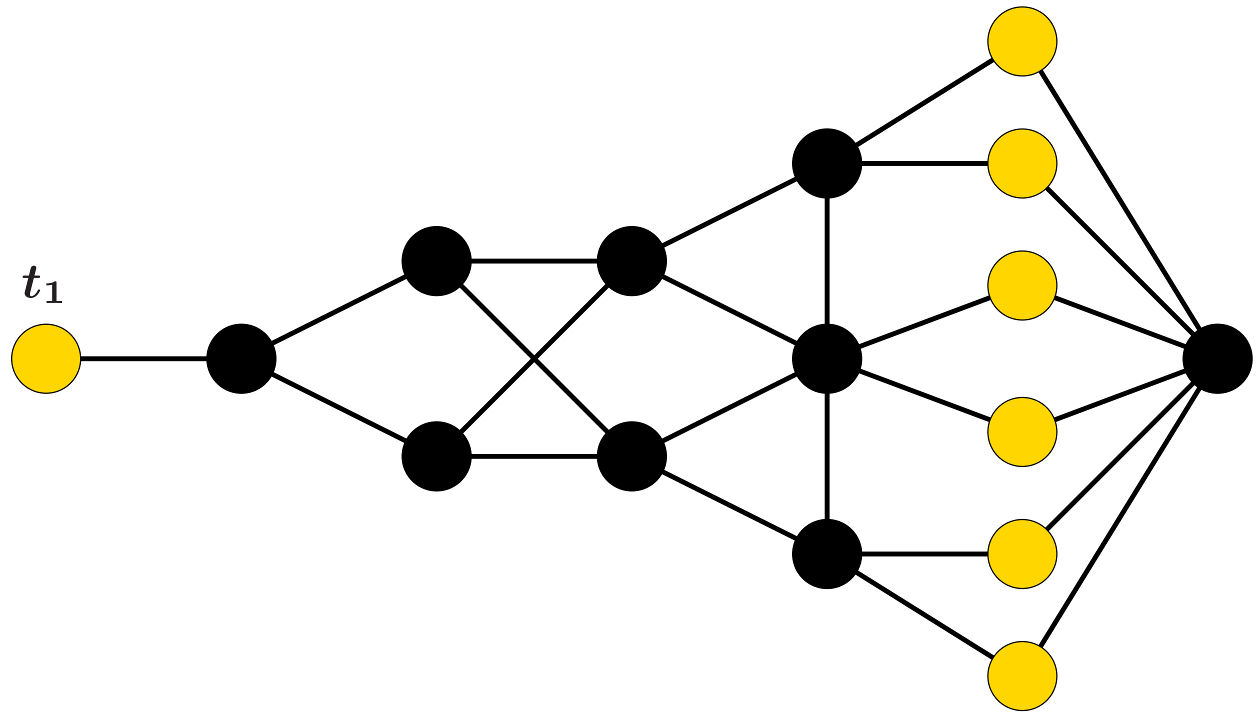
Theorem: MINIMUM TERMINAL SEPARATION is fixed-parameter tractable with parameter k.

(Follows from graph minors theory, but here we give a direct proof.)

Main idea

We try to separate t_1 from the rest of the terminals:



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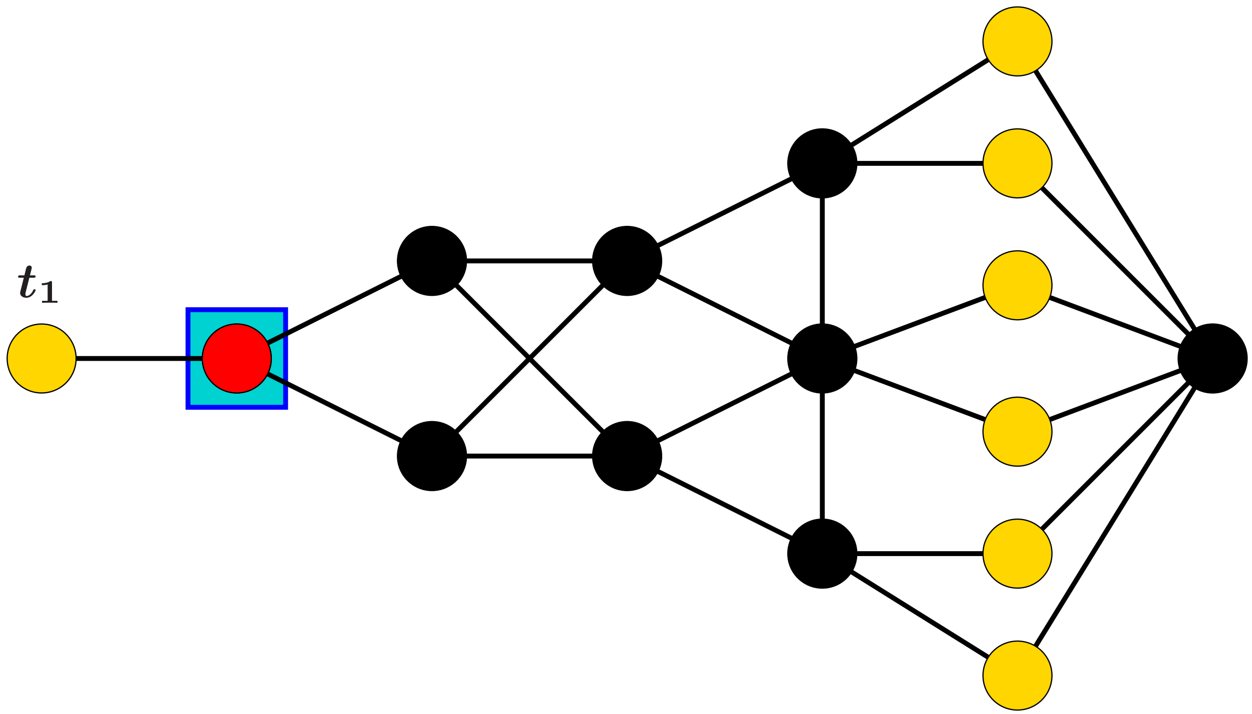

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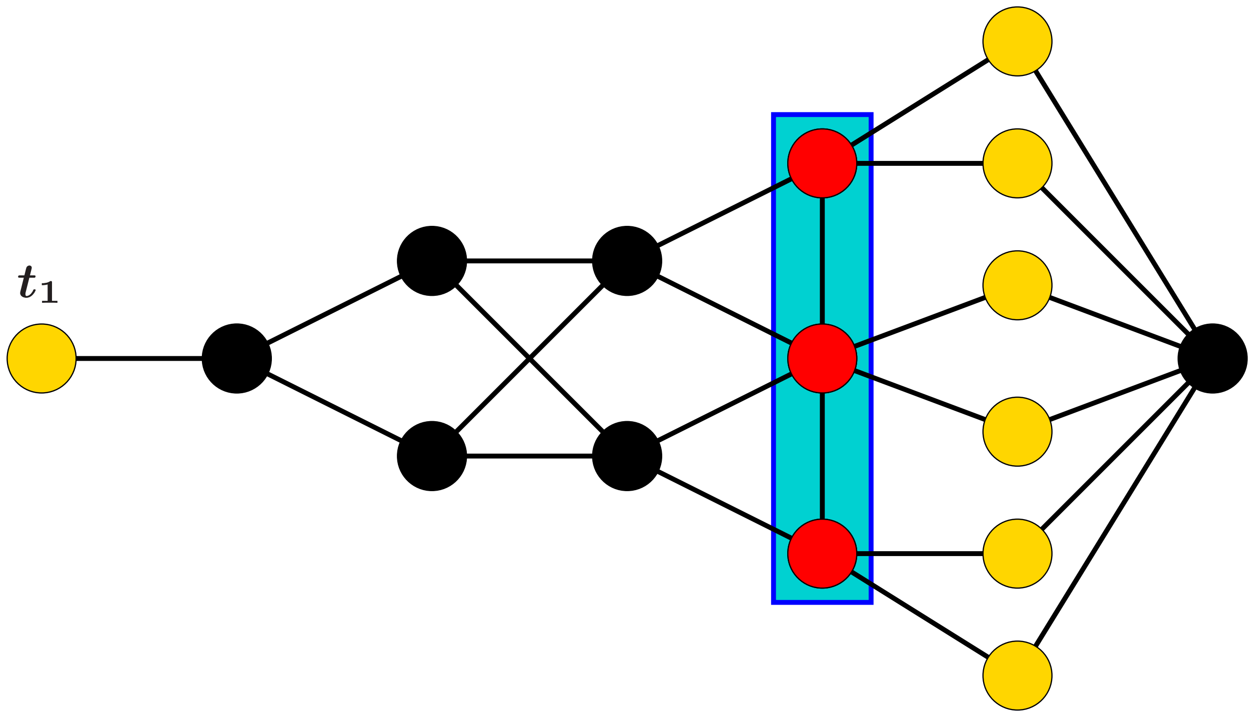

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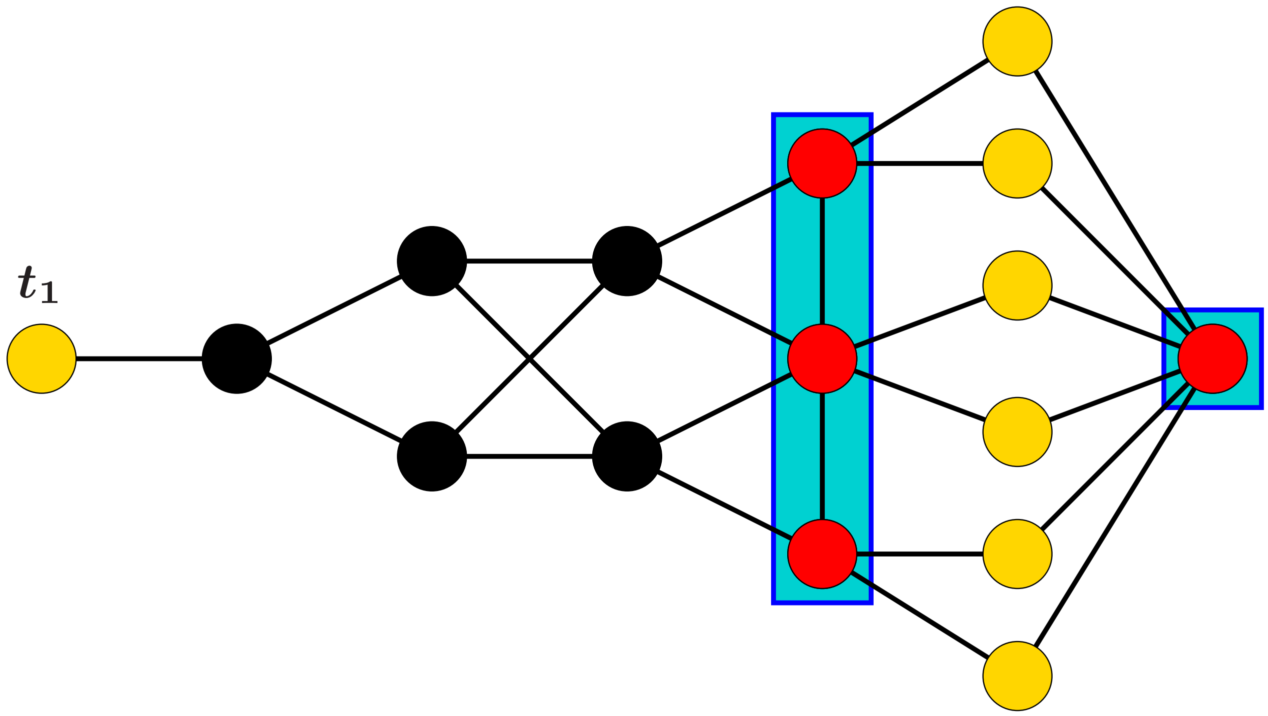

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Important separators

Definition: S is an **(X,Y)-separator** if it separates every vertex of X from every vertex of Y.

Definition: an (X,Y)-separator R **dominates** an (X,Y)-separator S if

- $|R| \leq |S|$,
- every vertex reachable from X in $G \setminus S$ is reachable from X in $G \setminus R$.

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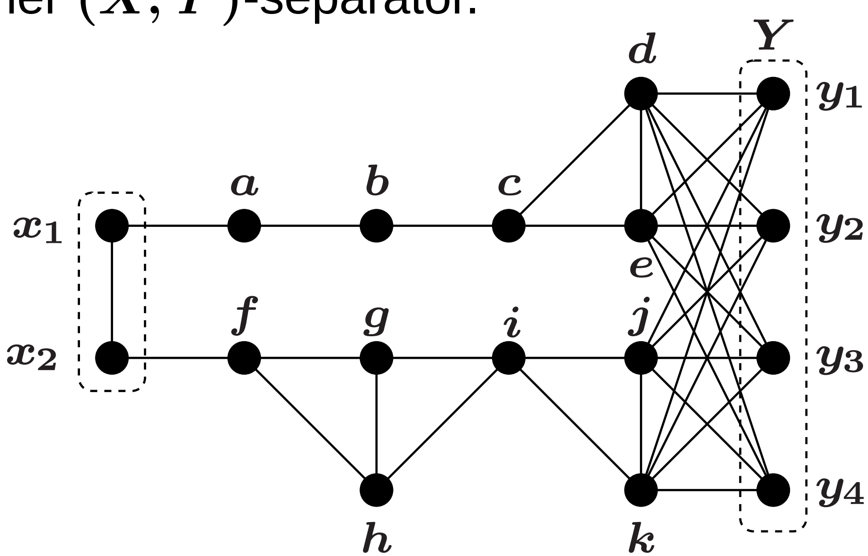

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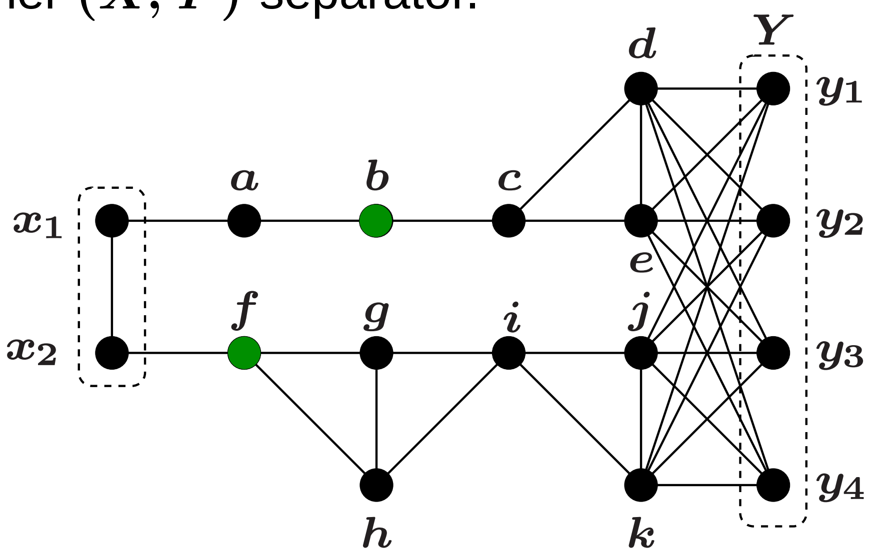

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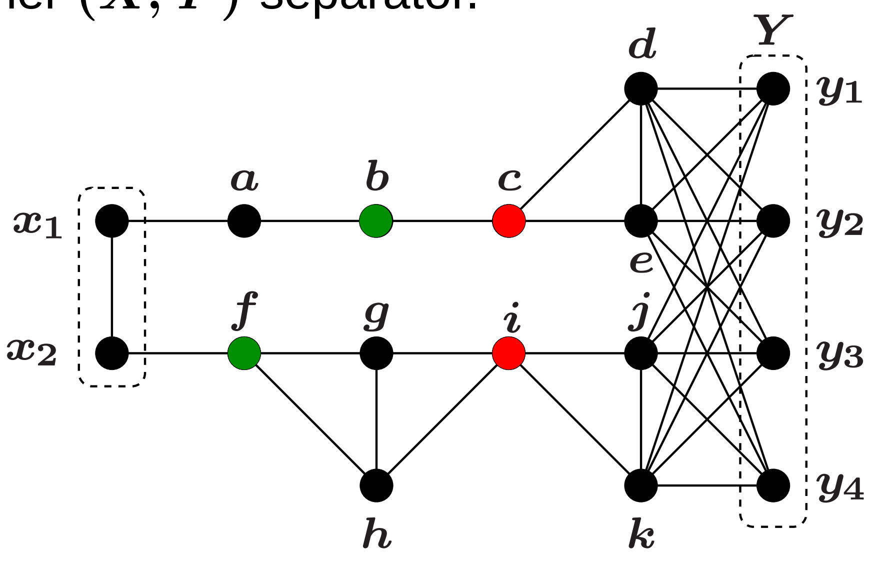

Separator $\{b, f\}$ is dominated by $\{c, i\}$.

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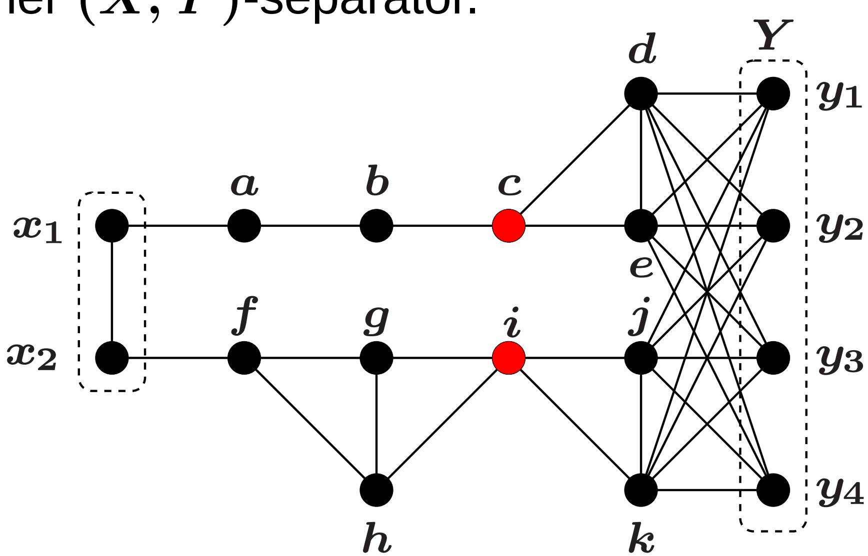

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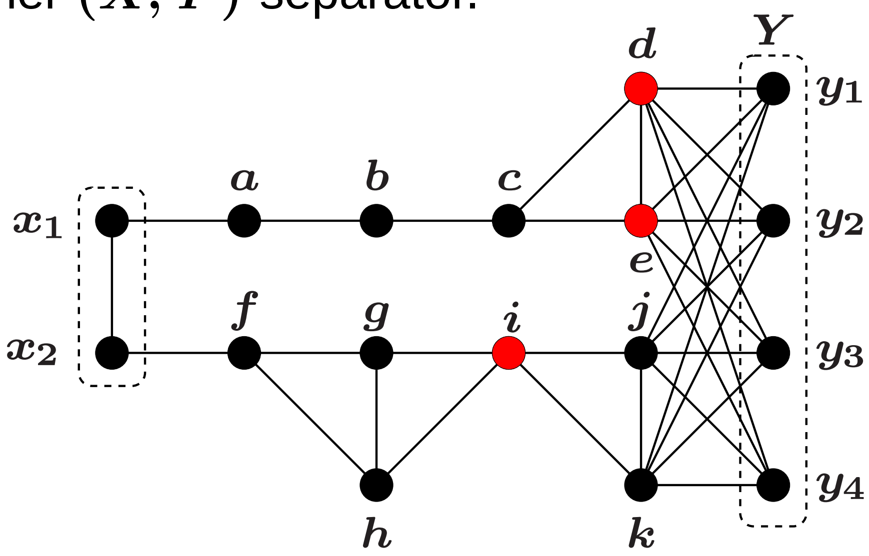

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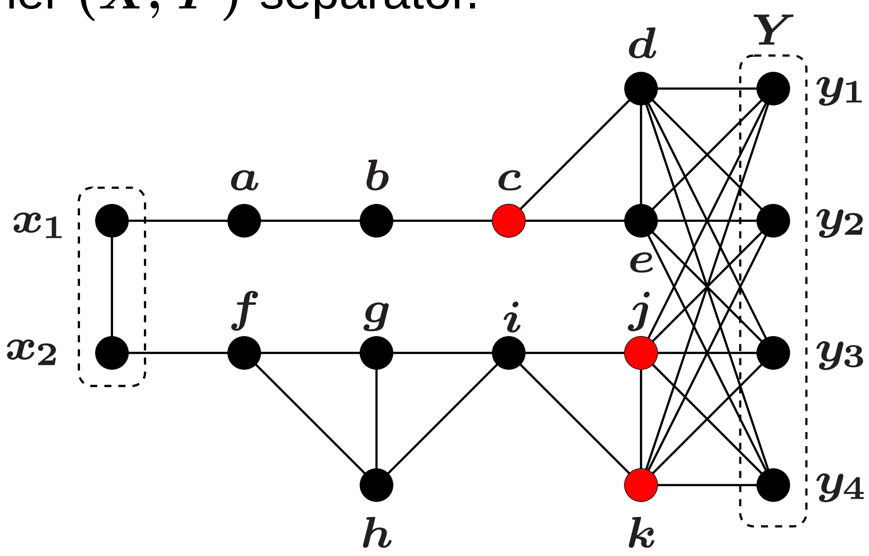

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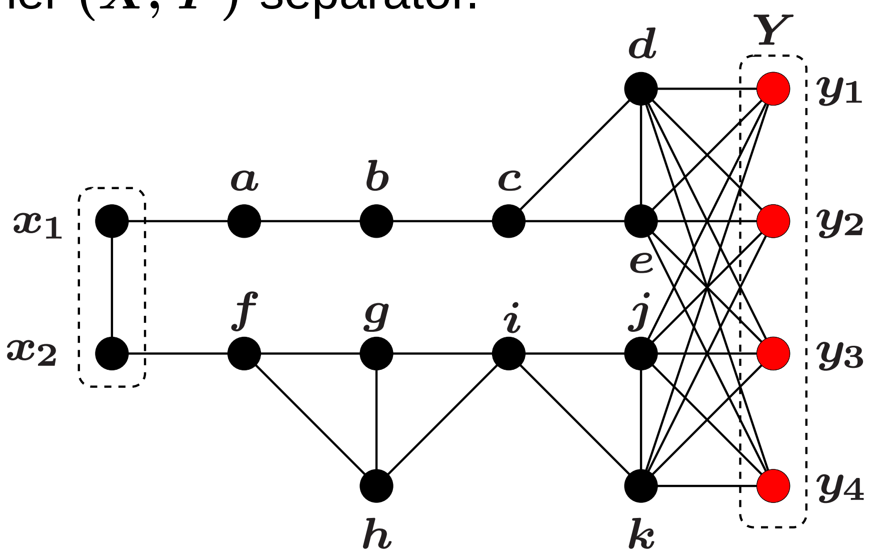

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$\{y_1, y_2, y_3, y_4\}$

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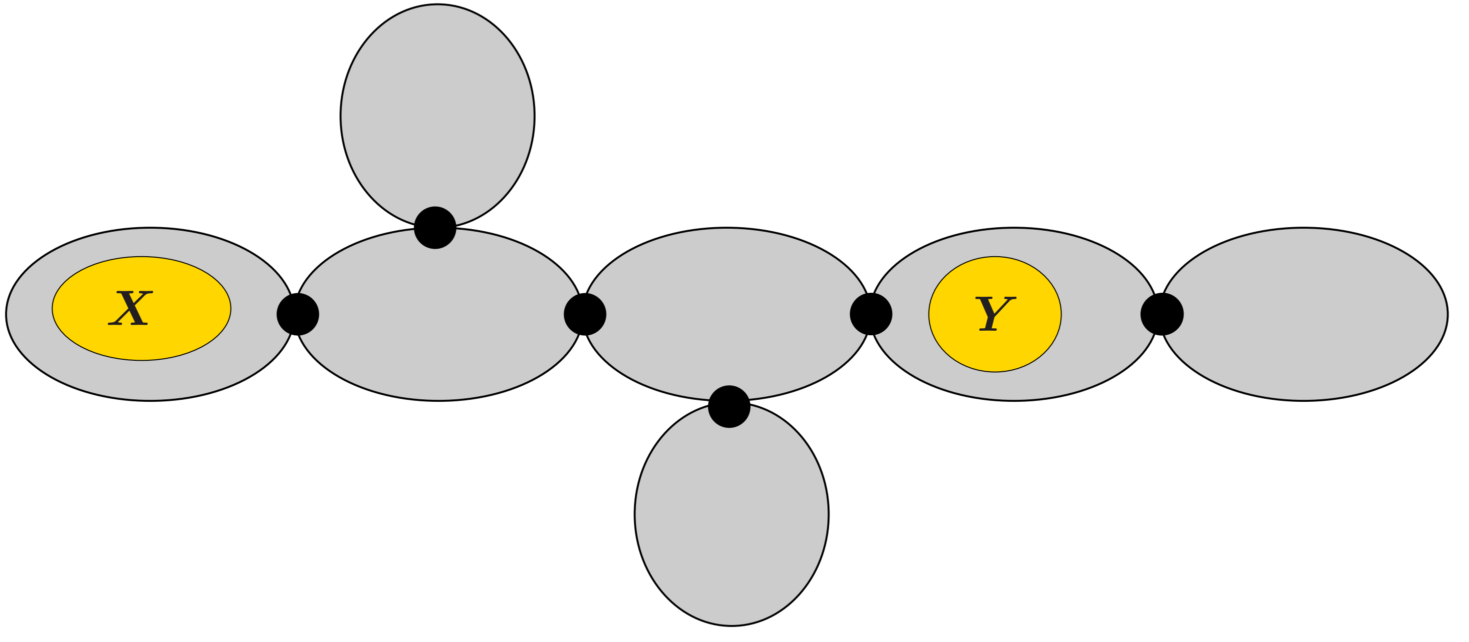

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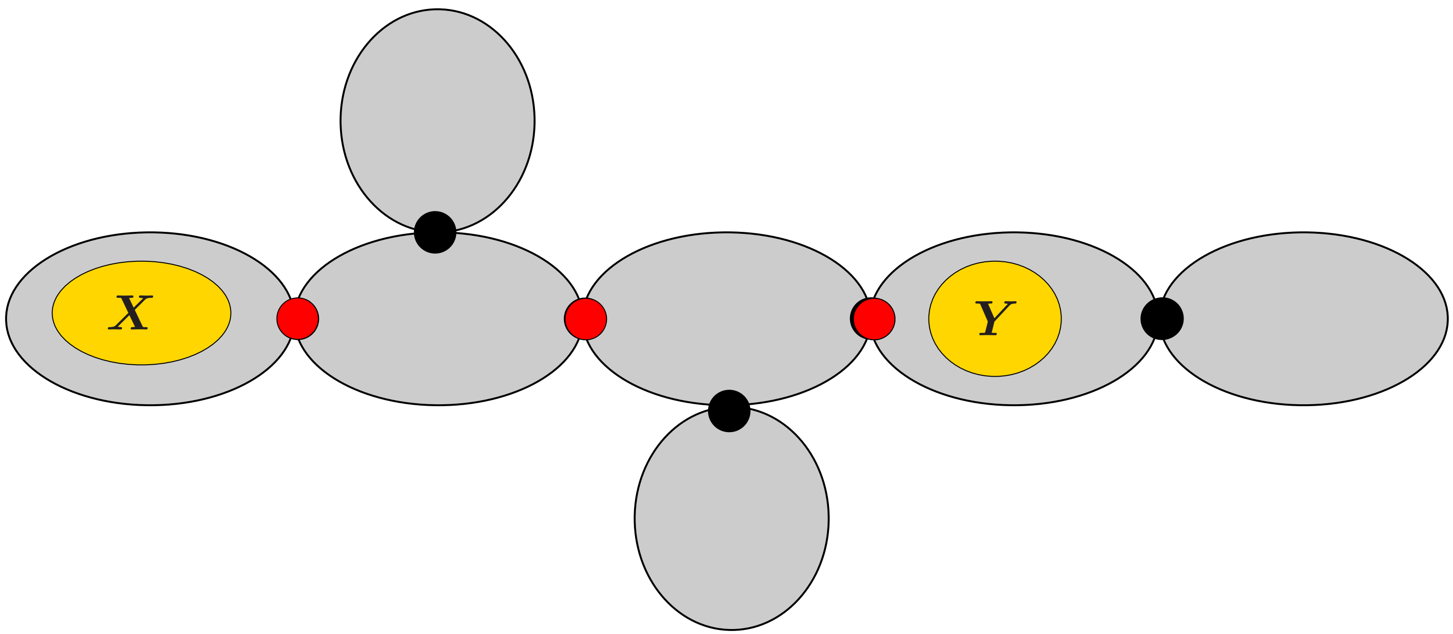

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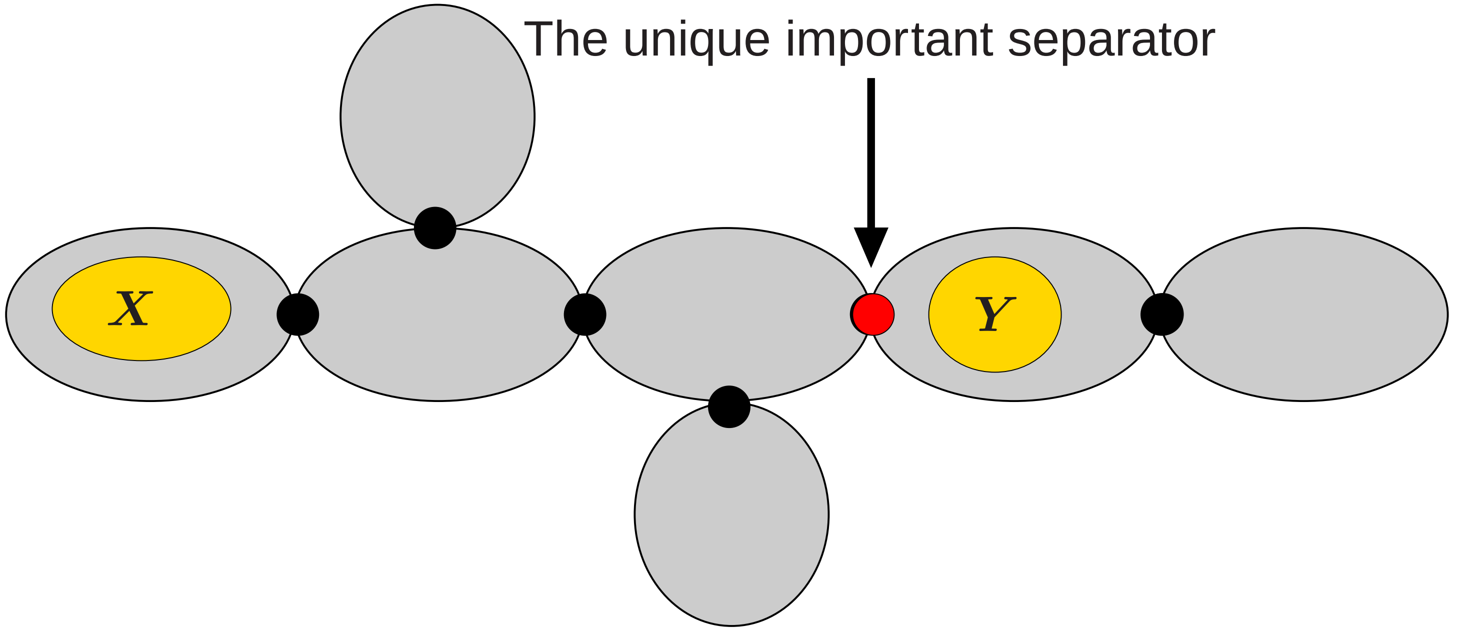

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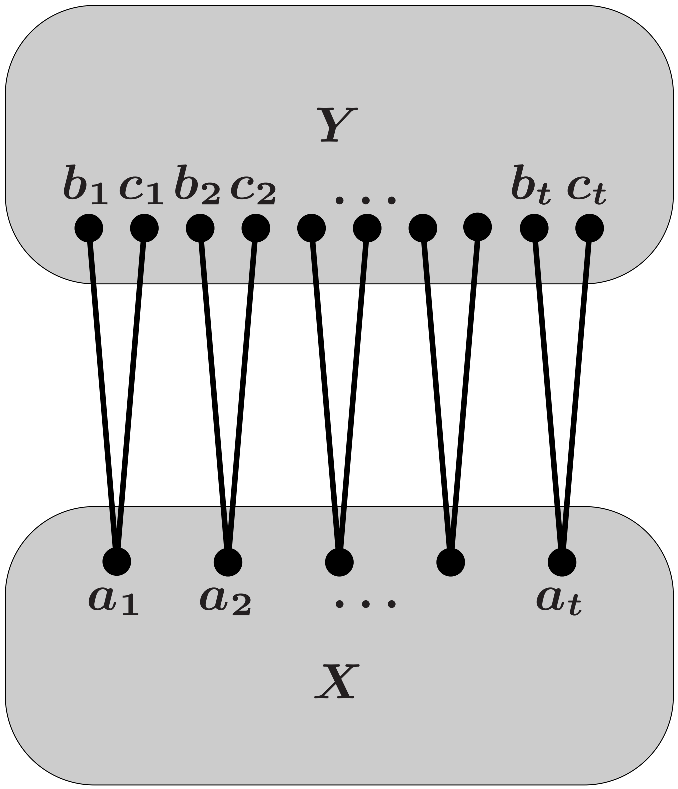

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- a_i or
- both b_i and c_i.

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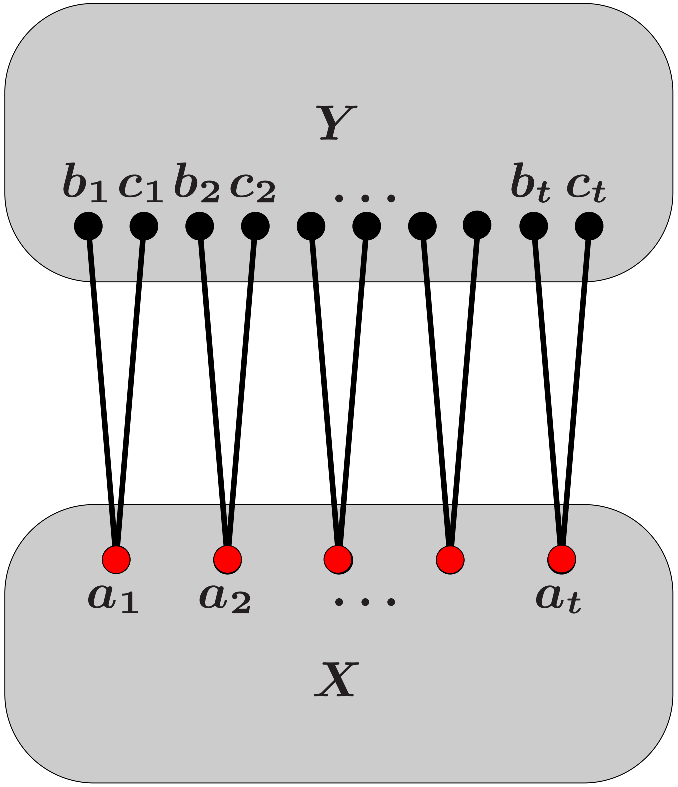

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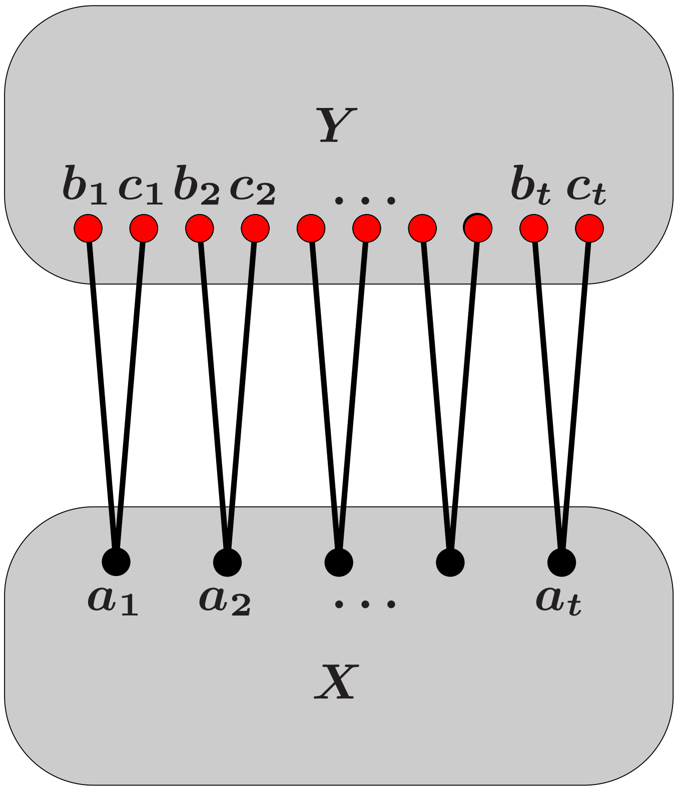

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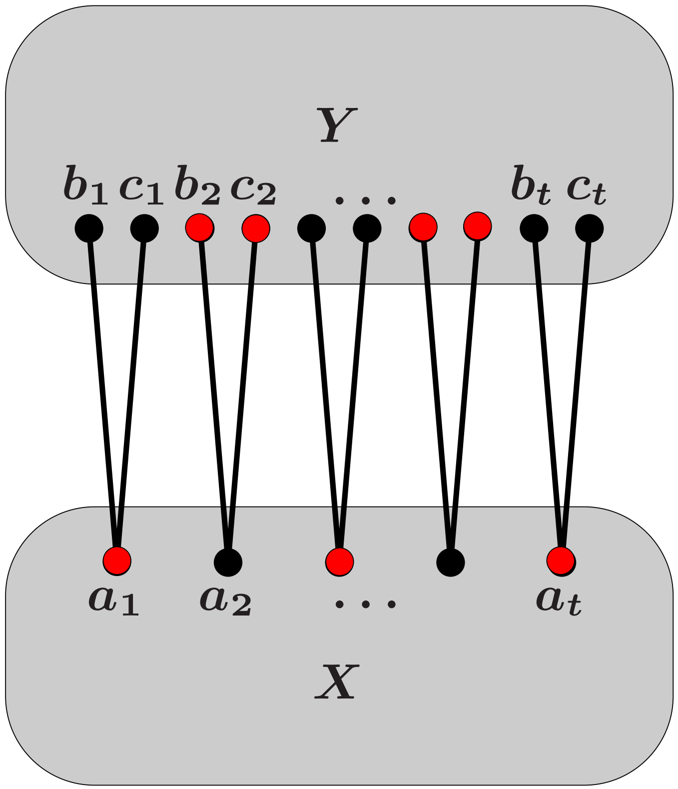

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Every combination gives an important separator $\Rightarrow$ there are 2^t important separators of size at most $2t$.

The two key lemmas

Lemma 1: There are at most 4^{k^2} important (X, Y)-separators of size $\leq k$.

Lemma 2: If the terminals t_1, t_2, $\ldots$, t_ℓ can be separated by deleting k vertices, then there is a solution that contains an important $(\{t_1\}, \{t_2, \ldots, t_\ell\})$-separator.

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Algorithm:

1. Enumerate all the important $(\{t_1\}, \{t_2, \ldots, t_\ell\})$-separators of size at most k.
2. Delete one of them from the graph.
3. Decrease the parameter k, and go to Step 1.

Bounded search tree: branch factor is at most 4^{k^2}, height is at most k.

Proof of Lemma 1

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Proof: By induction on k, we have seen the case $k = 1$. Let S be an important separator. How many other important separators are possible?

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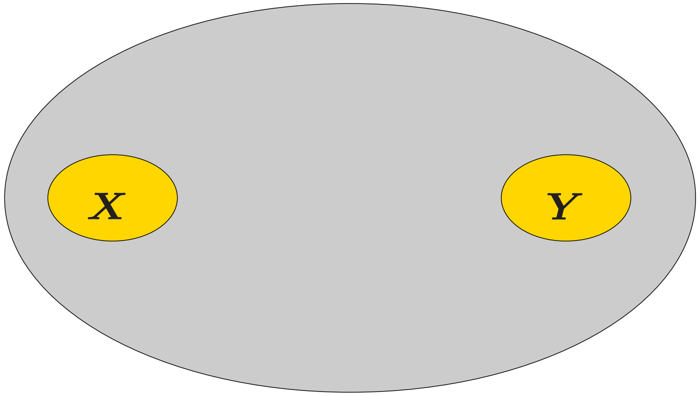

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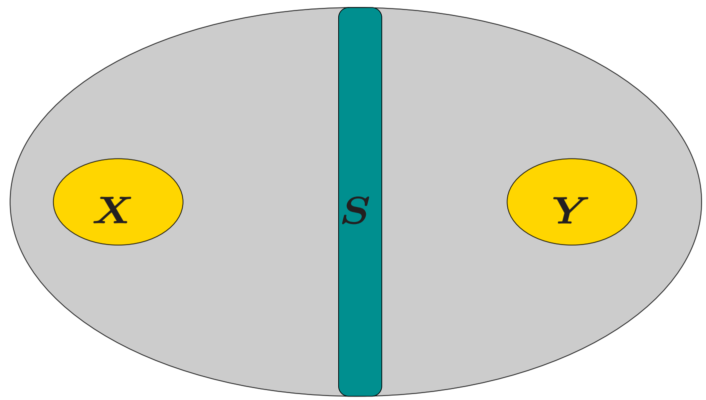

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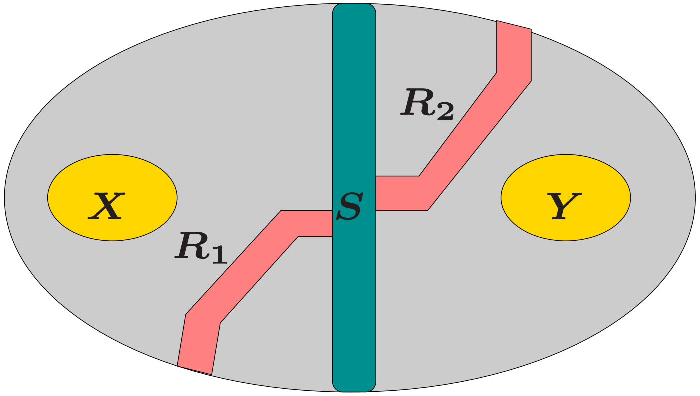

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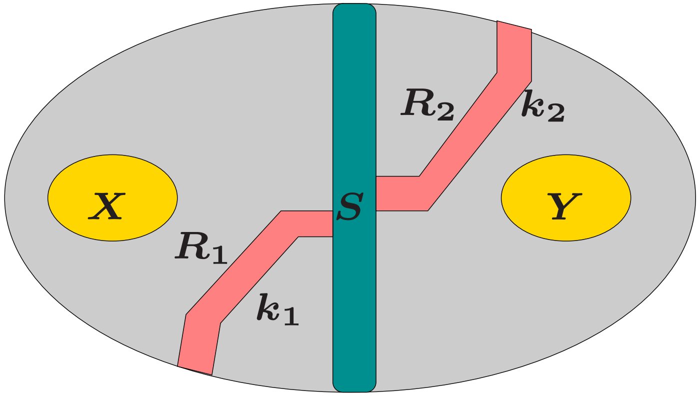

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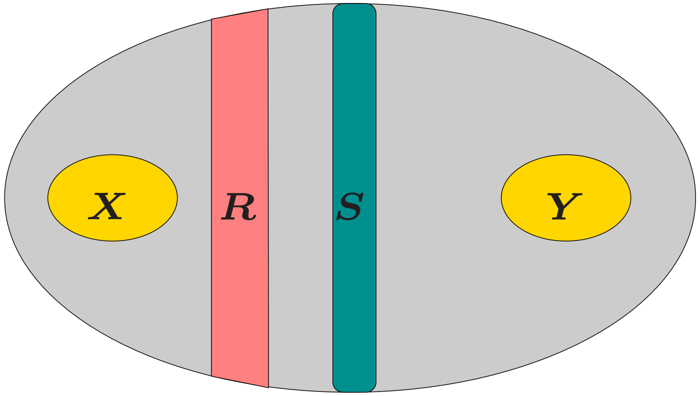

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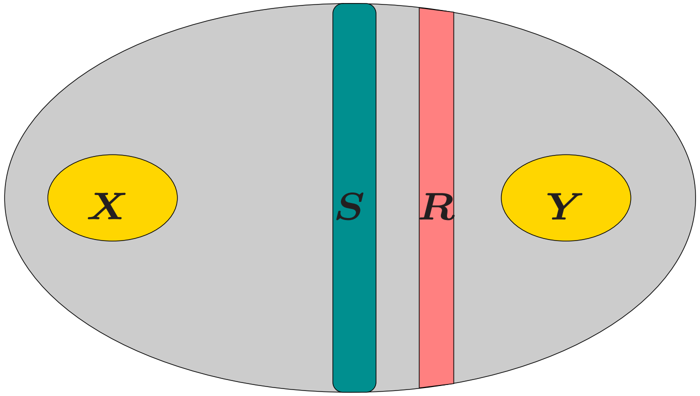

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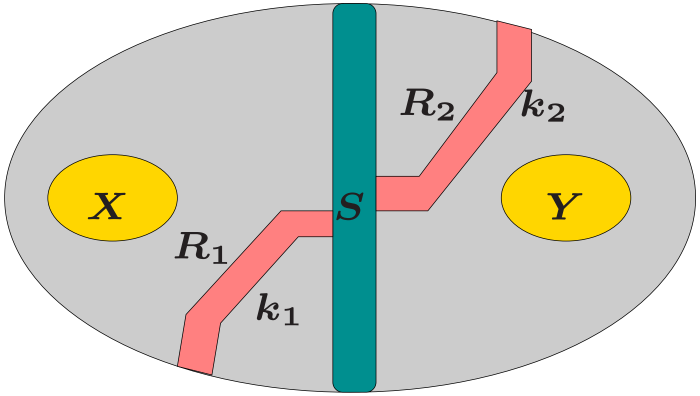

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It can be shown that R_1 (resp. R_2) is an important (X', Y')-separator (for some X', Y') $\Rightarrow$ constant number of possibilities for R_1 and R_2 $\Rightarrow$ constant number of possibilities for R.

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Lemma 2: If the terminals $t_1, t_2, \ldots, t_\ell$ can be separated by deleting k vertices, then there is a solution that contains an important $(\{t_1\}, \{t_2, \ldots, t_\ell\})$-separator.

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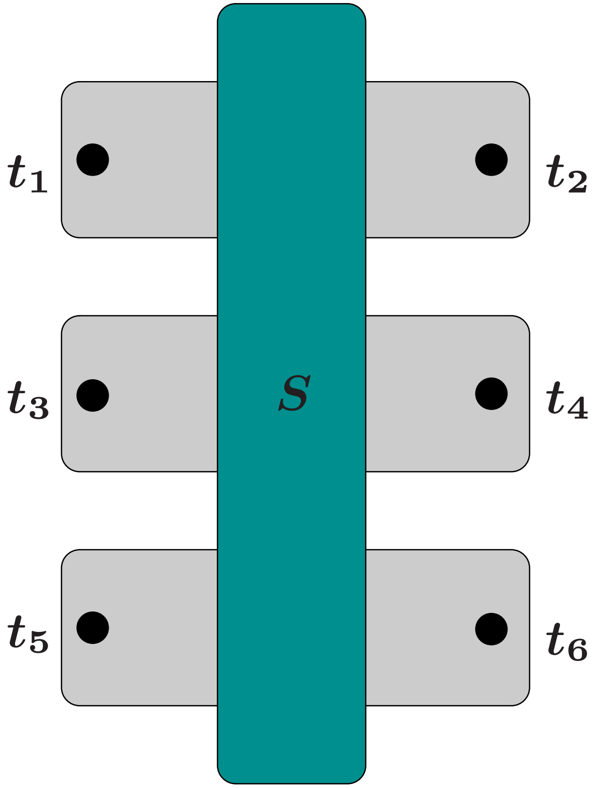

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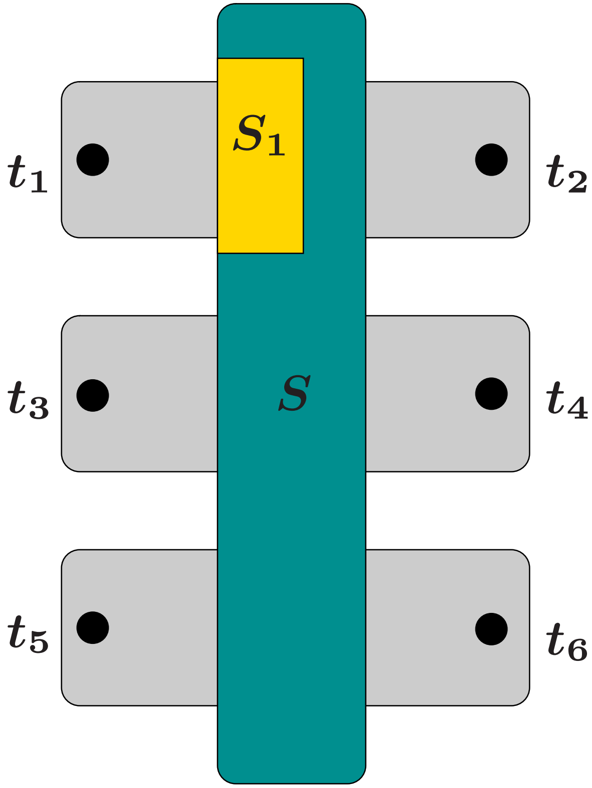

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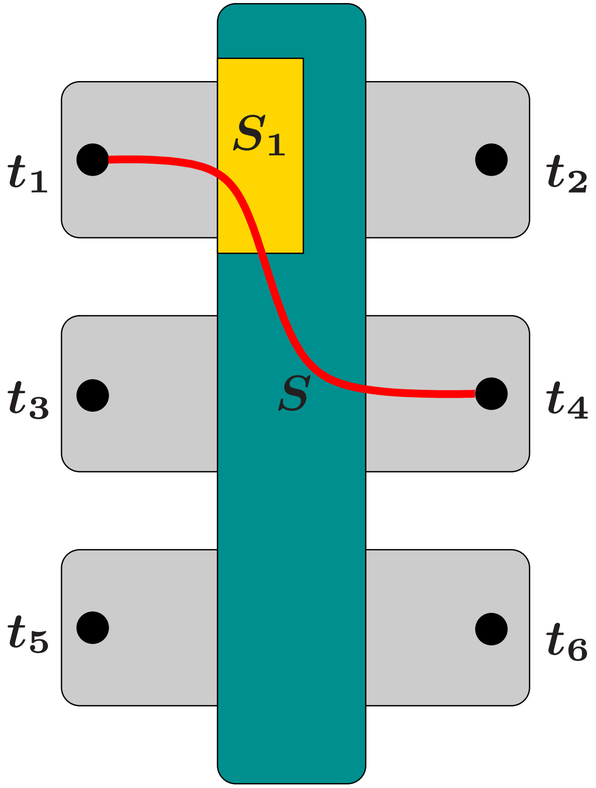

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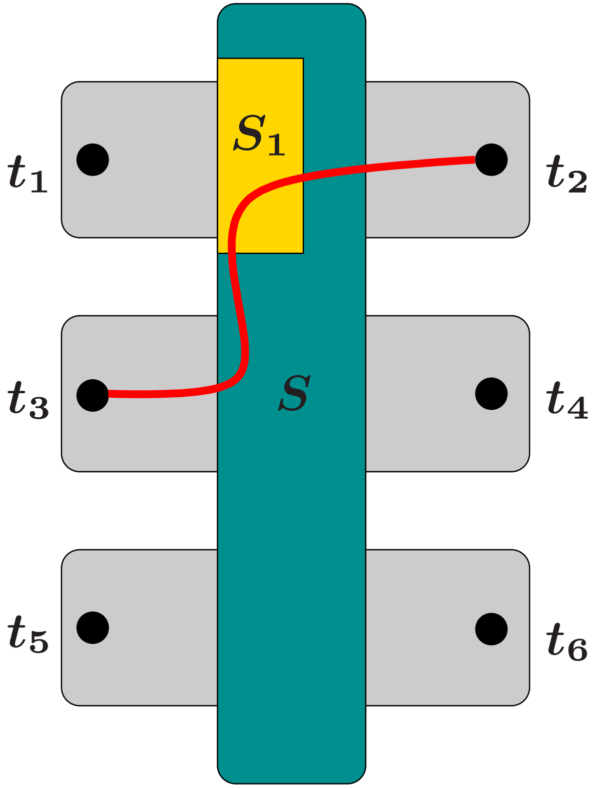

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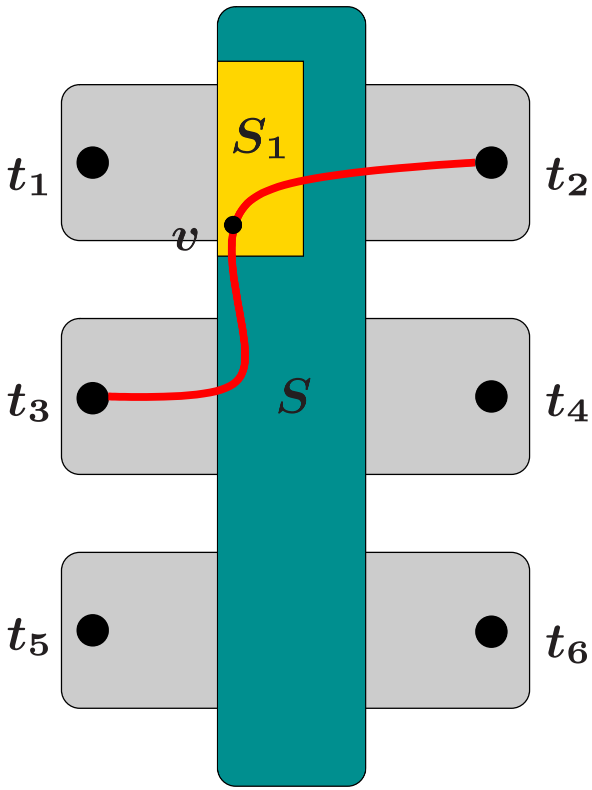

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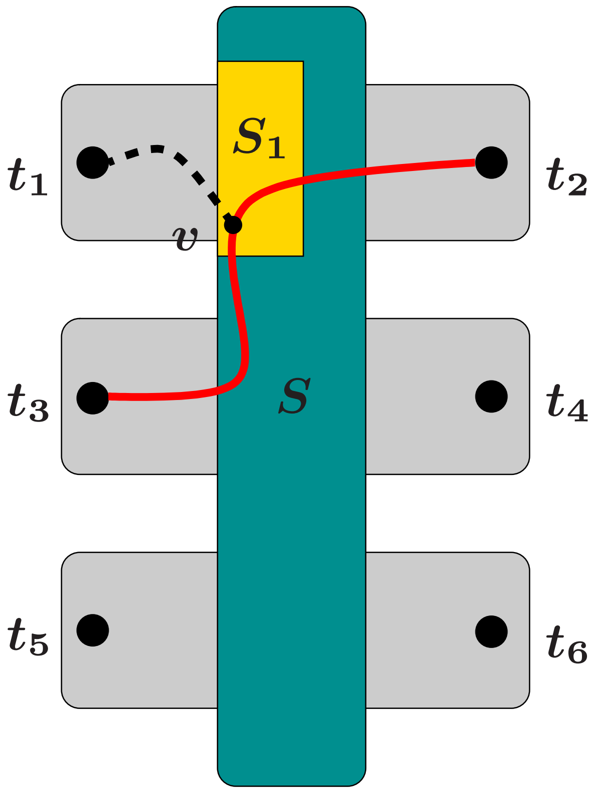

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- If t_2 and t_3 are connected, then the path has to go through some $v \in S_1 \Rightarrow t_1$ is connected to both t_2 and t_3, a contradiction.

Terminal pairs

MINIMUM TERMINAL PAIR SEPARATION

- **Given:** a graph G, an integer k, and ℓ pairs of terminals (s_1, t_1), $\dots$, (s_ℓ, t_ℓ).
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Open: What is the complexity of the problem if only k is the parameter?