The Complexity of Nonrepetitive Coloring

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Abstract

A coloring of a graph is *nonrepetitive* if the graph contains no path that has a color pattern of the form xx (where x is a sequence of colors). We show that determining whether a particular coloring of a graph is nonrepetitive is **coNP**-hard, even if the number of colors is limited to four. The problem becomes fixed-parameter tractable, if we only exclude colorings xx up to a fixed length k of x.

1 Squares and Nonrepetitive Colorings

In 1906 Axel Thue published his paper "Über unendliche Zeichenreihen" which showed the remarkable result that there is an infinite word over the alphabet $\Sigma = \{0, 1, 2\}$ that does not contain a *square*, namely a subword of the form xx:

$$010210120102120210120102101202 12\cdots$$

Remarkable, because over a binary alphabet there are only six square-free words: 0, 1, 01, 10, 010, 101. Remarkable also, because it is a rare instance of a pattern avoidance theorem: a counter-example to Ramsey theory published when Ramsey was three years old. Thue's result points in two directions: the study of patterns in words and the study of repetition. Combinatorics on words has become an active research field, not least through its importance to computer science [11, 12, 13]. In this paper we want to follow the second direction studying repetition in structures more general than words. There are recent surveys by Grytczuk [8] and Currie [4] on avoiding repetition in various areas of mathematics including graph theory, geometry, and number theory.

One natural generalization of a word is a circular words, that is, a word whose last letter is adjacent to its first letter. Currie [4] showed that there are square-free circular words of every length $n \geq 18$ on the alphabet $\{0, 1, 2\}$. Currie's result can be rephrased as saying that the cycle C_n on $n \geq 18$ vertices can be colored using 3 colors so that no subpath of C_n has a coloring of the form xx. We call such a coloring *nonrepetitive*. The coloring point of view was introduced by Alon, Grytczuk, Hałuszczak, and Riordan in a 2002 paper [1], which also contained the definition of the *Thue number* of a graph, $\pi(G)$, as the smallest number of colors needed in a nonrepetitive coloring of G. In this terminology, Currie proved that $\pi(C_n) = 3$ for $n \geq 18$.

Many problems related to the Thue number are still open. For example, it is not yet known whether $\pi(G)$ is bounded by some constant for all planar graphs G, a particularly intriguing problem. Kündgen and Pelsmajer [10] showed that graphs of treewidth at most k have Thue number at most 4^k, settling the special case of outerplanar graphs. It is also true that $\pi(G) \leq 36\Delta^2$, as was shown by Alon, Grytczuk, Hałuszczak, and Riordan [1]. It is also known that every graph has a subdivision whose Thue number is at most 4 (shown by Gryztcuk [7] for 5 and Barát and Wood for 4 [7, 9]).

We look at the Thue number from the point of view of computational complexity. Deciding whether $\pi(G) \leq k$ is an $\exists\forall$-question: is there a coloring such that no subpath of the graph has a square coloring. Deciding a question of this form belongs to the complexity class $\mathbf{\Sigma_2^p} = \mathbf{NP^{NP}}$, the second level of the polynomial-time hierarchy (see [14] for more information on the polynomial-time hierarchy). We conjecture that the Thue number problem is complete for that class. As a first result towards settling this conjecture we show in Section 2 that determining whether a *given* coloring of a graph is nonrepetitive is **coNP**-complete (in other words, deciding whether a coloring is repetitive is **NP**-complete). Indeed, the problem remains **coNP**-complete even when restricted to four colors, as we show in Section 3. As an illustration of our technique, we obtain a new proof of the Gryztcuk-Barát-Wood result that every graph has a subdivision with Thue number at most 4.

Since deciding whether a two-coloring of a graph is nonrepetitive is trivial, this raises the question of how hard it is to determine whether a coloring of a graph with three colors is nonrepetitive. This problem looks difficult; for example, by Currie's result, we can take a word w that is square-free as a circular word of any length $n \geq 18$. Then a path of length $2n$ with coloring ww is not square-free, but we have to look at a block of length n to find this out.

This example suggests studying nonrepetitiveness with restricted block-lengths. Let $\pi_k(G)$ be the smallest number of colors in a coloring of G which does not contain a path of length at most $2k$ with a repetitive coloring. This is a natural parameterization of the problem, $\pi_1(G)$ equals the chromatic number of G, and $\pi_2(G)$ is the *star-chromatic* number of G, introduced by Vince [15].

We complement the result that deciding the nonrepetitiveness of a coloring is **coNP**-hard, by showing how to decide in time $k^{O(k)}n^5 \log n$ whether a coloring of a graph on n vertices contains a path of length at most $2k$ with a repetitive coloring. Using the terminology of parameterized complexity [5, 6], for bounded block-lengths, nonrepetitiveness of a coloring is *fixed-parameter tractable:* the exponent of the polynomial running time does not depend on the parameter k.

2 Nonrepetitiveness of a Coloring

A word x is a *square* if $x = ww$ for some word w. A word is *nonrepetitive* if it does not contain a square as a subword. A *repetitive sequence* in a graph with a vertex-coloring is a path in the graph whose coloring, as read along the path, is a square. A graph coloring is *nonrepetitive* if it does not contain a repetitive sequence.

Theorem 2.1 *Deciding whether a coloring of a graph is nonrepetitive is* **coNP***-complete.*

Proof We reduce from the Hamiltonian Path problem. Let $G = (V, E)$ be a graph with $V = \{v_1, \ldots, v_n\}$. We construct a graph H and a coloring that is nonrepetitive if and only if G *does not* have a Hamiltonian path. The graph H consists of two parts. In the first part, for each v_i take a $K_{2,n}$ and color the two element partition using colors a and b, and the n-element partition using colors $c_{i,j}$ (for $1 \leq j \leq n$). Next, for every $i \neq j$ we introduce a new vertex colored $d_{i,j}$ and connect it to the b vertex of the $K_{2,n}$ belonging to v_i and the a vertex belonging to v_j. Also, we connect all the vertices colored b to a new vertex colored c. We construct the second part of H as follows: for each $1 \leq i, j \leq n$, we take a path $P_{i,j}$ on three vertices, coloring the vertices on $P_{i,j}$ by $a, c_{i,j}, b$. We connect the vertex colored by c to the a vertices of the paths $P_{i,1}$ $(1 \leq i \leq n)$. For every $P_{i,j}$ $(1 \leq i \leq n,\ 1 \leq j < n)$ and every edge $v_i v_{i'} \in E$ we add a new vertex of color $d_{i,i'}$ and connect it to the b vertex of $P_{i,j}$ and the a vertex of $P_{i',j+1}$. Finally, we connect all the b-vertices of $P_{i,n}$ to a new vertex colored c $(1 \leq i \leq n)$.

This finishes the construction of H and its coloring (for an example see Figure 1, where G is the diamond, i.e. $K_4 - e$). We claim that G contains a

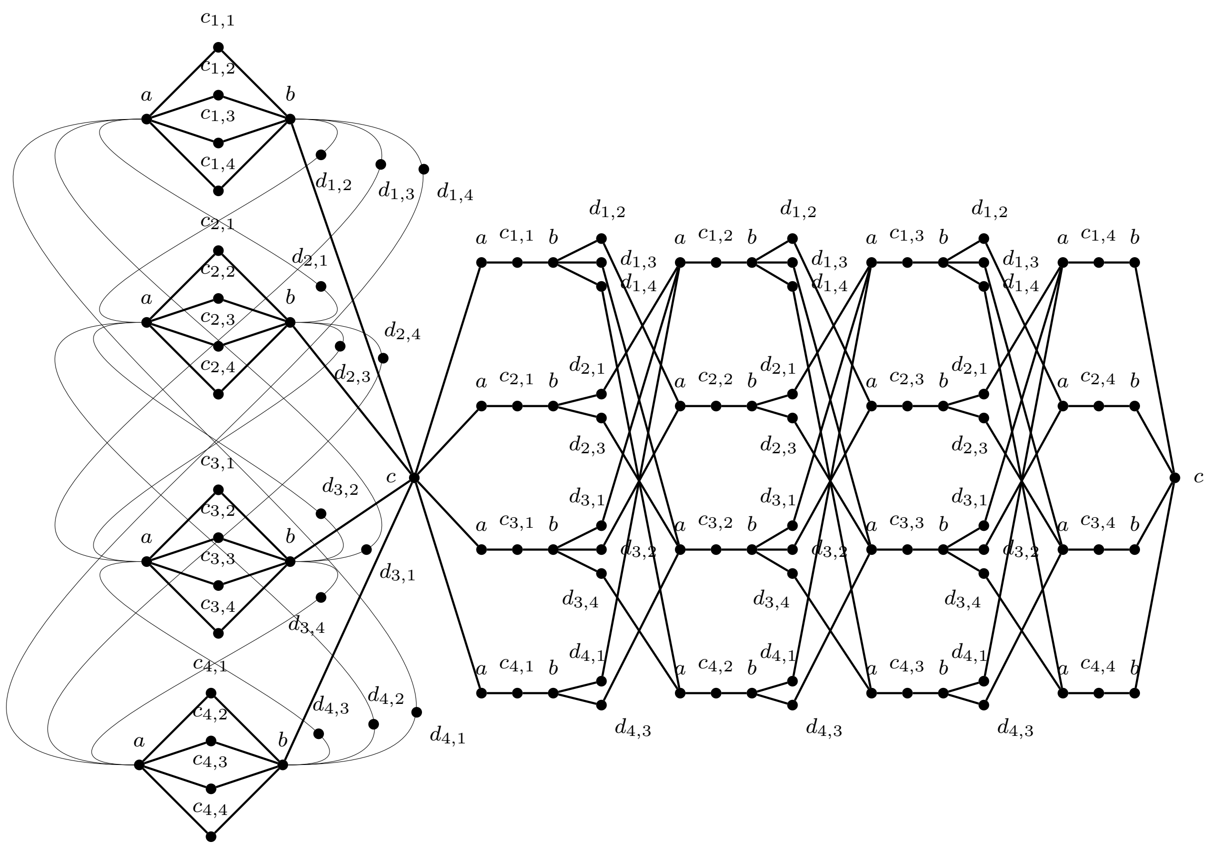

Figure 1: The graph H corresponding to the graph $(\{1,2,3,4\}, \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{3,4\}\})$.

Hamiltonian path if and only if the coloring of H we constructed is repetitive. This implies that deciding the nonrepetitiveness of a graph coloring is **coNP**-complete.

To prove the claim, let us first assume that G has a Hamiltonian path $v_{\pi(1)}, \ldots, v_{\pi(n)}$. Consider the following path through H: we start at the $K_{2,n}$ associated with $v_{\pi(1)}$, traversing it so we see colors $a, c_{\pi(1),1}, b$. We continue via the vertex colored $d_{\pi(1),\pi(2)}$ to the $K_{2,n}$ associated with $v_{\pi(2)}$, traversing it as $a, c_{\pi(2),2}, b$, etc. until we reach the b vertex in the $K_{2,n}$ belonging to $v_{\pi(n)}$. We then continue to the vertex colored c, and traverse the second half of H as follows: $P_{\pi(1),1}$, vertex colored $d_{\pi(1),\pi(2)}$, $P_{\pi(2),2}$, vertex colored $d_{\pi(2),\pi(3)}$, etc. finishing with $P_{\pi(n),n}$ and the vertex colored c. Since $v_{\pi(1)}, \ldots, v_{\pi(n)}$ is a Hamiltonian path, this traversal of H is possible, and, comparing the colors in the two halves of H, we see that they are the same, and, therefore, the coloring is repetitive.

For the reverse direction, assume that H contains a path P such that the colors along P are of the form ww for some word w. Let us first suppose that w does not contain the color c. Then P is entirely contained within the first or the second half of H. In either case we can argue that no repetition is possible, since all the colors except a and b are unique and vertices with colors a and b are not adjacent. We can therefore assume that w contains c. Consequently, P must contain both vertices z, z' colored c (let z be the vertex connecting the two halves). Without loss of generality, we can assume that P starts in the first half of H, and thus there are paths Q, Q', and Q'' such that $P = QzQ'z'Q''$. The first vertex of Q' has color a, while all neighbors of z' have color b, which means that Q'' is empty, and, therefore, $P = QzQ'z'$. Let m be the number of vertices in Q' having some color $c_{i,j}$; then $m \geq n$ and Q' has at least $4n-1$ vertices. On the other hand, Q can contain at most $4n$ vertices of which at most n can have some color $c_{i,j}$; therefore, $m = n$, and Q and Q' have length $4n-1$. Since Q' has length $4n-1$, for every i there is a j such that $c_{i,j}$ occurs on Q'. Similarly, along Q there is for every j an i such that $c_{i,j}$ occurs on Q. In other words, there is a permutation π such that $c_{\pi(j),j}$ occurs on Q. By the construction of the second half of H, $v_{\pi(1)}, \ldots, v_{\pi(n)}$ is a Hamiltonian path of G. □

We note that the proof used an unbounded number of colors to achieve the coding. This can be remedied as we will see in the next section.

3 The Case of 4 Colors

We reduce the number of colors by replacing colors with long nonrepetitive sequences on a fixed set of colors. As an illustration, we first prove a simple graph-theoretic result.

Proposition 3.1 (Grytczuk, Barát and Woods) *Every graph has a subdivision which can be nonrepetitively colored with at most 4 colors.*

Remark Grytczuk [7] proved that every graph has a subdivision which can be colored with at most 5 colors; Barát and Woods improved his result to 4 colors [9]. Our construction is closer in spirit to Grytczuk's original proof.

The following lemma constructs a family of nonrepetitive sequences with useful properties. We write x^R for the reverse of the sequence x.

Lemma 3.2 *We can in polynomial time construct m nonrepetitive sequences of length $O(m)$ on colors 1, 2 and 3 so that*

- (i) *for any two sequences x and y, if we split each sequence into two halves of equal length, $x = x_1x_2$ and $y = y_1y_2$, then $x_i \neq y_i$ and $x_i \neq y^R_{3-i}$ (for $i = 1, 2$),*
- (ii) *all sequences begin 31 and end 13, and*
- (iii) *all sequences have the same length.*

To see that the lemma is true, take a nonrepetitive sequence x of length $1764m+13$ and permute the colors so it starts with 31. We claim that every subword of 14 letters has to contain the sequences 13 and 31, a claim we will verify later. So if we let x_i be the subword of x that starts with the i-th 31 in x, and ends with the first 13 at least $1176m-1$ positions later, we know that $1176m \leq |x_i| \leq 1176m+13$. In this fashion we can pick $42m$ sequences x_i from x ($1 \leq i \leq 42m$), since x_{42m} ends no later than position $42m \cdot 14 + 1173m + 13 = 1764m + 13$. Note that any two of these sequences y and z overlap in at least $588m + 13$ positions in x, because x_{42m} must contain position $14 \cdot (42m - 1) + 1 = 588m - 13$ and x_1 ends no earlier than position $1176m$, so there is a string of length $588m + 13$ common to *all* x_i). Since y and z half length at most $1176m + 13$ the overlap of length at least $588m + 13$ between them forces their first halves, as well as their second halves to overlap. Therefore, the first halves of y and z must differ from each other, as must the second halves (otherwise, x would contain a square). Among the $42m$ sequences, we can pick $3m$ sequences of the same length. While it is possible that for two of these sequences y and z, the first half of y equals the reverse of the second half of z, it is not possible that the first half of y equals the reverse of the second half of two other sequences z and z', since in that case the second halves of z and z' would be identical, which we excluded. Similarly, the second half of y can be equal to the reverse of at most one other sequence. Hence we can pick $m = 3m/3$ sequences fulfilling condition (i).

We are left with the proof of the claim that any nonrepetitive sequence of length 14 contains the subsequence 13, and, consequently, every other two-digit subsequence. So let x be a nonrepetitive 14-digit string over the alphabet $\{1, 2, 3\}$. A 1 must occur within the first four digits of x. If that 1 is followed by a 3 we are done, so we know that there is a sequence 12 starting within the first four positions of x. Suppose that sequence continued with a 1, i.e. we see 121. Then the next digit cannot be 2 again, so we have 1213, and, therefore, a 13 within the first seven digits of x. In other words, we know that there is a sequence 123 starting within the first four positions of x. There are two cases: suppose the next digit after 123 is 1, i.e. we have 1231, the next digit has to be 2 (otherwise we have a 13), followed by 1 (since the word is nonrepetitive): 123121. The next digit cannot be a 2, since the word is nonrepetitive, so it has to be a 3 and we are done, since we have found a 13 within the first nine positions of x. In the second case, we have 1232. To avoid repetition, this sequence needs to continue 12321. If the next digit is a 3, we are done, so we can assume we see 123212, which cannot be followed by 1 (repetition), so we have 1232123, which cannot be followed by 2 (repetition), giving us 12321231 followed by 2 (otherwise we have a 13), followed by 1 (repetition), yielding 1232123121. Finally, this string cannot be followed by a 2, so we see 12321231213, which means a 13 within x.

Proof of Proposition 3.1 It is enough to prove the theorem for the case $G = K_n$. Let $(x_i)_{i=1}^m$ be a family of $m = \binom{n}{2}$ nonrepetitive sequences as described in Lemma 3.2. Replace the i-th edge of G with a path of length $|x_i| + 7$ and color it $210x_i012$. Also, give each vertex of G color 0. We claim that this coloring of a subdivision G' of G is nonrepetitive.

Suppose, to the contrary, that G' contains a path P with a coloring of the form ww. P has to contain the color 0, since otherwise ww would be a subword of some x_i which is not possible (as the x_i's are nonrepetitive). There are two types of vertices colored 0: the vertices of G, all of whose neighbors are colored 2, and the vertices introduced in the subdivision, all of whose neighbors are colored 1 and 3. Hence, for a repetition, P must contain two vertices colored 0 of the same type, and that is only possible if P contains a whole path Q between two vertices of G. It is not possible that the coloring of Q is a subword of w, since the colorings of the paths (and their reverses) are unique. Hence, Q must contain the border between the two halves of P. In other words, ww has to contain the following string:

$$0210v0120,$$

where $v = x_i$ for some i (if $v = x_i^R$ we reverse P), and the boundary of ww occurs within v. Assuming that the boundary occurs in the second half of v (the other case being similar), the first half of v must coincide with the prefix or the reverse of a suffix of some other x_j. This possibility, however, we excluded by the choice of sequences. □

Corollary 3.3 *Deciding whether a coloring of a graph is nonrepetitive is* **coNP***-complete even for colorings with at most* 4 *colors.*

Proof We will show how to replace the colors in the graph H constructed in the proof of Theorem 2.1 with just 4 colors. Using Lemma 3.2 we obtain sequences x_i, one for each of the colors c, $c_{k,j}$, and $d_{k,j}$. If a vertex has color $c_{k,j}$ or $d_{k,j}$, and it has been assigned sequence x_i, replace the vertex with a path of length $|x_i| + 7$ and color it $210x_i012$. For the two vertices colored c, we proceed similarly, but in this case the vertex is replaced with a path colored $130x_i031$; call the two paths replacing the c vertices C and C' (where C is the path connecting the two halves of G). Finally, recolor vertices with colors a or b to have color 0. This construction uses colors $0, 1, 2, 3$ only.

We claim that the coloring of the resulting graph will be nonrepetitive if and only if the original graph G did not have a Hamiltonian path.

The proof of one direction remains unchanged: a Hamiltonian path in G still corresponds to a repetitive coloring, since we just replaced colors by color sequences.

Suppose then that G contains a path P colored ww. As we argued earlier, P has to contain the color 0, since otherwise ww would be a subword of some x_i, which is nonrepetitive.

We have four types of vertices colored 0: those with neighbors $1, 3$, those with neighbors $1, 2$, those with two neighbors colored 2 and those with two neighbors colored 3. Let us look at the last type first.

Suppose P does not contain the sequence 303 (which occurs exactly four times: twice on each of the paths replacing c. In that case P cannot traverse C (or C'), and is therefore caught within one of the two halves of G. We claim that this is impossible.

First of all, observe that P does have to contain at least one vertex from C or C', since otherwise we argue as in the proof of Proposition 3.1 that the two halves of the graph obtained by removing C and C' do not contain a square. (That part of the proof of Theorem 2.1 did not use the fact that a and b are different colors.) Suppose then that P contains exactly one vertex from C or C'. That vertex must be one of the end-vertices of C or C' colored 1. Then P must contain the sequence 201. If P lies in the left half of G, it can contain at most one 201 (since all occurrences of 201 overlap in the 1). Hence, the middle of P has to occur either at $2|01$ or $20|1$. In the first case, P must contain two 010, which is impossible, in the later case it has to contain two 102, which is also impossible. If P lies in the right half, the argument is similar: there has to be an occurrence of 201. To match it either as 201 or $2|01$ or $20|1$, the path P needs to contain vertices from both C and C', implying that ww contains a string of the form $210x_i012$. As we argued in Proposition 3.1 this is impossible by the construction of the x_i.

Consequently, P must contain at least two vertices from C or C'; since we assumed that it does not contain 303, P must end, or begin, in C or C' with 013 or 0130. Both sequences, however, do not occur a second time in a half without overlapping the earlier occurrence, so this is not possible.

We conclude that P must contain the sequence 303. This sequence occurs exactly four times, twice in C and C'. The two occurrences in the same path C or C' cannot match with each other, since one begins $3031x_i$, and the other 30310 (and the x_i's do not contain zeroes). Hence a 303 from C must match with a 303 from C'. But then either all of C or all of C', and therefore both must belong to P.

From this point on, we can argue as in the original proof. □

4 Bounded-Length Sequences

Checking whether a coloring of a graph is nonrepetitive for block-lengths up to some fixed value k can be done in polynomial time: we have to check all the $O(n^{2k})$ paths of length at most $2k$. Here we present an algorithm that is significantly more efficient than brute force: we show that the problem is *fixed-parameter tractable*, i.e., it can be solved in time $O(f(k)n^c)$. This means that the exponent of n does not increase as k increases.

Theorem 4.1 *Given a vertex-colored graph $G(V, E)$, it can be checked in time $k^{O(k)} \cdot |V|^5 \log |V|$ whether G has a repetitive sequence of length $2k$.*

Proof The algorithm is based on color-coding, introduced by Alon et al. [2]. Assign a random label from $\{1, \dots, 2k\}$ to each vertex of G independently with uniform distribution. Assume that we have a polynomial-time algorithm for checking whether there is a repetitive sequence $v_1, \dots, v_{2k}$ where vertex v_i has label i (below we will present such an algorithm). If the graph has a repetitive sequence, then the sequence receives the labels $1, \dots, 2k$ with probability $1/(2k)^{2k}$, hence the algorithm finds such a repetitive sequence with probability $1/(2k)^{2k}$. If the graph has no repetitive sequence, then of course no such sequence is found by the algorithm. Therefore, the algorithm produces a correct answer with probability $1/(2k)^{2k}$, which can be increased to a constant by repeating the algorithm $(2k)^{2k}$ times. Randomized algorithms based on color-coding can be derandomized using standard techniques, see [2] and [5, Section 8.3].

We still need to show how to check whether there is a repetitive sequence $v_1, \dots, v_{2k}$ where vertex v_i has label i. For a given labeling $\lambda : V \to \{1, \dots, 2k\}$ of the vertices, we proceed as follows. For a given vertex x, the algorithm below checks whether there is a repetitive sequence $v_1, \ldots, v_{2k}$ where $\lambda(v_i) = i$ and $v_k = x$. Therefore, the algorithm has to be repeated for every possible choice of x, i.e., $|V|$ times.

We build a directed graph $D(U, A)$ where the U is a subset of $V \times V$. For $v, v' \in V$, the pair (v, v') is a vertex of D only if

- v and v' have the same color in G,
- $\lambda(v') = \lambda(v) + k$,
- if $\lambda(v) = k$, then $v = x$, and
- if $\lambda(v') = k + 1$, then v' is a neighbor of x in G.

There is an arc from (v, v') to (u, u') in D if and only if

- u is a neighbor of v,
- u' is a neighbor of v', and
- $\lambda(u) = \lambda(v) + 1$.

Note that, by the properties of the vertices in D, the last requirement also implies $\lambda(u') = \lambda(v') + 1$.

It is easy to see that D is acyclic, hence the length of the longest directed path can be determined in time $O(|A|)$ using standard techniques. We claim that D has a directed path on k vertices if and only if G has a repetitive sequence on $2k$ vertices. Indeed, if (v_1, v'_1), (v_2, v'_2), $\ldots$, (v_k, v'_k) is a directed path in D, then $v_1, \ldots, v_k, v'_1, \ldots, v'_k$ is a path in G. Notice that the i-th vertex of the path in G has label i, thus a vertex cannot appear twice in the sequence. Furthermore, v_i and v'_i have the same color in G, hence the path is repetitive. The converse statement is also easy to see: if $v_1, \ldots, v_{2k}$ is a repetitive sequence such that $\lambda(v_i) = i$ and $v_k = x$, then the vertices (v_1, v_{k+1}), (v_2, v_{k+2}), $\ldots$, (v_k, v_{2k}) exist in D and they form a directed path.

The directed graph D contains at most $|V|^2$ vertices and hence at most $|V|^4$ edges. Finding the longest path in the acyclic graph D can be done in linear time. The algorithm has to be repeated for every possible vertex x, thus the running time is $|V|^5$ for a given labeling. The derandomization adds a factor $O(\log |V|)$ to the running time. □

The case $k = 2$ is of special interest. Graphs that do not have repetitive sequences of length at most 4 are often called *star-free* or *apathic*. For apathic coloring, the complexity of the coloring problem is settled:

Proposition 4.2 (Coleman, Moré [3]) *Deciding whether a graph has a star-free coloring with three colors is* **NP***-complete, even if the graph is bipartite.*

The proof is quite simple: replace each edge of a graph G with three paths of length 2. Then the original graph is 3-colorable, if and only if the resulting (bipartite) graph has a star-free 3-coloring. The result was proved by Coleman and Moré in the context of computing sparse Hessian matrices.

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