Important separators and parameterized algorithms

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Methods for Discrete Structures

February 7, 2011

Overview

Main message: Small separators in graphs have interesting extremal properties that can be exploited in combinatorial and algorithmic results.

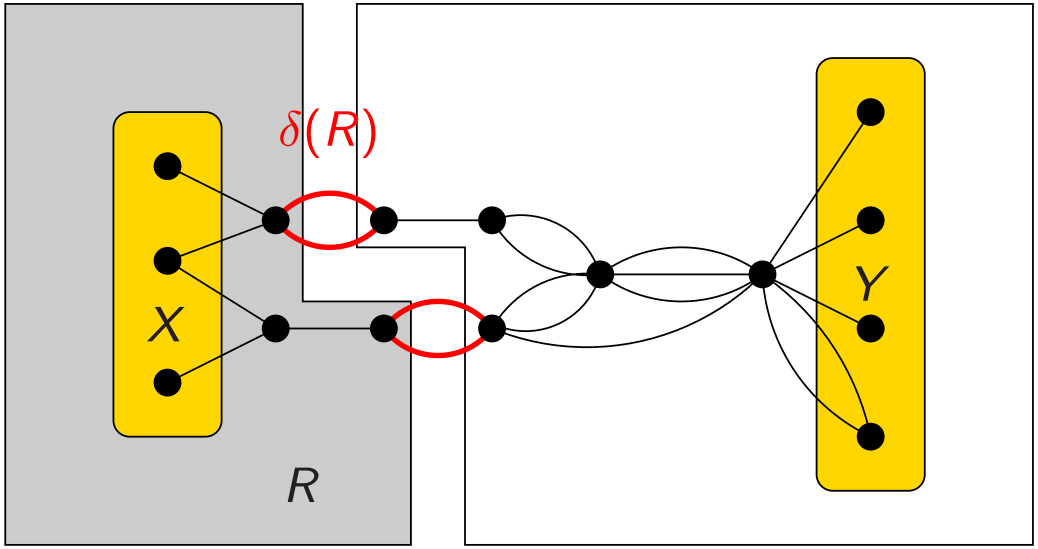
- Bounding the number of “important” separators.
- Combinatorial application: Erdős-Pósa property for “spiders.”
- Algorithmic applications: FPT algorithm for multiway cut and a directed feedback vertex set.

Important separators

Definition: $\delta(R)$ is the set of edges with exactly one endpoint in R.

Definition: A set S of edges is an (X, Y)**-separator** if there is no $X - Y$ path in $G \setminus S$ and no proper subset of S breaks every $X - Y$ path.

Observation: Every (X, Y)-separator S can be expressed as $S = \delta(R)$ for some $X \subseteq R$ and $R \cap Y = \emptyset$.



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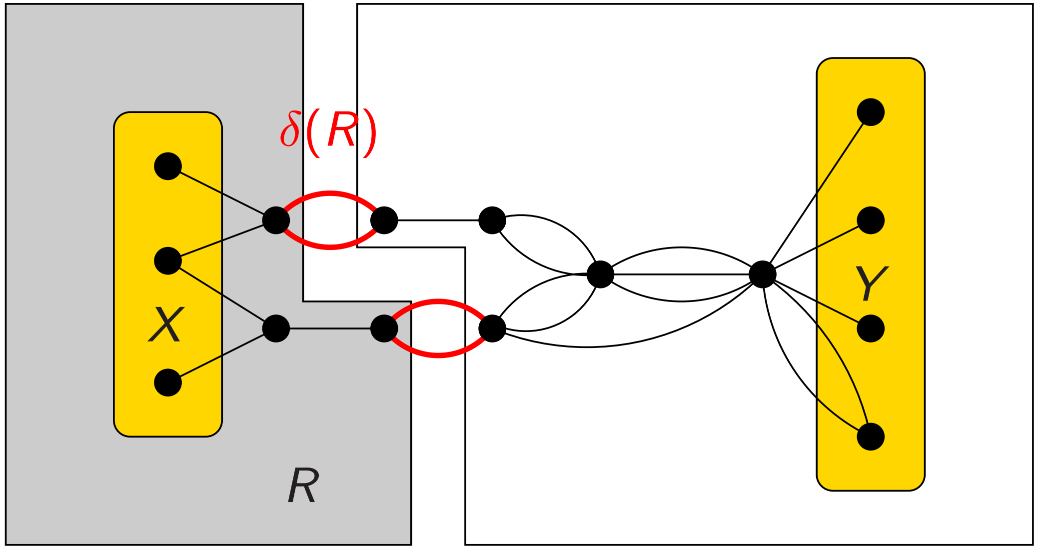
Definition: An (X, Y)-separator $\delta(R)$ is **important** if there is no (X, Y)-separator $\delta(R')$ with $R \subset R'$ and $|\delta(R')| \leq |\delta(R)|$.

Note: Can be checked in polynomial time if a separator is important.



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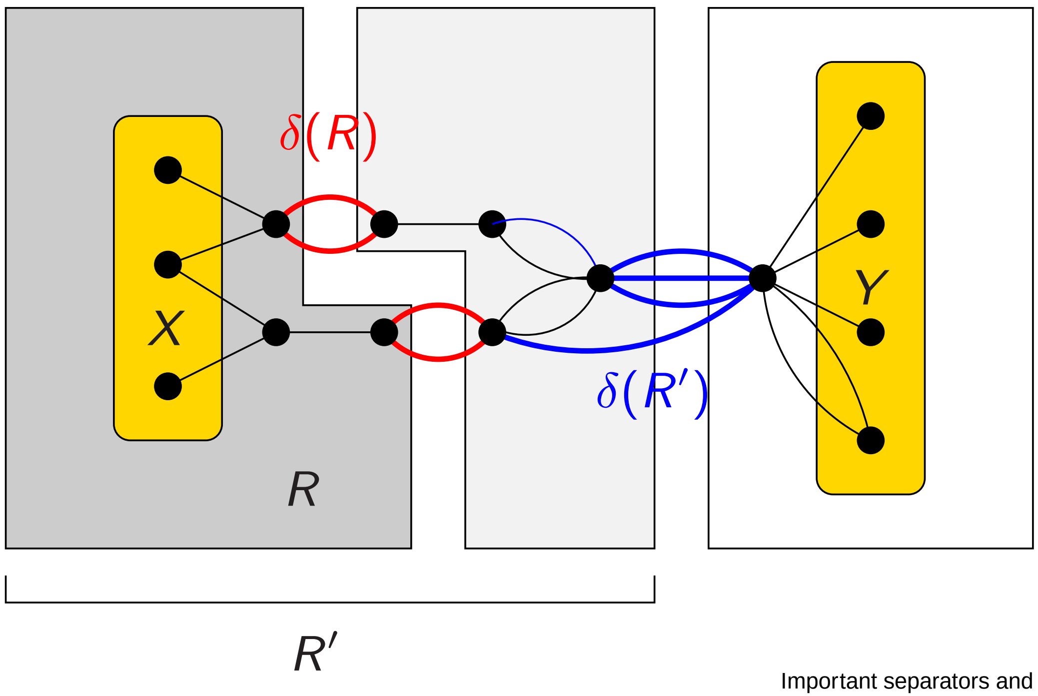

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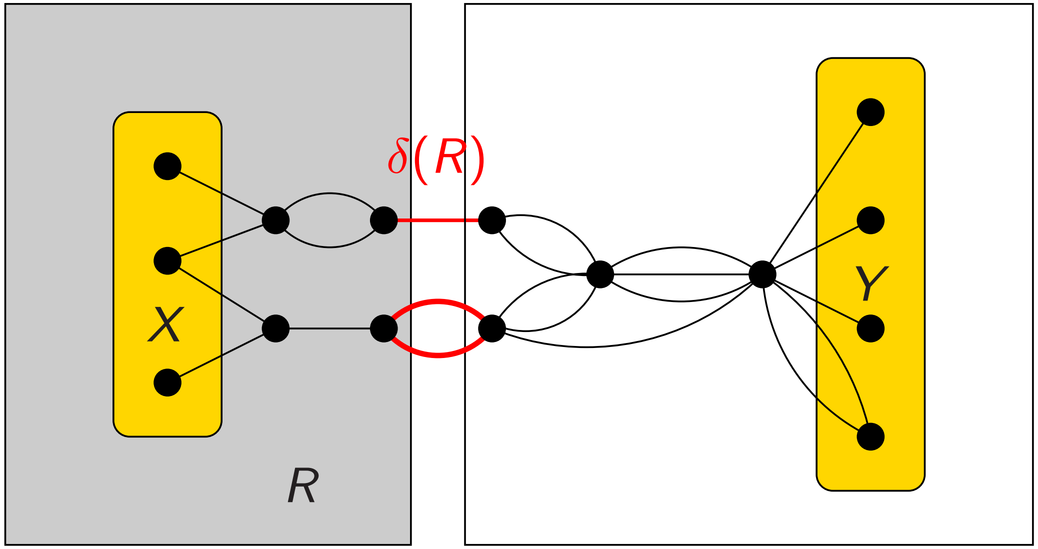
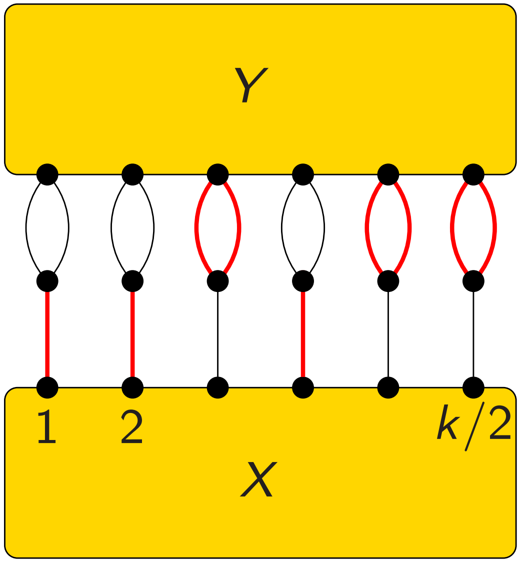


Important separators

The number of important separators can be exponentially large.

Example:



This graph has exactly $2^{k/2}$ important (X, Y)-separators of size at most k.

Theorem: There are at most 4^k important (X, Y)-separators of size at most k.
(Proof is implicit in [Chen, Liu, Lu 2007], worse bound in [M. 2004].)

Submodularity

Fact: The function δ is **submodular:** for arbitrary sets A, B,

$$|\delta(A)| + |\delta(B)| \geq |\delta(A \cap B)| + |\delta(A \cup B)|$$

Consequence: Let λ be the minimum (X, Y)-separator size. There is a unique maximal $R_{\max} \supseteq X$ such that $\delta(R_{\max})$ is an (X, Y)-separator of size λ.

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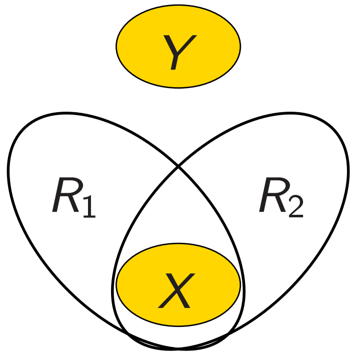
Proof: Let $R_1, R_2 \supseteq X$ be two sets such that $\delta(R_1), \delta(R_2)$ are (X, Y)-separators of size λ.

$$|\delta(R_1)| + |\delta(R_2)| \geq |\delta(R_1 \cap R_2)| + |\delta(R_1 \cup R_2)|$$

$$\lambda \qquad \lambda \qquad \geq \lambda$$

$$\Rightarrow |\delta(R_1 \cup R_2)| \leq \lambda$$



Note: Analogous result holds for a unique minimal $R_{\min}$.

Important separators

Theorem: There are at most 4^k important (X, Y)-separators of size at most k.

Proof: Let λ be the minimum (X, Y)-separator size and let $\delta(R_{\max})$ be the unique important separator of size λ such that $R_{\max}$ is maximal.

First we show that $R_{\max} \subseteq R$ for every important separator $\delta(R)$.

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By the submodularity of δ:

$$|\delta(R_{\max})| + |\delta(R)| \geq |\delta(R_{\max} \cap R)| + |\delta(R_{\max} \cup R)|$$

$$\lambda \qquad\qquad\qquad \geq \lambda$$

$$\Downarrow$$

$$|\delta(R_{\max} \cup R)| \leq |\delta(R)|$$

$$\Downarrow$$

If $R \neq R_{\max} \cup R$, then $\delta(R)$ is not important.

Thus the important (X, Y)- and $(R_{\max}, Y)$-separators are the same.
$\Rightarrow$ We can assume $X = R_{\max}$.

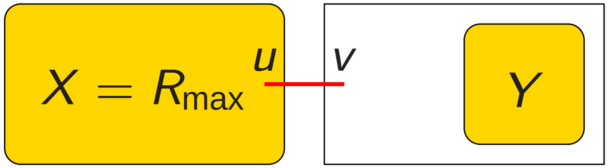
Important separators

Lemma: There are at most 4^k important (X, Y)-separators of size at most k.

Search tree algorithm for enumerating all these separators:

An (arbitrary) edge uv leaving $X = R_{\max}$ is either in the separator or not.

Branch 1: If $uv \in S$, then $S \setminus uv$ is an important (X, Y)-separator of size at most $k - 1$ in $G \setminus uv$.



Branch 2: If $uv \notin S$, then S is an important $(X \cup v, Y)$-separator of size at most k in G.

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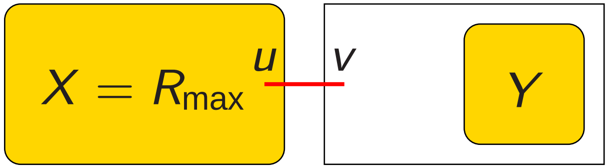
$\Rightarrow$ k decreases by one, λ decreases by at most 1.



Branch 2: If $uv \notin S$, then S is an important $(X \cup v, Y)$-separator of size at most k in G.

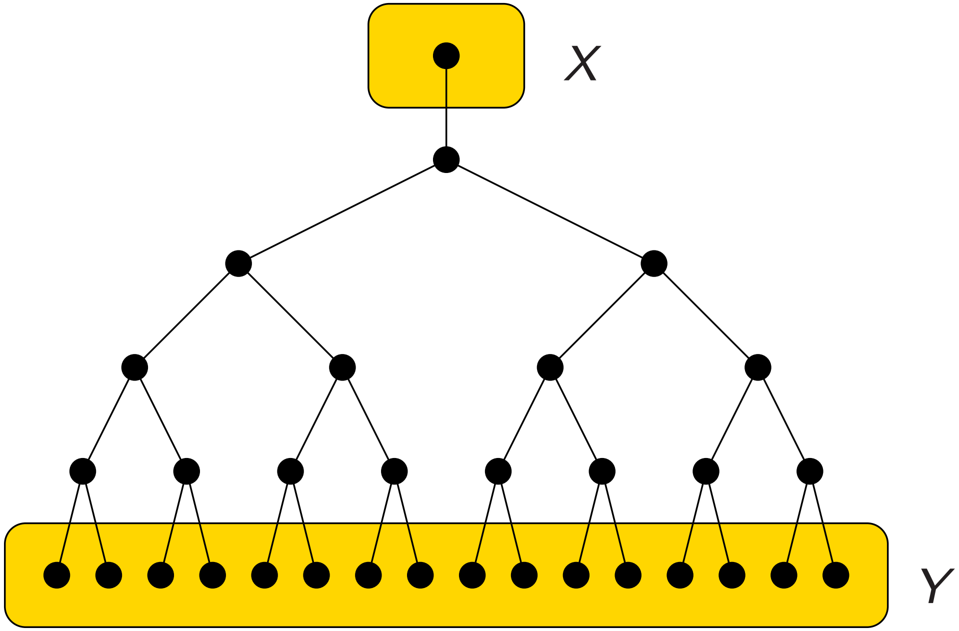
$\Rightarrow$ k remains the same, λ increases by 1.

The measure $2k - \lambda$ decreases in each step.

$\Rightarrow$ Height of the search tree $\leq 2k \Rightarrow \leq 2^{2k}$ important separators of size $\leq k$.

Important separators

Example: The bound 4^k is essentially tight.



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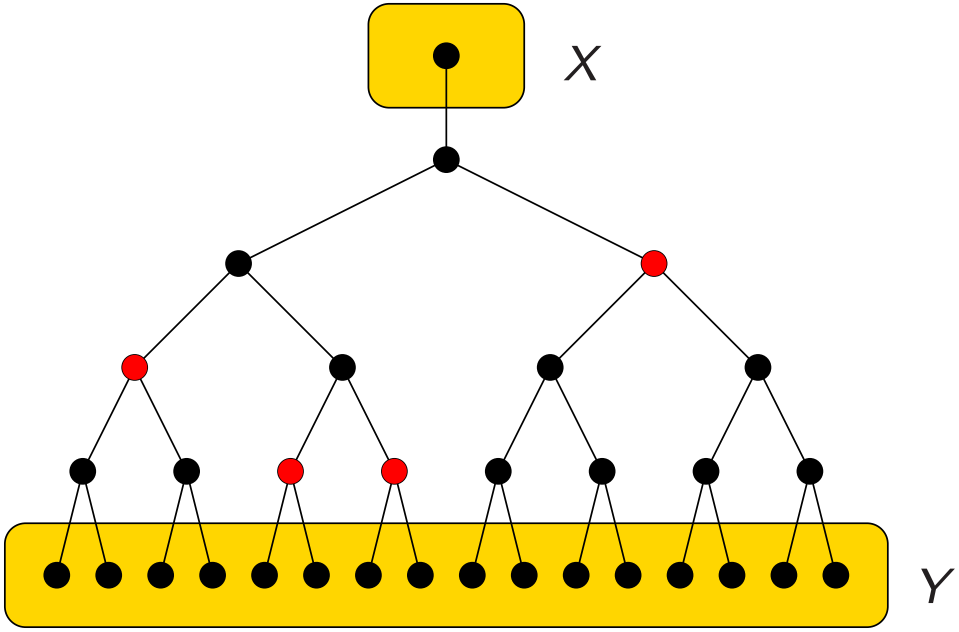

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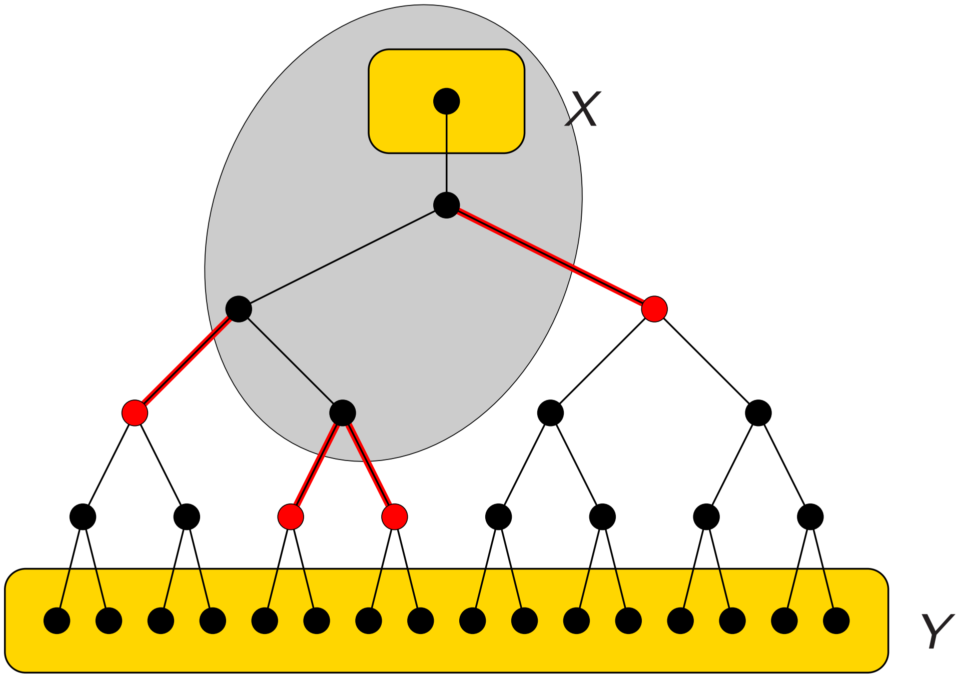

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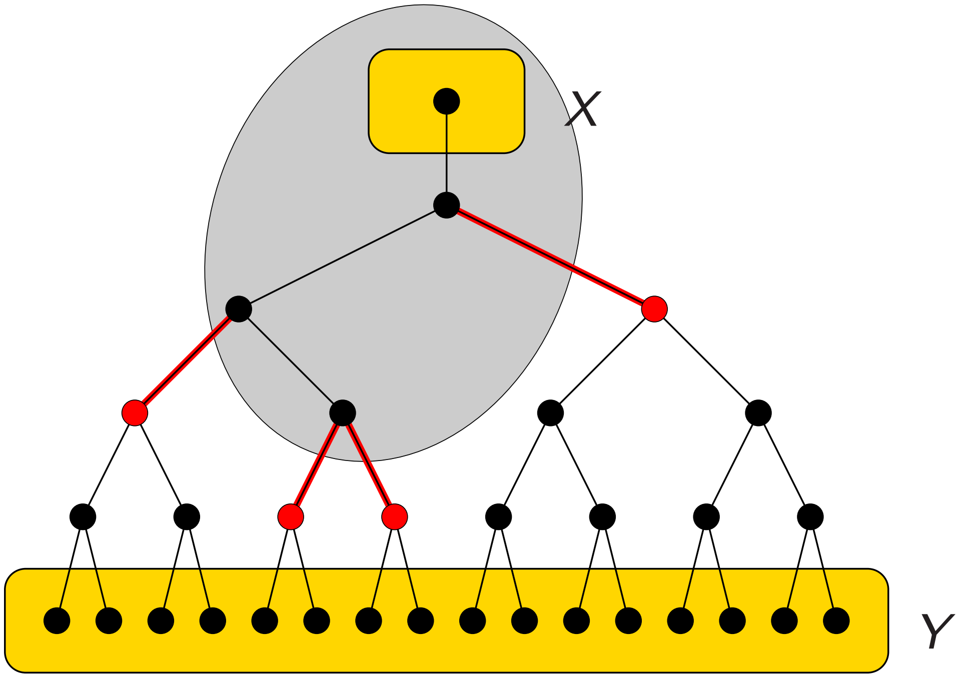

Any subtree with k leaves gives an important (X, Y)-separator of size k.

The number of subtrees with k leaves is the Catalan number

$$C_{k-1} = \frac{1}{k}\binom{2k-2}{k-1} \geq 4^k/\text{poly}(k).$$

Simple application

Lemma: At most $k \cdot 4^k$ edges incident to t can be part of an inclusionwise minimal $s - t$ cut of size at most k.

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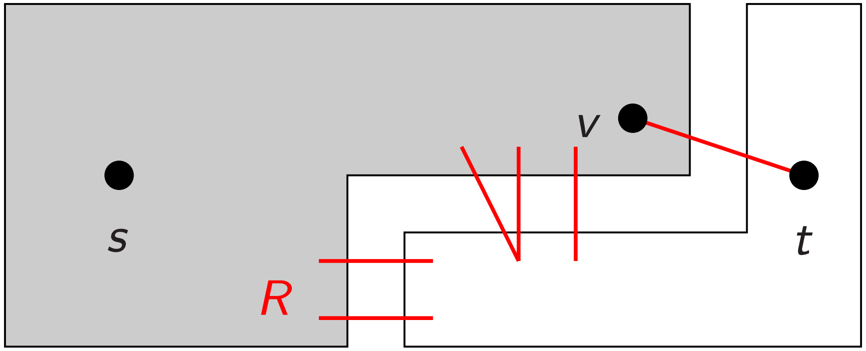
Proof: We show that every such edge is contained in an important (s, t)-separator of size at most k.



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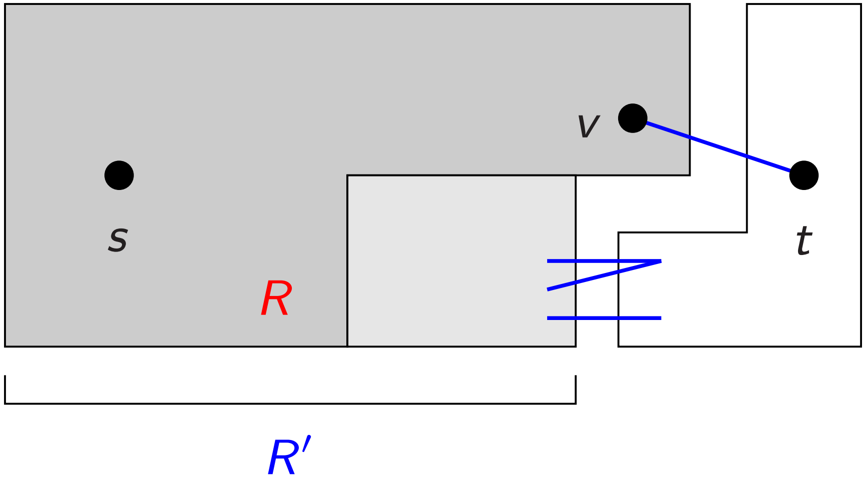


Suppose that $vt \in \delta(R)$ and $|\delta(R)| = k$.

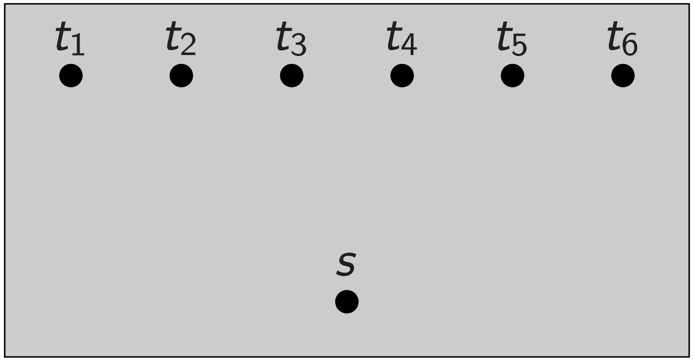
There is an important (s, t)-separator $\delta(R')$ with $R \subseteq R'$ and $|\delta(R')| \leq k$.

Clearly, $vt \in \delta(R')$: $v \in R$, hence $v \in R'$.

Anti isolation

Let $s, t_1, \ldots, t_n$ be vertices and $S_1, \ldots, S_n$ be sets of at most k edges such that S_i separates t_i from s, but S_i **does not** separate t_j from s for any $j \neq i$.

It is possible that n is "large" even if k is "small."



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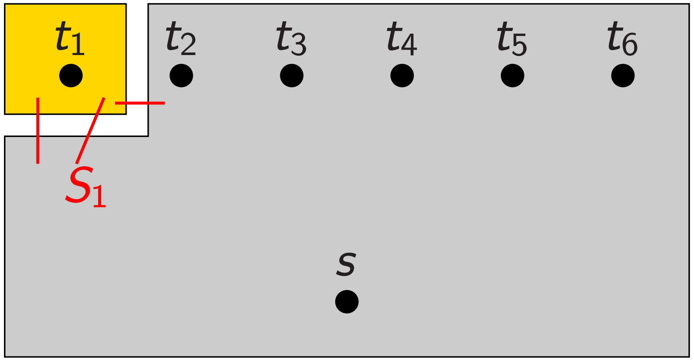

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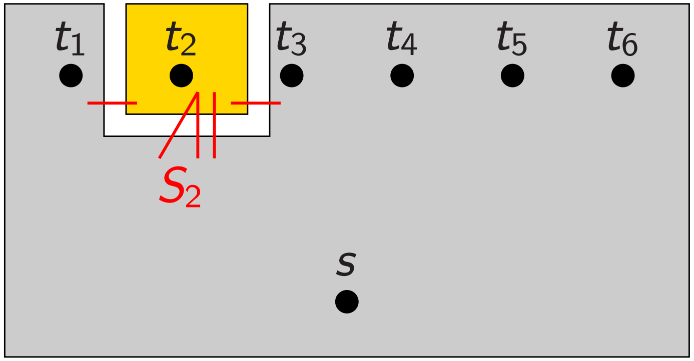

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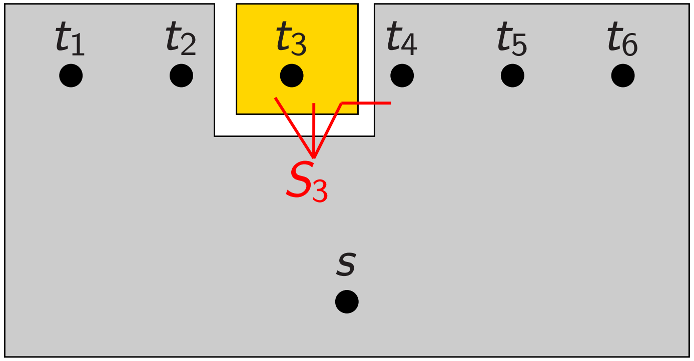

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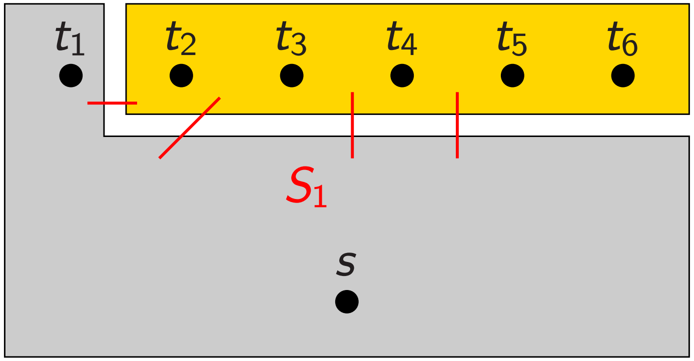

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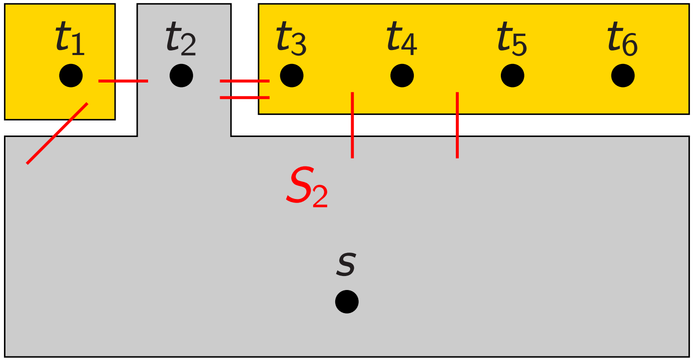

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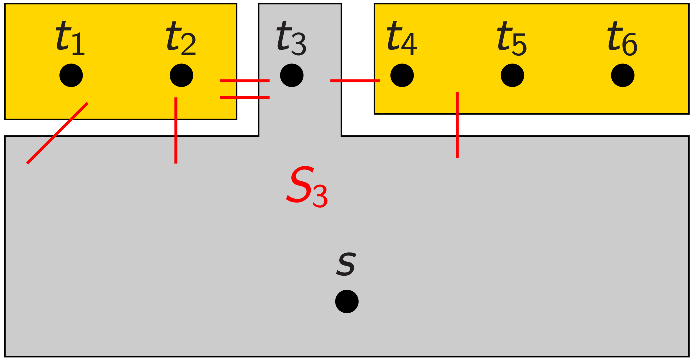

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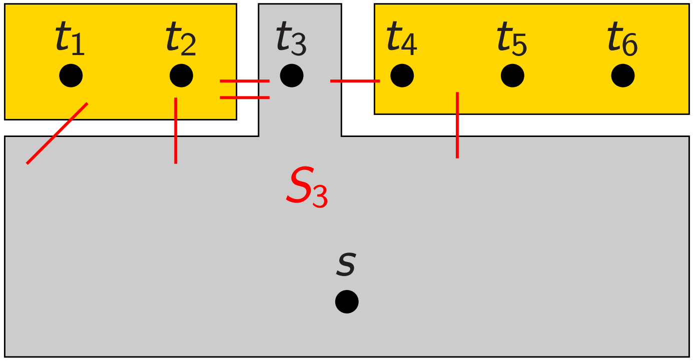

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Lemma: If S_i separates t_j from s if and only $j \neq i$ and every S_i has size at most k, then $n \leq (k+1) \cdot 4^{k+1}$.

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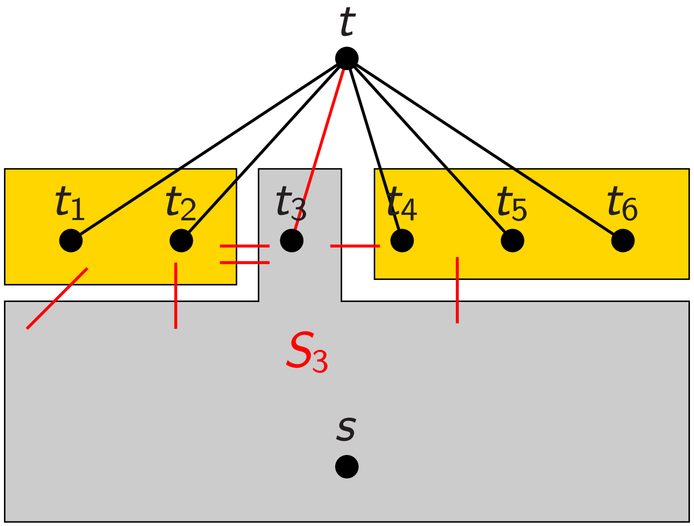

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Proof: Add a new vertex t. Every edge tt_i is part of an (inclusionwise minimal) (s,t)-separator of size at most $k+1$. Use the previous lemma.

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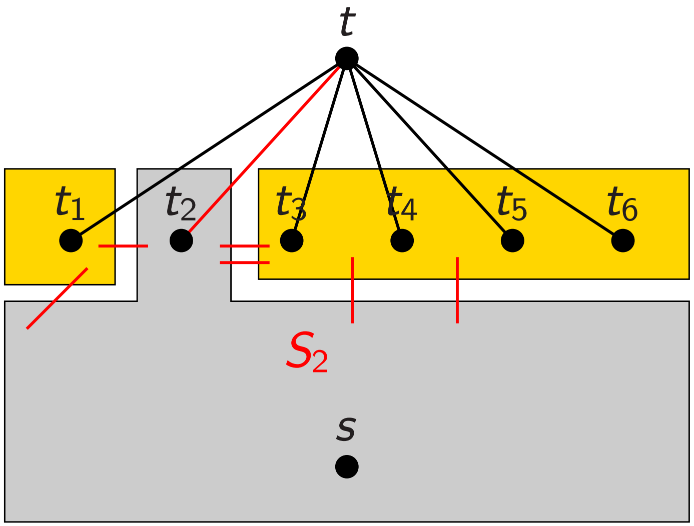

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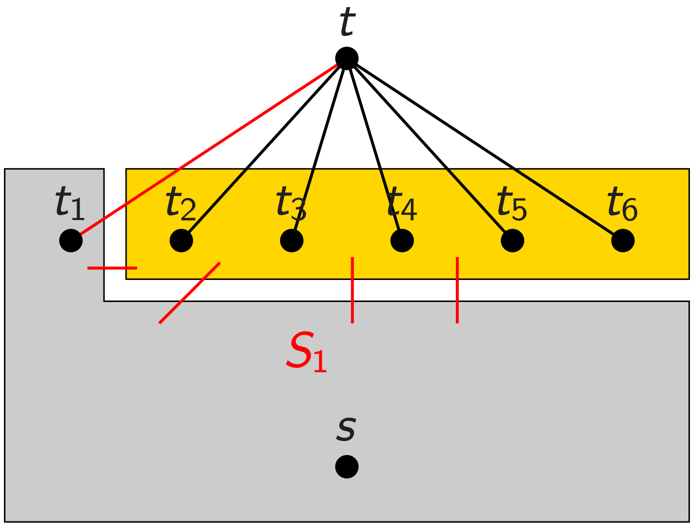

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Erdős-Pósa property

Theorem: [Erdős-Pósa 1965] There is a function $f(k) = O(k \log k)$ such that for every undirected graph G and integer k, either

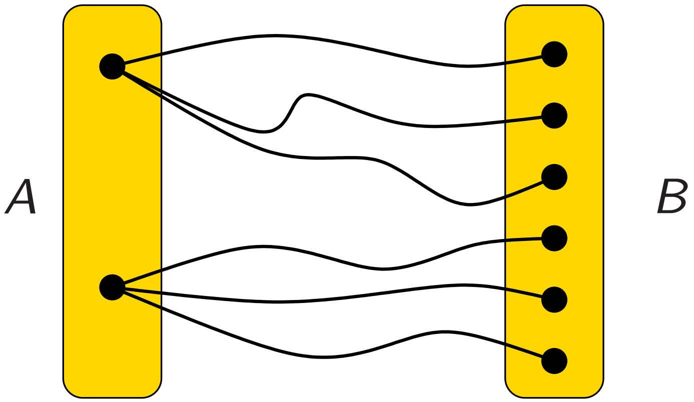
- G has k vertex-disjoint cycles, or
- G has a set S of at most $f(k)$ vertices such that $G \setminus S$ is acyclic.

More generally: A set of objects has the Erdős-Pósa property if the covering (hitting number) can be bounded by a function of the packing number.

Spiders

Let A and B be two disjoint sets of vertices in G. A d**-spider** with center v is a set of d edge disjoint paths connecting $v \in A$ with B.

Suppose for simplicity that every vertex of A has degree exactly d.



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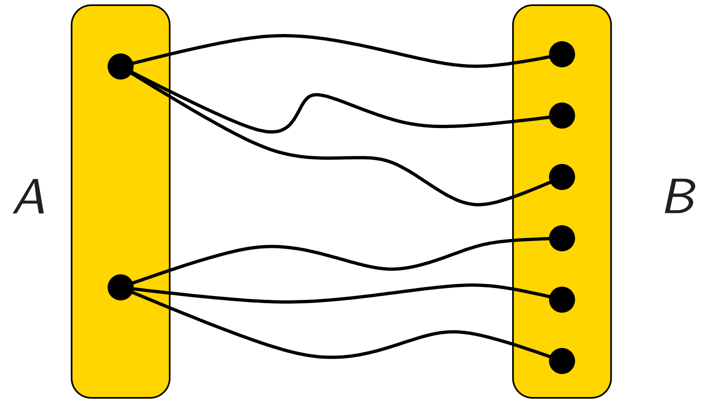

Theorem: There is a function $f(k, d) = 2^{O(kd)}$ such that for every graph G and disjoint sets A, B either

- there are k edge-disjoint d-spiders, or
- there is a set D of at most $f(k, d)$ edges that intersects every d-spider.

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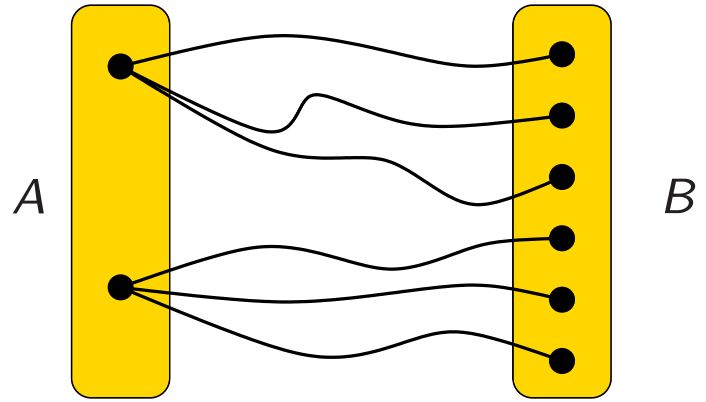

Proved by Robertson and Seymour in Graph Minors XXIII:

7.2. *Let $\mathcal{T}$ be a tangle in a hypergraph G, and let $W \subseteq V(G)$ be free relative to $\mathcal{T}$, with $|W| \leqslant w$. Let $h \geqslant 1$ be an integer, and let $\mathcal{T}$ have order $\geqslant (w+h)^{h+1} + h$. Then there exists $W' \subseteq V(G)$ with $W \subseteq W'$ and $|W'| \leqslant (w+h)^{h+1}$ such that for every $(C, D) \in \mathcal{T}$ of order $< |W| + h$ with $W \subseteq V(C)$, there exists $(A', B') \in \mathcal{T}$ with $W' \subseteq V(A' \cap B')$, $|V(A' \cap B') \setminus W'| < h$, $C \subseteq A'$ and $E(B') \subseteq E(D)$.*

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- there is a set D of at most $f(k, d)$ edges that intersects every d-spider.

Proof: Assuming that there are no k edge-disjoint d-spiders,

1. we construct a set D and
2. show that D intersects every d-spider.

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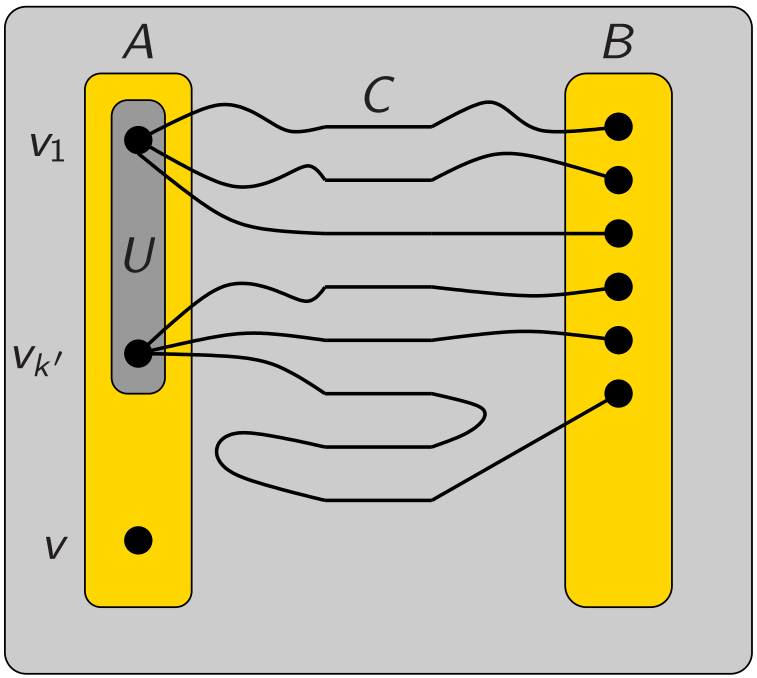
Proof: Suppose that there are $k' < k$ disjoint d-spiders with centers $U = \{v_1, \ldots, v_{k'}\}$, but there are no $k' + 1$ disjoint spiders.

Let D be the union of all the important (v_i, B)-separators of size at most kd for $1 \leq i \leq k'$.
$\Rightarrow$ size of D is at most $f(k, d) := k \cdot 4^{kd} \cdot kd$.

We claim that D intersects every d-spider.

Spiders

Remember: D contains every important (v_i, B)-separator of size $\leq kd$.



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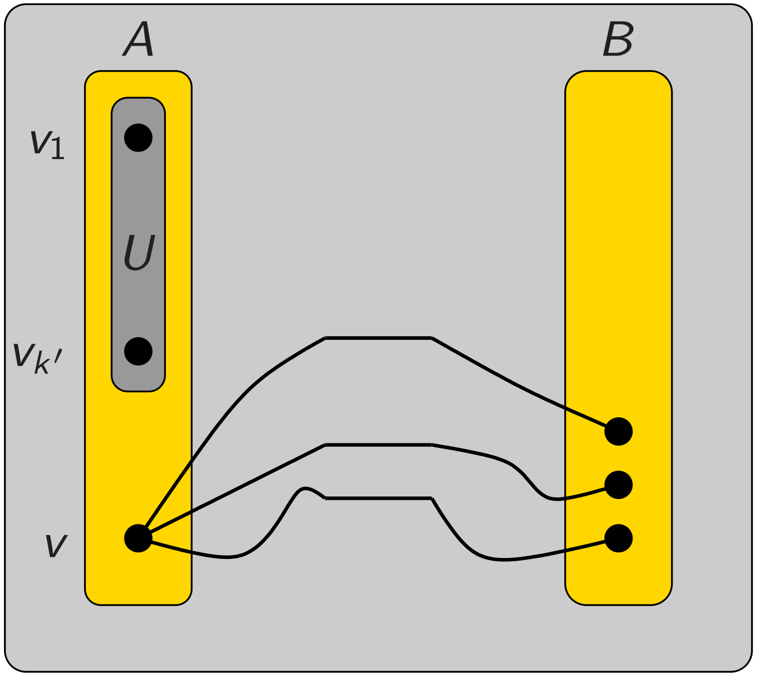
Remember: D contains every important (v_i, B)-separator of size $\leq kd$.

Consider a spider S with center v. As there are no $k'+1$ spiders with centers $U \cup v$, there is a $(U \cup v, B)$-separator C with $|C| < (k'+1)d$.



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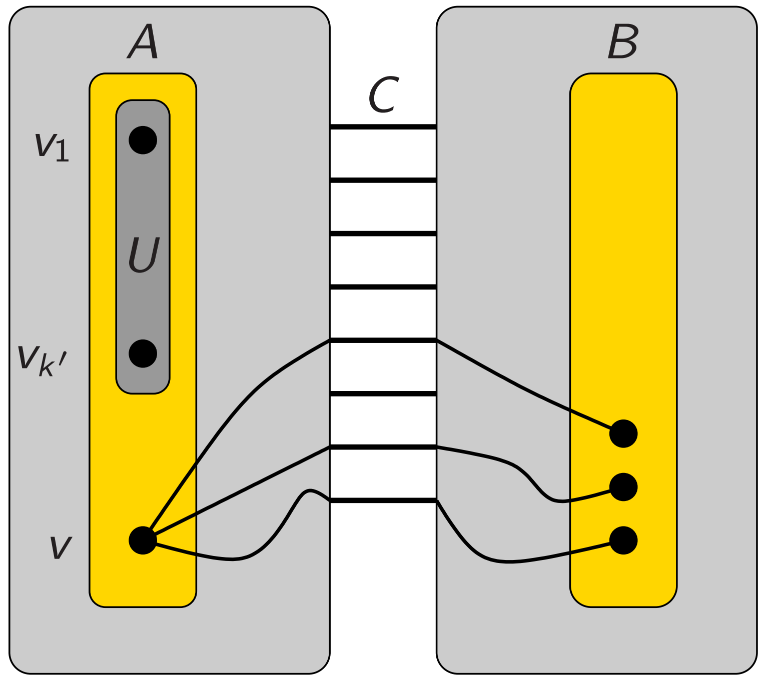

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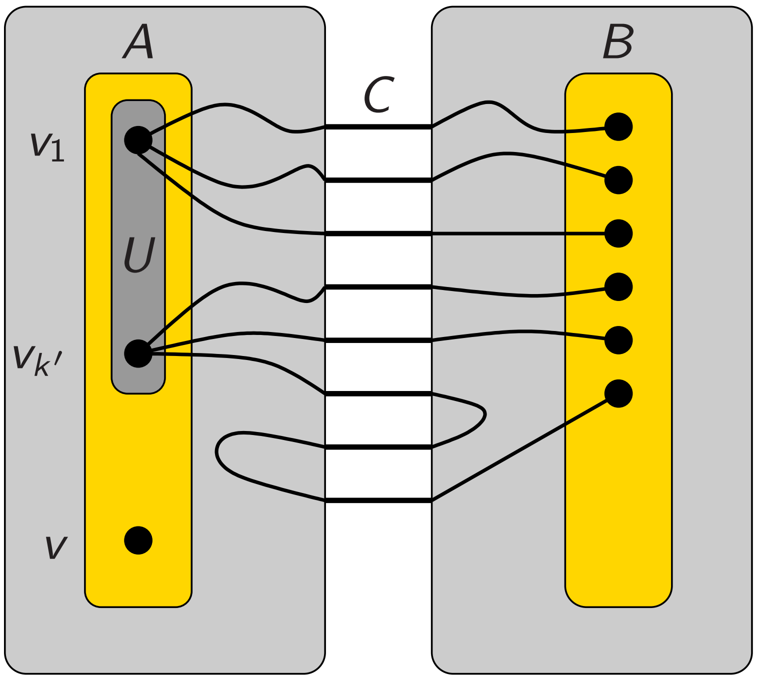

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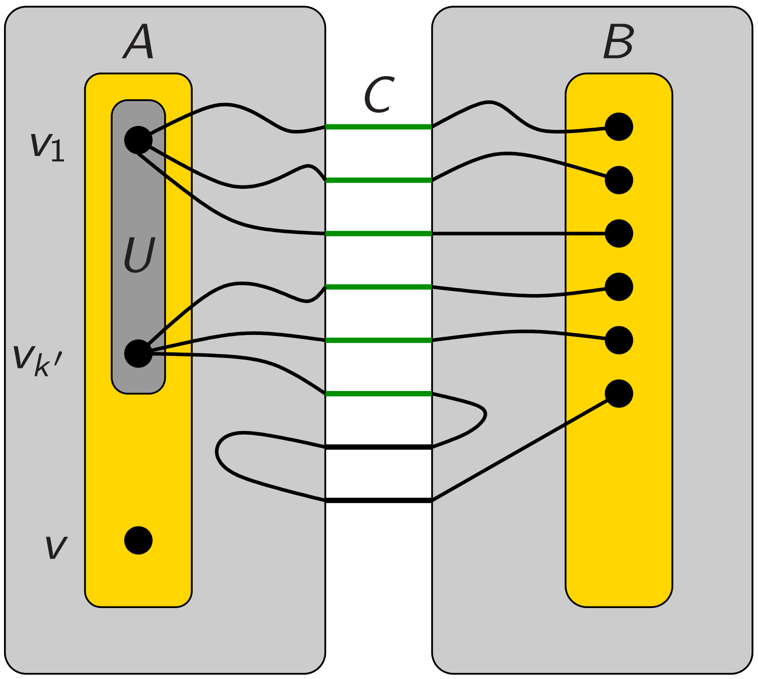
An edge of C is green if it is the first edge in C of any of the paths of the k' spiders

$\Rightarrow$ there are $k'd$ green edges.

$\Rightarrow$ there are $\leq d-1$ non-green edges.



Spiders

Remember: D contains every important (v_i, B)-separator of size $\leq kd$.

Consider a spider S with center v. As there are no $k'+1$ spiders with centers $U \cup v$, there is a $(U \cup v, B)$-separator C with $|C| < (k'+1)d$.

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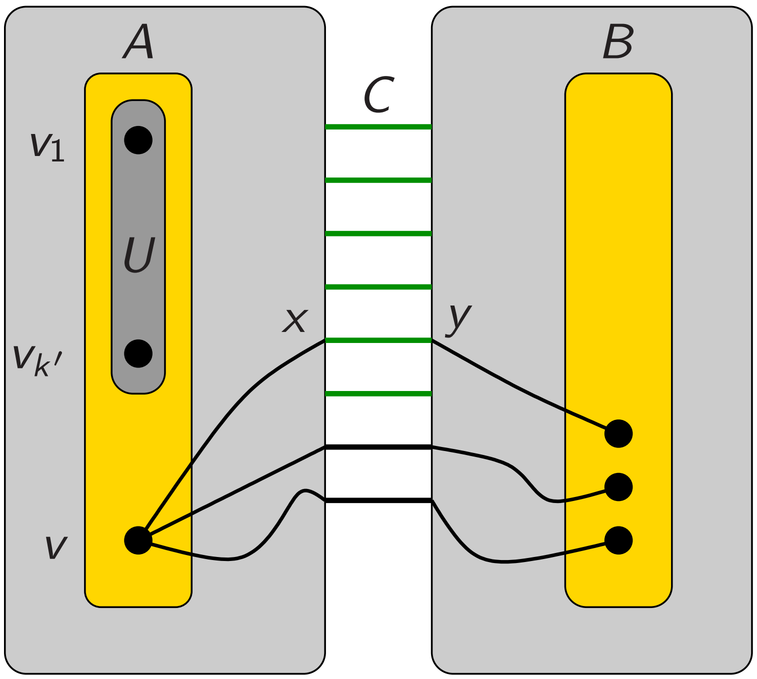

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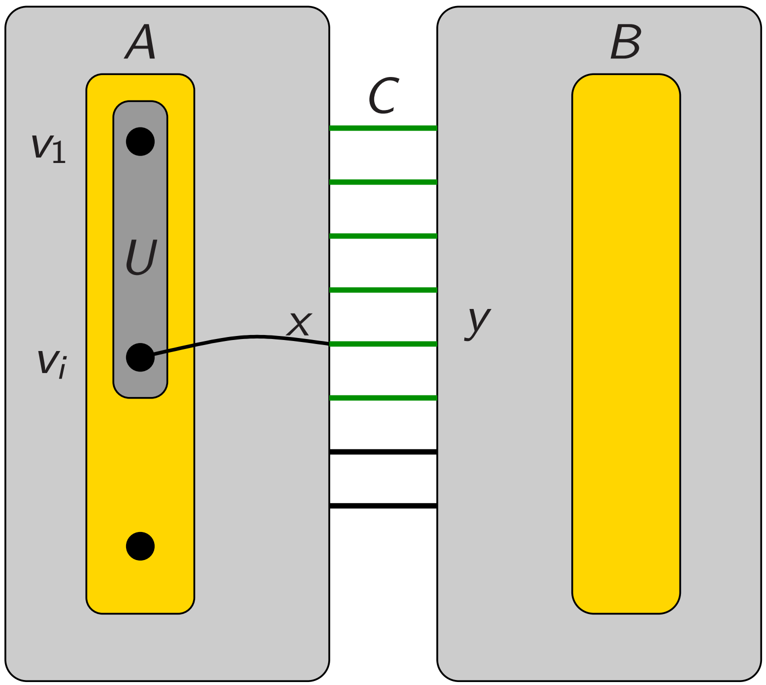

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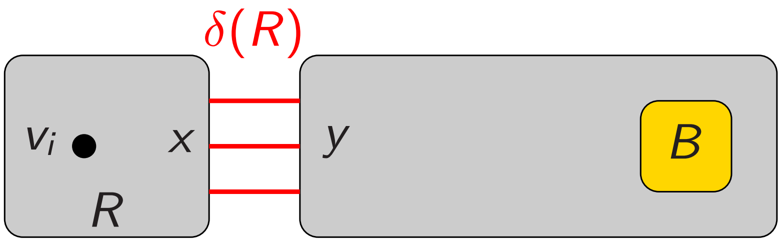

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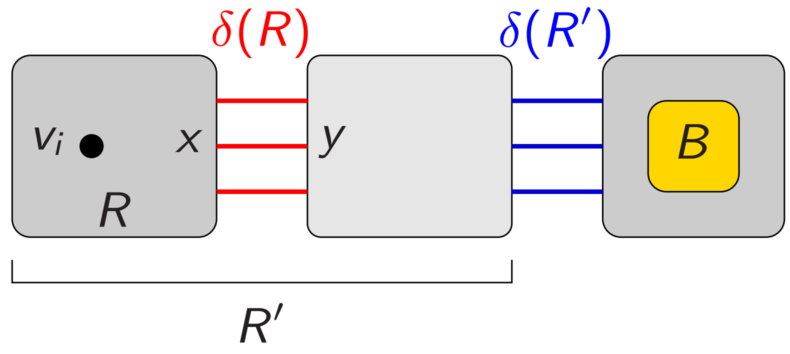
$\delta(R)$ is a (v_i, B)-separator of size $< kd$
$\Rightarrow$ D contains a separator $\delta(R')$ with $R \subseteq R'$.

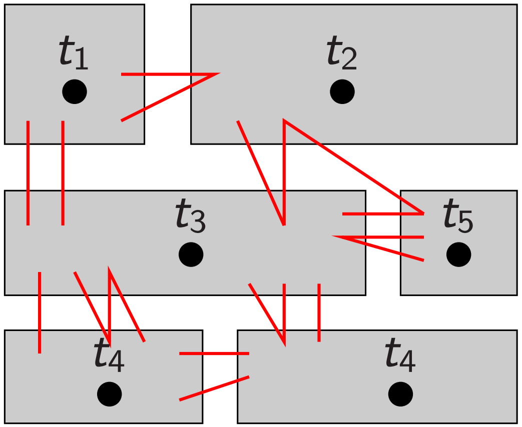


$x \in R' \Rightarrow \delta(R')$ separates x and B
$\Rightarrow D \supseteq \delta(R')$ intersects the spider S.

MULTIWAY CUT

Definition: A **multiway cut** of a set of terminals T is a set S of edges such that each component of $G \setminus S$ contains at most one vertex of T.

MULTIWAY CUT	
Input:	Graph G, set T of vertices, integer k
Find:	A **multiway cut** S of at most k edges.



Polynomial for $|T| = 2$, but NP-hard for any fixed $|T| \geq 3$ [Dalhaus et al. 1994].

Trivial to solve in polynomial time for fixed k (in time $n^{O(k)}$).

MULTIWAY CUT

Central notion of parameterized complexity:

> **Definition:** A problem is **fixed-parameter tractable (FPT)** parameterized by k if it can be solved in time $f(k) \cdot n^{O(1)}$ for some function $f(k)$ depending only on k.

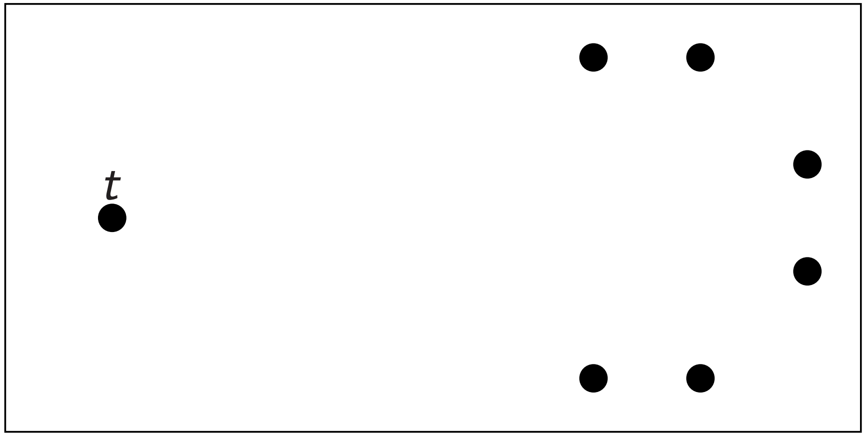
FPT means that the k can be removed from the exponent of n and the combinatorial explosion can be restricted to k.

If $f(k)$ is e.g., 1.2^k, then this can be actually an efficient algorithm!

Theorem: MULTIWAY CUT can be solved in time $4^k \cdot n^{O(1)}$, i.e., it is fixed-parameter tractable (FPT) parameterized by k.

MULTIWAY CUT

Intuition: Consider a $t \in T$. A subset of the solution S is a $(t, T \setminus t)$-separator.



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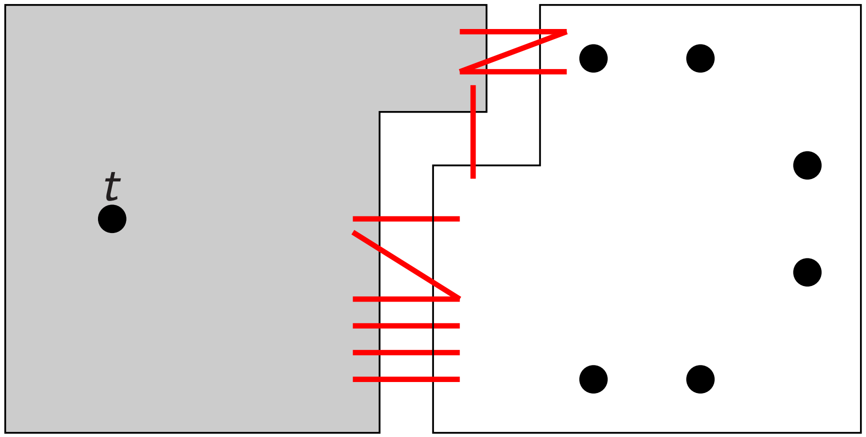

There are many such separators.

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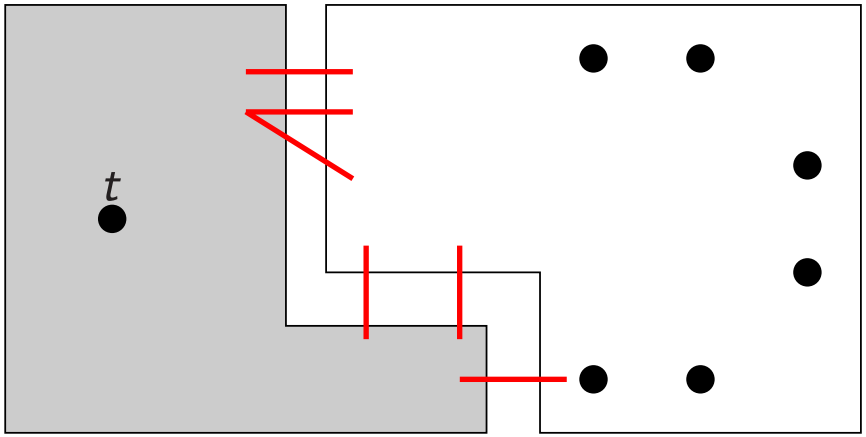

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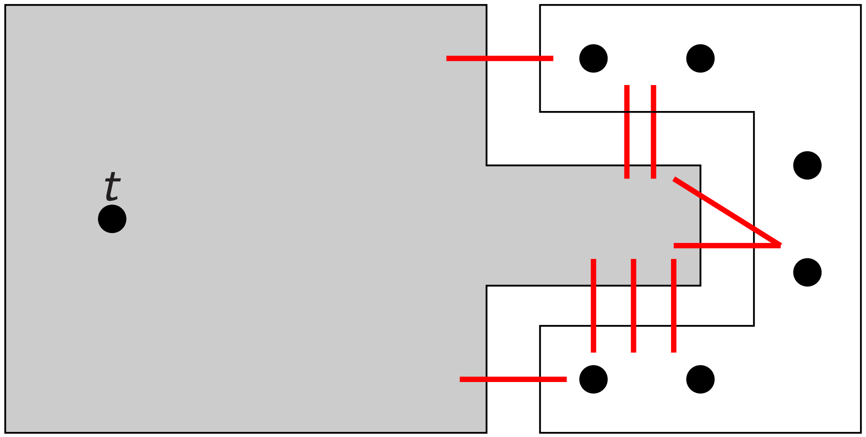

There are many such separators.

But a separator farther from t and closer to $T \setminus t$ seems to be more useful.

MULTIWAY CUT *and important separators*

Pushing Lemma: Let $t \in T$. The MULTIWAY CUT problem has a solution S that contains an important $(t, T \setminus t)$-separator.

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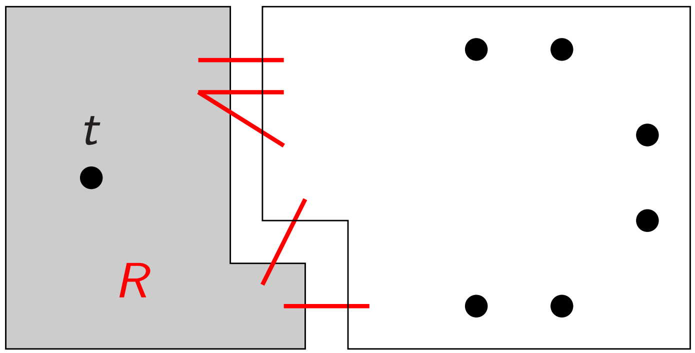

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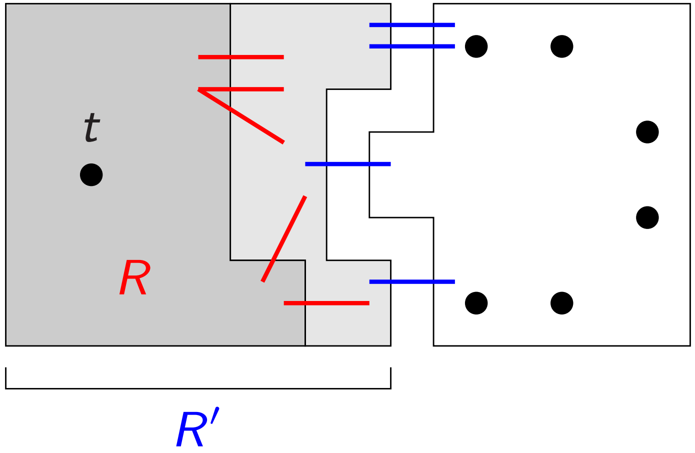

If $\delta(R)$ is not important, then there is an important separator $\delta(R')$ with $R \subset R'$ and $|\delta(R')| \leq |\delta(R)|$. Replace S with $S' := (S \setminus \delta(R)) \cup \delta(R') \Rightarrow |S'| \leq |S|$

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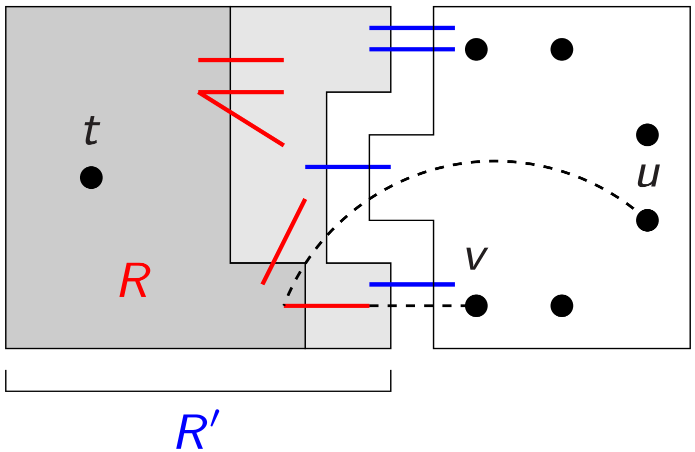

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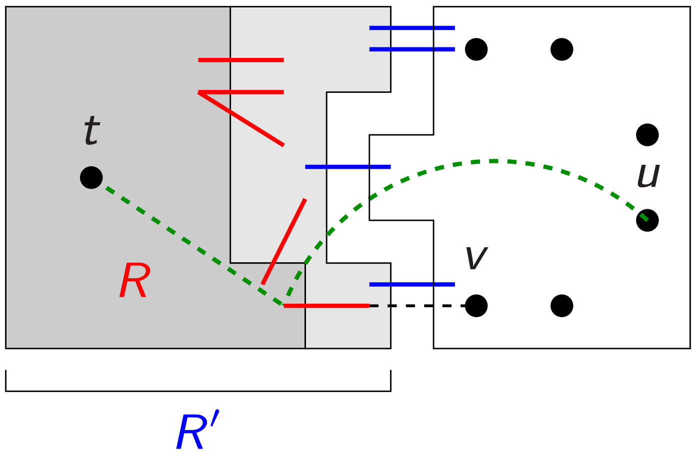

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Algorithm for MULTIWAY CUT

1. If every vertex of T is in a different component, then we are done.
2. Let $t \in T$ be a vertex with that is not separated from every $T \setminus t$.
3. Branch on a choice of an important $(t, T \setminus t)$ separator S of size at most k.
4. Set $G := G \setminus S$ and $k := k - |S|$.
5. Go to step 1.

We branch into at most 4^k directions at most k times.

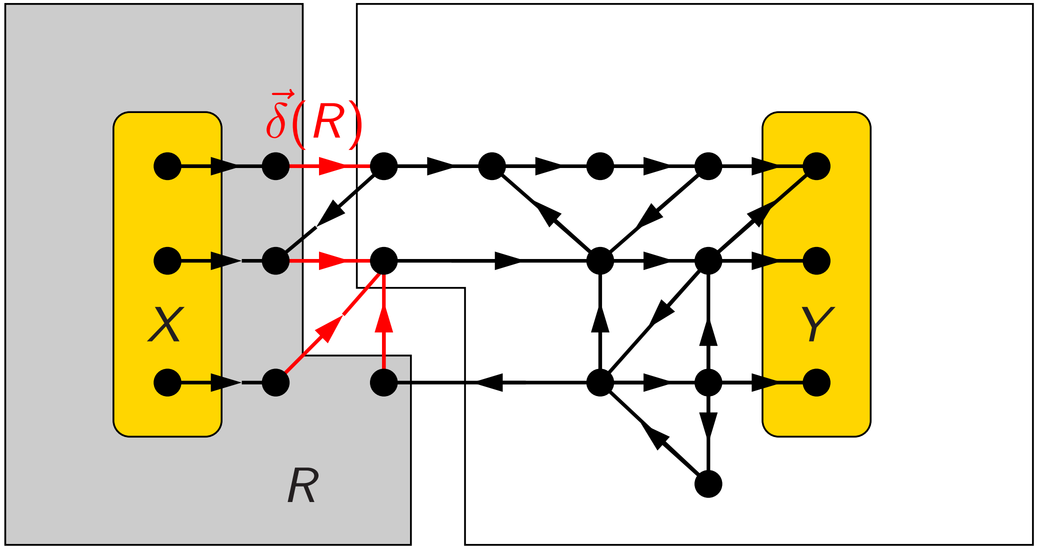
(Better analysis gives 4^k bound on the size of the search tree.)

Directed graphs

Definition: $\vec{\delta}(R)$ is the set of edges leaving R.

Observation: Every inclusionwise-minimal directed (X, Y)-separator S can be expressed as $S = \vec{\delta}(R)$ for some $X \subseteq R$ and $R \cap Y = \emptyset$.

Definition: An (X, Y)-separator $\vec{\delta}(R)$ is **important** if there is no (X, Y)-separator $\vec{\delta}(R')$ with $R \subset R'$ and $|\vec{\delta}(R')| \leq |\vec{\delta}(R)|$.



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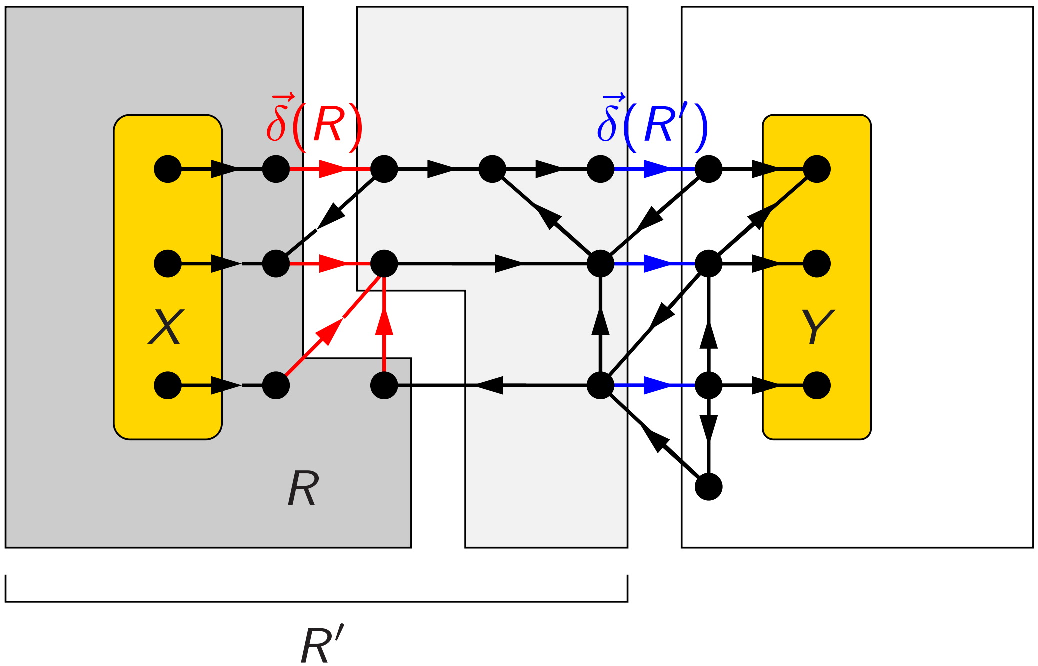

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The proof for the undirected case goes through for the directed case:

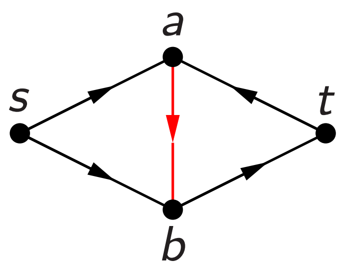
Theorem: There are at most 4^k important directed (X, Y)-separators of size at most k.

Directed Multiway Cut

It is open [?] whether Directed Multiway Cut is FPT or not. The approach for undirected graphs does not work: the pushing lemma is not true.

Pushing Lemma: [for undirected graphs] Let $t \in T$. The Multiway Cut problem has a solution S that contains an important $(t, T \setminus t)$-separator.

Directed counterexample:



Unique solution with $k = 1$ edges, but it is not an important separator (boundary of $\{s, a\}$, but the boundary of $\{s, a, b\}$ is of the same size).

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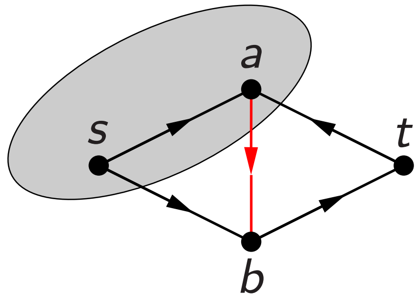

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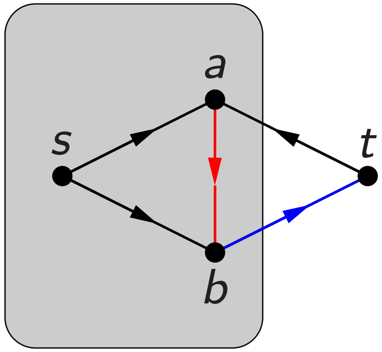

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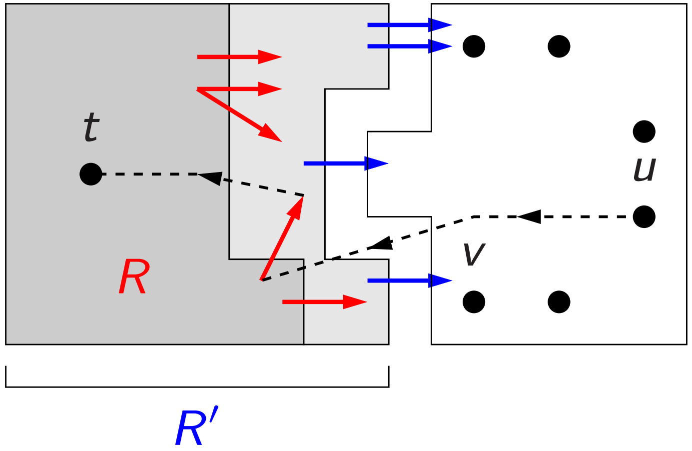
Problem in the undirected proof:



Replacing R by R' cannot create a $t \to u$ path, but can create a $u \to t$ path.

SKEW MULTICUT

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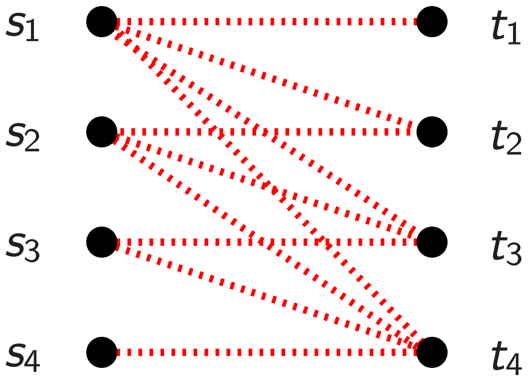
Input: Graph G, pairs $(s_1, t_1), \ldots, (s_\ell, t_\ell)$, integer k

Find: A set S of k directed edges such that $G \setminus S$ contains no $s_i \to t_j$ path for any $i \leq j$.



Pushing Lemma: SKEW MULTCUT problem has a solution S that contains an important $(s_1, \{t_1, \ldots, t_\ell\})$-separator.

Theorem: [Chen et al. 2008] SKEW MULTICUT can be solved in time $4^k \cdot n^{O(1)}$.

Directed Feedback Vertex Set

Directed Feedback Vertex/Edge Set	
Input:	Directed graph G, integer k
Find:	A set S of k vertices/edges such that $G \setminus S$ is acyclic.

Note: Edge and vertex versions are equivalent, we will consider the edge version here.

Theorem: [Chen et al. 2008] Directed Feedback Edge Set is FPT parameterized by k.

Solution uses the technique of **iterative compression** introduced by [Reed et at. 2004].

The compression problem

DIRECTED FEEDBACK EDGE SET COMPRESSION

Input:	Directed graph G, integer k, a set of $k+1$ edges such that $G \setminus S'$ is acyclic,
Find:	A set S of k edges such that $G \setminus S$ is acyclic.

Easier than the original problem, as the extra input S' gives us useful structural information about G.

Lemma: The compression problem is FPT parameterized by k.

The compression problem

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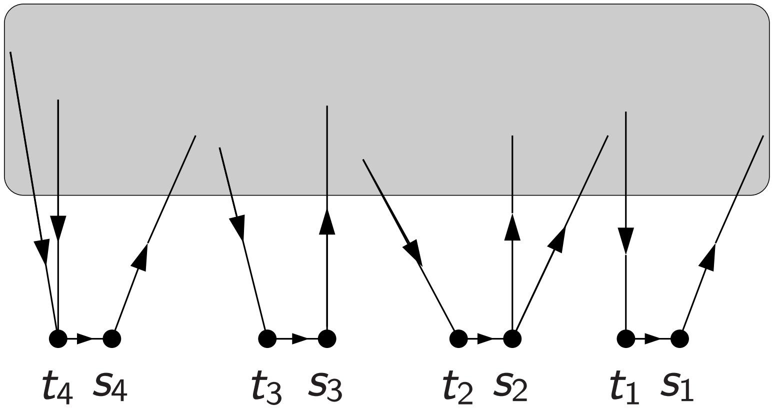
Proof: Let $S' = \{\overrightarrow{t_1 s_1}, \ldots, \overrightarrow{t_{k+1} s_{k+1}}\}$.



- By guessing and removing $S \cap S'$, we can assume that S and S' are disjoint [2^{k+1} possibilities].
- By guessing the order of $\{s_1, \ldots, s_{k+1}\}$ in the acyclic ordering of $G \setminus S$, we can assume that $s_{k+1} < s_k < \cdots < s_1$ in $G \setminus S$ [$(k+1)!$ possibilities].

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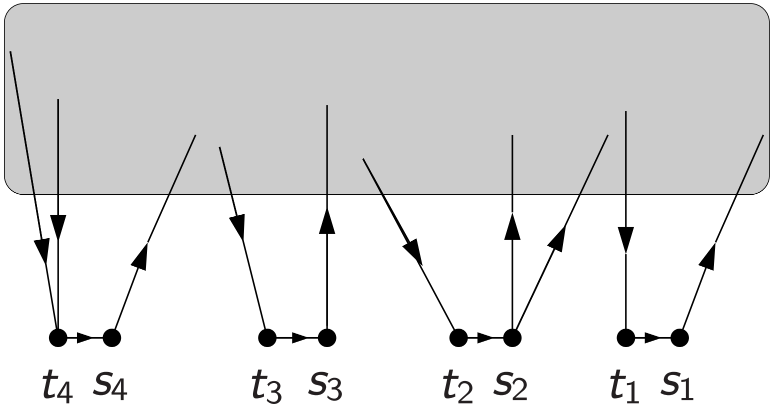

Claim: Suppose that $S' \cap S = \emptyset$.

$G \setminus S$ is acyclic and has an ordering with $s_{k+1} < s_k < \cdots < s_1$

$\Updownarrow$

S covers every $s_i \to t_j$ path for every $i \le j$

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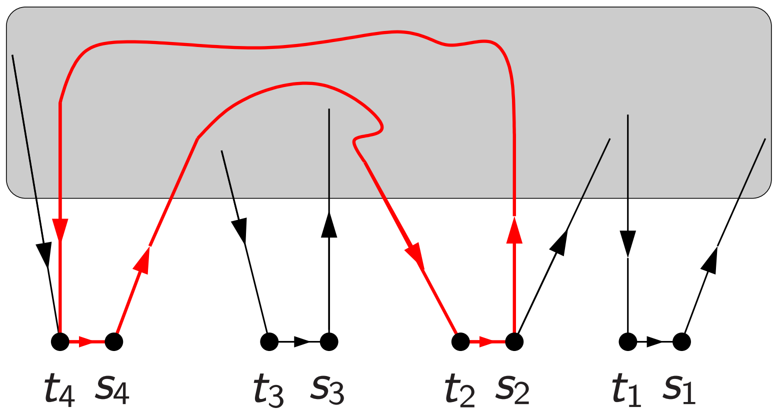

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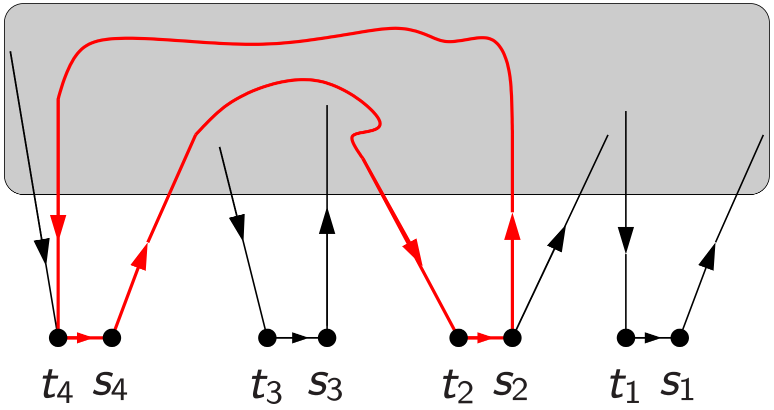

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$$S \text{ covers every } s_i \rightarrow t_j \text{ path for every } i \leq j$$

$\Rightarrow$ We can solve the compression problem by $2^{k+1} \cdot (k+1)!$ applications of SKEW MULTICUT.

Iterative compression

We have given a $f(k)n^{O(1)}$ algorithm for the following problem:

DIRECTED FEEDBACK EDGE SET COMPRESSION	
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Nice, but how do we get a solution S' of size $k+1$?

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We get it for free!

Useful trick: **iterative compression** (introduced by [Reed, Smith, Vetta 2004] for BIPARTITE DELETION).

Iterative compression

Let $e_1, \dots, e_m$ be the edges of G and let G_i be the subgraph containing only the first i edges (and all vertices).

For every $i = 1, \dots, m$, we find a set S_i of k edges such that $G_i \setminus S_i$ is acyclic.

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- For $i = k$, we have the trivial solution $S_i = \{e_1, \dots, e_k\}$.
- Suppose we have a solution S_i for G_i. Then $S_i \cup \{e_{i+1}\}$ is a solution of size $k+1$ in the graph G_{i+1}
- Use the compression algorithm for G_{i+1} with the solution $S_i \cup \{e_{i+1}\}$.
  - If the there is no solution of size k for G_{i+1}, then we can stop.
  - Otherwise the compression algorithm gives a solution S_{i+1} of size k for G_{i+1}.

We call the compression algorithm m times, everything else is polynomial.
$\Rightarrow$ DIRECTED FEEDBACK EDGE SET is FPT.

Conclusions

- A simple (but essentially tight) bound on the number of important separators.
- Combinatorial result: Erdős-Pósa property for spiders. Is the function $f(k,d)$ really exponential?
- Algorithmic results: FPT algorithms for
  - MULTIWAY CUT in undirected graphs,
  - SKEW MULTICUT in directed graphs, and
  - DIRECTED FEEDBACK VERTEX/EDGE SET.