Important separators and parameterized algorithms

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Overview

Main message

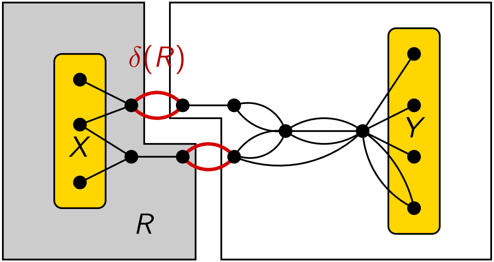
Small separators in graphs have interesting extremal properties that can be exploited in combinatorial and algorithmic results.

- Bounding the number of “important” cuts.
- Edge/vertex versions, directed/undirected versions.
- Algorithmic applications: FPT algorithm for
  - Multiway cut,
  - Directed Feedback Vertex Set, and
  - (p, q)-Clustering.

Important cuts

Definition: $\delta(R)$ is the set of edges with exactly one endpoint in R.
Definition: A set S of edges is a **minimal (X,Y)-cut** if there is no $X-Y$ path in $G \setminus S$ and no proper subset of S breaks every $X-Y$ path.

Observation: Every minimal (X,Y)-cut S can be expressed as $S = \delta(R)$ for some $X \subseteq R$ and $R \cap Y = \emptyset$.



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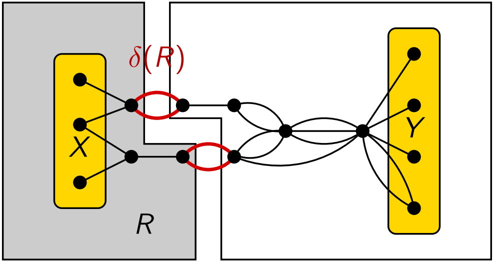
Definition: A minimal (X, Y)-cut $\delta(R)$ is **important** if there is no (X, Y)-cut $\delta(R')$ with $R \subset R'$ and $|\delta(R')| \leq |\delta(R)|$.

Note: Can be checked in polynomial time if a cut is important.



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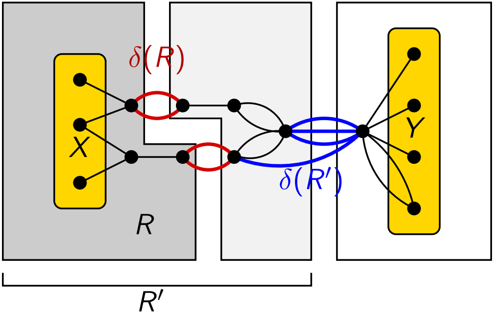

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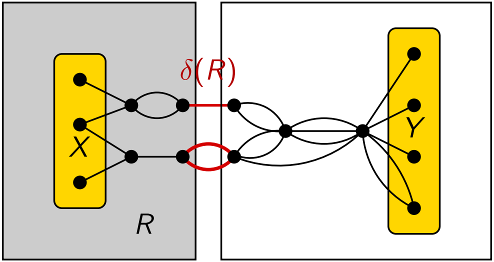
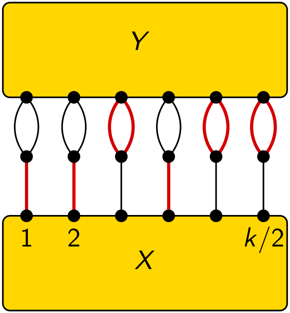


Important cuts

The number of important cuts can be exponentially large.

Example:



This graph has $2^{k/2}$ important (X, Y)-cuts of size at most k.

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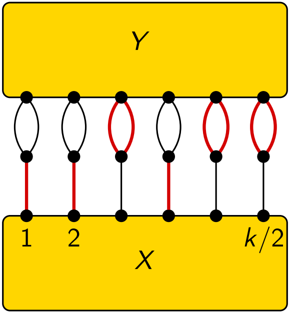

This graph has $2^{k/2}$ important (X, Y)-cuts of size at most k.

Theorem

There are at most 4^k important (X, Y)-cuts of size at most k.

(Proof is implicit in [Chen, Liu, Lu 2007], worse bound in [M. 2004].)

Submodularity

Fact: The function δ is **submodular:** for arbitrary sets A, B,

$$|\delta(A)| \quad + \quad |\delta(B)| \quad \geq \quad |\delta(A \cap B)| \quad + \quad |\delta(A \cup B)|$$

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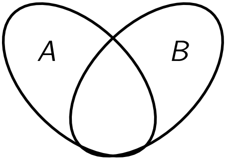
Proof: Determine separately the contribution of the different types of edges.



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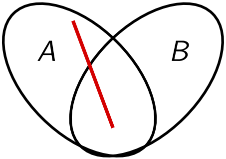

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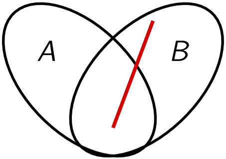

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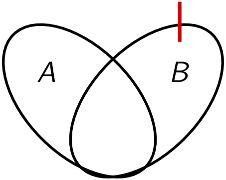

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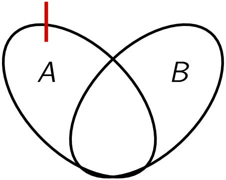

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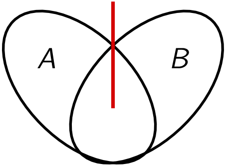

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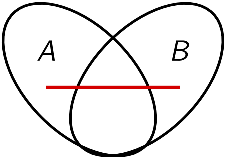

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Lemma

Let λ be the minimum (X, Y)-cut size. There is a unique maximal $R_{\max} \supseteq X$ such that $\delta(R_{\max})$ is an (X, Y)-cut of size λ.

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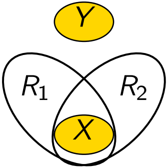
Proof: Let $R_1, R_2 \supseteq X$ be two sets such that $\delta(R_1), \delta(R_2)$ are (X, Y)-cuts of size λ.

$$\underbrace{|\delta(R_1)|}_{\lambda} + \underbrace{|\delta(R_2)|}_{\lambda} \geq \underbrace{|\delta(R_1 \cap R_2)|}_{\geq \lambda} + |\delta(R_1 \cup R_2)|$$

$$\Rightarrow |\delta(R_1 \cup R_2)| \leq \lambda$$



Note: Analogous result holds for a unique minimal $R_{\min}$.

Important cuts

Theorem

There are at most 4^k important (X, Y)-cuts of size at most k.

Proof: Let λ be the minimum (X, Y)-cut size and let $\delta(R_{\max})$ be the unique important cut of size λ such that $R_{\max}$ is maximal.

(1) We show that $R_{\max} \subseteq R$ for every important cut $\delta(R)$.

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By the submodularity of δ:

$$\underbrace{|\delta(R_{\max})|}_{\lambda} + |\delta(R)| \geq \underbrace{|\delta(R_{\max} \cap R)|}_{\geq \lambda} + |\delta(R_{\max} \cup R)|$$

$$\Downarrow$$

$$|\delta(R_{\max} \cup R)| \leq |\delta(R)|$$

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If $R \neq R_{\max} \cup R$, then $\delta(R)$ is not important.

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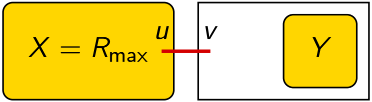
If $R \neq R_{\max} \cup R$, then $\delta(R)$ is not important.

Thus the important (X, Y)- and $(R_{\max}, Y)$-cuts are the same.
$\Rightarrow$ We can assume $X = R_{\max}$.

Important cuts

(2) Search tree algorithm for enumerating all these cuts:

An (arbitrary) edge uv leaving $X = R_{\max}$ is either in the cut or not.



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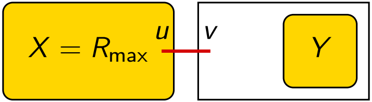

Branch 1: If $uv \in S$, then $S \setminus uv$ is an important (X, Y)-cut of size at most $k - 1$ in $G \setminus uv$.

Branch 2: If $uv \notin S$, then S is an important $(X \cup v, Y)$-cut of size at most k in G.

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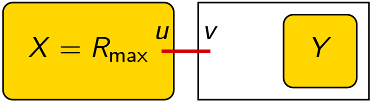

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$\Rightarrow$ k decreases by one, λ decreases by at most 1.

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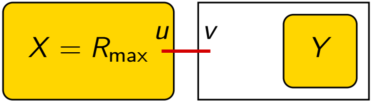

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The measure $2k - \lambda$ decreases in each step.

$\Rightarrow$ Height of the search tree $\leq 2k$

$\Rightarrow$ $\leq 2^{2k} = 4^k$ important cuts of size at most k.

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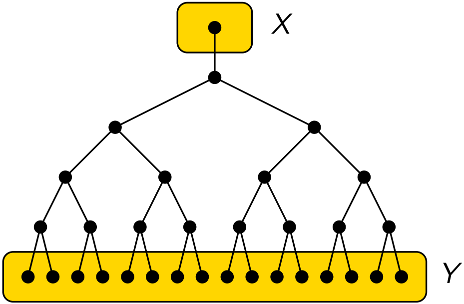
There are at most 4^k important (X, Y)-cuts of size at most k.

Example: The bound 4^k is essentially tight.



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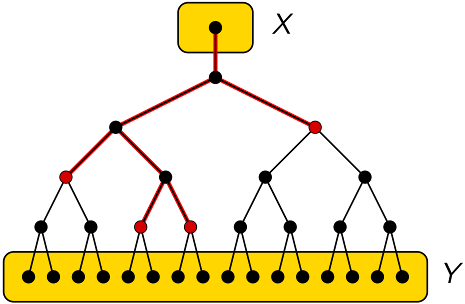

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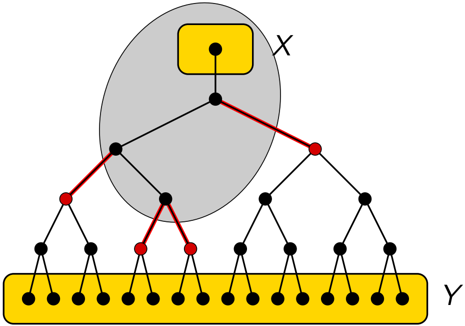

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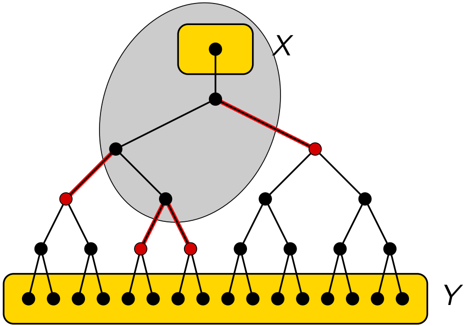


Any subtree with k leaves gives an important (X, Y)-cut of size k.
The number of subtrees with k leaves is the Catalan number

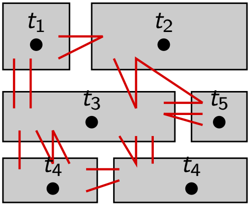
$$C_{k-1} = \frac{1}{k}\binom{2k-2}{k-1} \geq 4^k/\mathrm{poly}(k).$$

Multiway Cut

Definition: A **multiway cut** of a set of terminals T is a set S of edges such that each component of $G \setminus S$ contains at most one vertex of T.

Multiway Cut

Input:	Graph G, set T of vertices, integer k
Find:	A **multiway cut** S of at most k edges.



Polynomial for $|T| = 2$, but NP-hard for any fixed $|T| \geq 3$ [Dalhaus et al. 1994].

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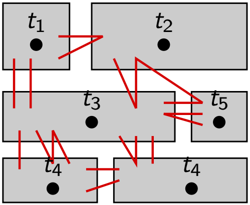


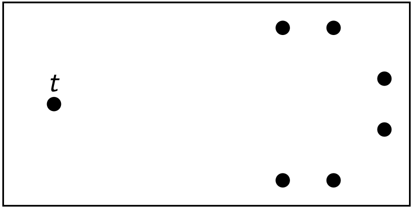
Trivial to solve in polynomial time for fixed k (in time $n^{O(k)}$).

Theorem

Multiway cut can be solved in time $4^k \cdot n^{O(1)}$, i.e., it is fixed-parameter tractable (FPT) parameterized by the size k of the solution.

Multiway Cut

Intuition: Consider a $t \in T$. A subset of the solution S is a $(t, T \setminus t)$-cut.



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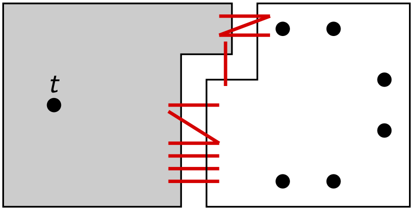

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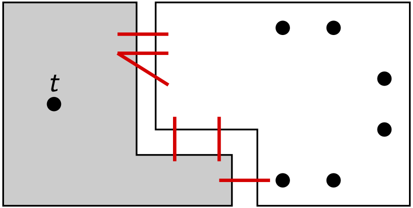

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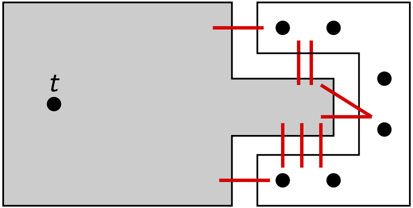

There are many such cuts.

But a cut farther from t and closer to $T \setminus t$ seems to be more useful.

Multiway Cut and important cuts

Pushing Lemma

Let $t \in T$. The Multiway Cut problem has a solution S that contains an important $(t, T \setminus t)$-cut.

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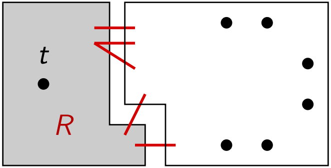

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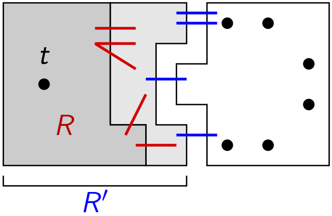

$\delta(R)$ is not important, then there is an important cut $\delta(R')$ with $R \subset R'$ and $|\delta(R')| \le |\delta(R)|$. Replace S with $S' := (S \setminus \delta(R)) \cup \delta(R') \Rightarrow |S'| \le |S|$

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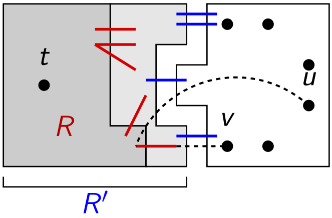

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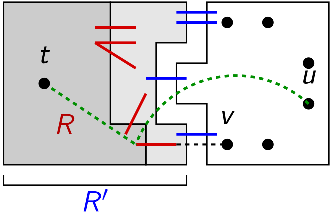
S' is a multiway cut: (1) There is no t-u path in $G \setminus S'$ and (2) a u-v path in $G \setminus S'$ implies a t-u path, a contradiction.

MULTIWAY CUT and important cuts

Pushing Lemma

Let $t \in T$. The MULTIWAY CUT problem has a solution S that contains an important $(t, T \setminus t)$-cut.

Proof: Let R be the vertices reachable from t in $G \setminus S$ for a solution S.



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Algorithm for MULTIWAY CUT

1. If every vertex of T is in a different component, then we are done.
2. Let $t \in T$ be a vertex that is not separated from every $T \setminus t$.
3. Branch on a choice of an important $(t, T \setminus t)$ cut S of size at most k.
4. Set $G := G \setminus S$ and $k := k - |S|$.
5. Go to step 1.

We branch into at most 4^k directions at most k times.

(Better analysis gives 4^k bound on the size of the search tree.)

Multicut

Multicut

Input: Graph G, pairs (s_1, t_1), ..., (s_ℓ, t_ℓ), integer k

Find: A set S of edges such that $G \setminus S$ has no s_i-t_i path for any i.

Theorem

Multicut can be solved in time $f(k, \ell) \cdot n^{O(1)}$ (FPT parameterized by combined parameters k and ℓ).

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Proof: The solution partitions $\{s_1, t_1, \dots, s_\ell, t_\ell\}$ into components. Guess this partition, contract the vertices in a class, and solve Multiway Cut.

Theorem [Bousquet, Daligault, Thomassé 2011] [M., Razgon 2011]

Multicut is FPT parameterized by the size k of the solution.

Directed graphs

Definition: $\vec{\delta}(R)$ is the set of edges leaving R.

Observation: Every inclusionwise-minimal directed (X, Y)-cut S can be expressed as $S = \vec{\delta}(R)$ for some $X \subseteq R$ and $R \cap Y = \emptyset$.

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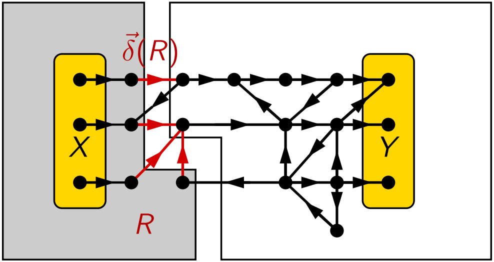

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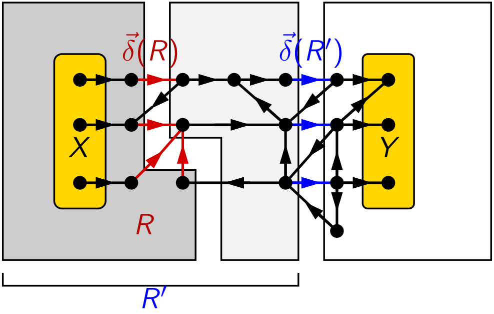

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The proof for the undirected case goes through for the directed case:

Theorem

There are at most 4^k important directed (X,Y)-cuts of size at most k.

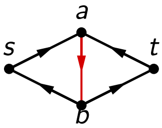
Directed Multiway Cut

The undirected approach does not work: the pushing lemma is not true.

Pushing Lemma (for undirected graphs)

Let $t \in T$. The Multiway Cut problem has a solution S that contains an important $(t, T \setminus t)$-cut.

Directed counterexample:



Unique solution with $k = 1$ edges, but it is not an important cut (boundary of $\{s, a\}$, but the boundary of $\{s, a, b\}$ has same size).

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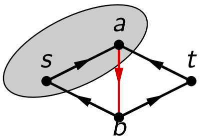

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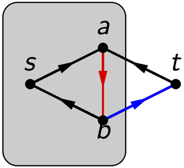

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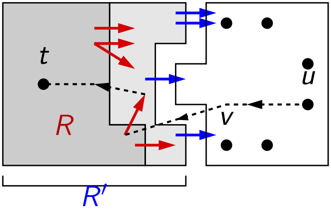
Problem in the undirected proof:



Replacing R by R' cannot create a $t \to u$ path, but can create a $u \to t$ path.

Directed Multiway Cut

The undirected approach does not work: the pushing lemma is not true.

Pushing Lemma (for undirected graphs)

Let $t \in T$. The Multiway Cut problem has a solution S that contains an important $(t, T \setminus t)$-cut.

Using additional techniques, one can show:

Theorem [Chitnis, Hajiaghayi, M. 2011]

Directed Multiway Cut is FPT parameterized by the size k of the solution.

Directed Multicut

Directed Multicut

Input: Graph G, pairs (s_1, t_1), ..., (s_ℓ, t_ℓ), integer k

Find: A set S of edges such that $G \setminus S$ has no $s_i \to t_i$ path for any i.

Theorem [M. and Razgon 2011]

Directed Multicut is W[1]-hard parameterized by k.

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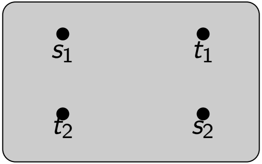

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Directed Multicut	
Input:	Graph G, pairs $(s_1, t_1), \ldots, (s_\ell, t_\ell)$, integer k
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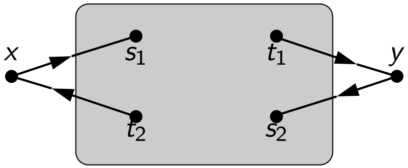

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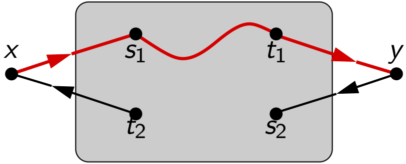

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Theorem [M. and Razgon 2011]

Directed Multicut is W[1]-hard parameterized by k.

Corollary

Directed Multicut with $\ell = 2$ is FPT parameterized by the size k of the solution.

?

Open: Is Directed Multicut with $\ell = 3$ FPT?

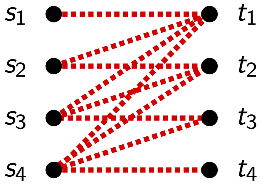
Open: Is there an $f(k, \ell) \cdot n^{O(1)}$ algorithm for Directed Multicut?

Skew Multicut

Skew Multicut

Input: Graph G, pairs $(s_1, t_1), \ldots, (s_\ell, t_\ell)$, integer k

Find: A set S of k directed edges such that $G \setminus S$ contains no $s_i \to t_j$ path for any $i \geq j$.



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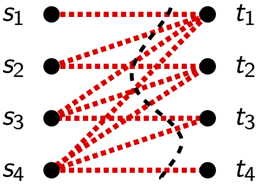

Pushing Lemma

Skew Multcut problem has a solution S that contains an important $(s_\ell, \{t_1, \ldots, t_\ell\})$-cut.

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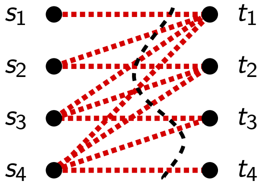

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Skew Multcut problem has a solution S that contains an important $(s_\ell, \{t_1, \ldots, t_\ell\})$-cut.

Theorem [Chen, Liu, Lu, O'Sullivan, Razgon 2008]

Skew Multicut can be solved in time $4^k \cdot n^{O(1)}$.

Directed Feedback Vertex Set

Directed Feedback Vertex/Edge Set	
Input:	Directed graph G, integer k
Find:	A set S of k vertices/edges such that $G \setminus S$ is acyclic.

Note: Edge and vertex versions are equivalent, we will consider the edge version here.

Theorem [Chen, Liu, Lu, O'Sullivan, Razgon 2008]

Directed Feedback Edge Set is FPT parameterized by the size k of the solution.

Solution uses the technique of **iterative compression** introduced by [Reed, Smith, Vetta 2004].

The compression problem

Directed Feedback Edge Set Compression	
Input:	Directed graph G, integer k, a set W of $k+1$ edges such that $G \setminus W$ is acyclic
Find:	A set S of k edges such that $G \setminus S$ is acyclic.

Easier than the original problem, as the extra input W gives us useful structural information about G.

Lemma

The compression problem is FPT parameterized by k.

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Directed Feedback Edge Set Compression	
Input:	Directed graph G, integer k, a set W of $k+1$ **vertices** such that $G \setminus W$ is acyclic
Find:	A set S of k edges such that $G \setminus S$ is acyclic.

Easier than the original problem, as the extra input W gives us useful structural information about G.

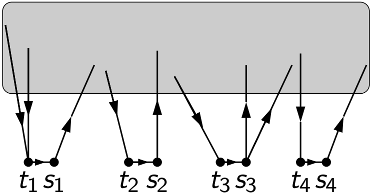
Lemma

The compression problem is FPT parameterized by k.

A useful trick for edge deletion problems: we define the compression problem in a way that a solution of $k+1$ **vertices** are given and we have to find a solution of k **edges**.

The compression problem

Proof: Let $W = \{w_1, \ldots, w_{k+1}\}$
Let us split each w_i into an edge $\overrightarrow{t_i s_i}$.



- By guessing the order of $\{w_1, \ldots, w_{k+1}\}$ in the acyclic ordering of $G \setminus S$, we can assume that $w_1 < w_2 < \cdots < w_{k+1}$ in $G \setminus S$ [$(k+1)!$ possibilities].

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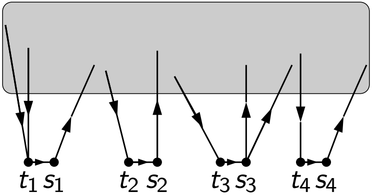

Claim:

$G \setminus S$ is acyclic and has an ordering with $w_1 < w_2 < \cdots < w_{k+1}$

$\Downarrow$

S covers every $s_i \to t_j$ path for every $i \geq j$

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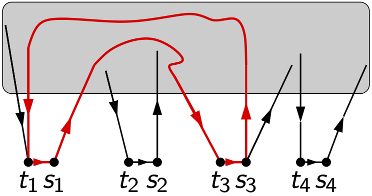

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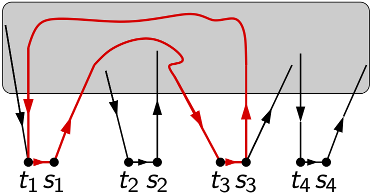

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$\Rightarrow$ We can solve the compression problem by $(k+1)!$ applications of Skew Multicut.

Iterative compression

We have given a $f(k)n^{O(1)}$ algorithm for the following problem:

Directed Feedback Edge Set Compression	
Input:	Directed graph G, integer k, a set W of $k+1$ vertices such that $G \setminus W$ is acyclic
Find:	A set S of k edges such that $G \setminus S$ is acyclic.

Nice, but how do we get a solution W of size $k+1$?

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Nice, but how do we get a solution W of size $k+1$?

We get it for free!

Powerful technique: **iterative compression** (introduced by [Reed, Smith, Vetta 2004] for BIPARTITE DELETION).

Iterative compression

Let v_1, ..., v_n be the edges of G and let G_i be the subgraph induced by $\{v_1, \ldots, v_i\}$.

For every $i = 1, \ldots, n$, we find a set S_i of at most k edges such that $G_i \setminus S_i$ is acyclic.

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- For $i = 1$, we have the trivial solution $S_i = \emptyset$.
- Suppose we have a solution S_i for G_i. Let W_i contain the head of each edge in S_i. Then $W_i \cup \{v_{i+1}\}$ is a set of at most $k + 1$ vertices whose removal makes G_{i+1} acyclic.
- Use the compression algorithm for G_{i+1} with the set $W_i \cup \{v_{i+1}\}$.
    - If there is no solution of size k for G_{i+1}, then we can stop.
    - Otherwise the compression algorithm gives a solution S_{i+1} of size k for G_{i+1}.

We call the compression algorithm n times, everything else is polynomial.

$\Rightarrow$ Directed Feedback Edge Set is FPT.

Outline

So far we have seen:

- Definition of important cuts.
- Combinatorial bound on the number of important cuts.
- Pushing argument: we can assume that the solution contains an important cut. Solves Multiway Cut, Skew Multiway Cut.
- Iterative compression reduces Directed Feedback Vertex Set to Skew Multiway Cut.

Next:

- Randomized sampling of important separators.

Randomized sampling of important cuts

A new technique used by several results:

- Multicut [M. and Razgon STOC 2011]
- Clustering problems [Lokshtanov and M. ICALP 2011]
- Directed Multiway Cut [Chitnis, Hajiaghayi, M. SODA 2012]
- Directed Multicut in DAGs [Kratsch, Pilipczuk, Pilipczuk, Wahlström ICALP 2012]
- Directed Subset Feedback Vertex Set [Chitnis, Cygan, Hajiaghayi, M. ICALP 2012]
- Parity Multiway Cut [Lokshtanov, Ramanujan ICALP 2012]
- List homomorphism removal problems [Chitnis, Egri, and M. ESA 2013]
- . . . more work in progress.

Clustering

We want to partition objects into clusters subject to certain requirements (typically: related objects are clustered together, bounds on the number or size of the clusters etc.)

(p, q)-CLUSTERING

Input: A graph G, integers p, q.

Find: A partition $(V_1, \dots, V_m)$ of $V(G)$ such that for every i

- $|V_i| \leq p$ and
- $\delta(V_i) \leq q$.

$\delta(V_i)$: number of edges leaving V_i.

Theorem [Lokshtanov and M. 2011]

(p, q)-CLUSTERING can be solved in time $2^{O(q)} \cdot n^{O(1)}$.

A sufficient and necessary condition

Good cluster: size at most p and at most q edges leaving it.

Necessary condition:

Every vertex is contained in a good cluster.

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Necessary condition:

Every vertex is contained in a good cluster.

But surprisingly, this is also a **sufficient condition!**

Lemma

Graph G has a (p, q)-clustering if and only if every vertex is in a good cluster.

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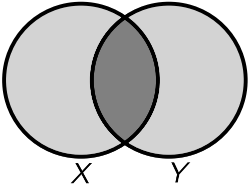
Proof: Find a collection of good clusters covering every vertex and having minimum total size. Suppose two clusters intersect.



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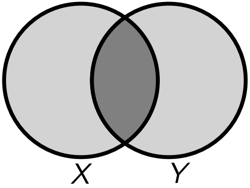

$$\delta(X) + \delta(Y) \geq \delta(X \setminus Y) + \delta(Y \setminus X)$$

(posimodularity)

$\Rightarrow$ either $\delta(X) \geq \delta(X \setminus Y)$ or $\delta(Y) \geq \delta(Y \setminus X)$ holds.

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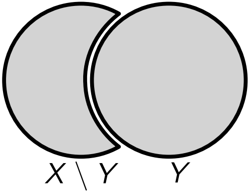

$$\delta(X) + \delta(Y) \geq \delta(X \setminus Y) + \delta(Y \setminus X)$$

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If $\delta(X) \geq \delta(X \setminus Y)$, replace X with $X \setminus Y$, strictly decreasing the total size of the clusters.

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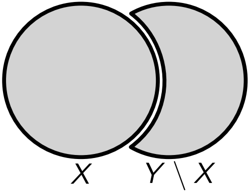

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(posimodularity)

If $\delta(Y) \geq \delta(Y \setminus X)$, replace Y with $Y \setminus X$, strictly decreasing the total size of the clusters. QED ■

Finding a good cluster

We have seen:

Lemma

Graph G has a (p, q)-clustering if and only if every vertex is in a good cluster.

All we have to do is to check if a given vertex v is in a good cluster. Trivial to do in time $n^{O(q)}$.

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We prove next:

Lemma

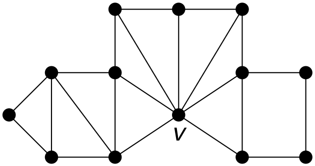
We can check in time $2^{O(q)} \cdot n^{O(1)}$ if v is in a good cluster.

Important sets

Definition

Fix a distinguished vertex v in a graph G. A set $X \subseteq V(G)$ is an **important set** if

- $v \notin X$,
- there is no set $X \subset X'$ with $v \notin X$ and $\delta(X') \le \delta(X)$.



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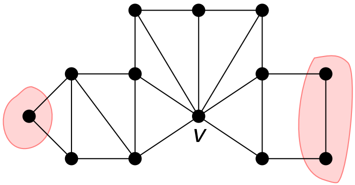

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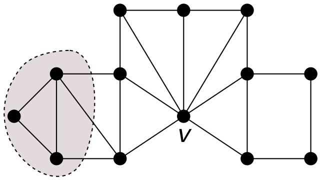

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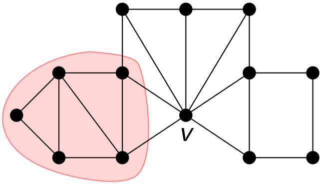

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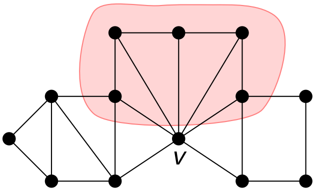

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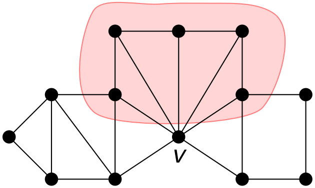
- there is no set $X \subset X'$ with $v \notin X$ and $\delta(X') \leq \delta(X)$.



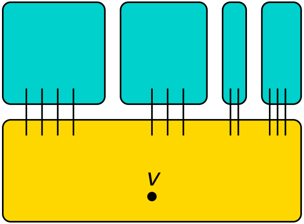
Observation: X is an important set if and only if $\delta(X)$ is an important (x, v)-cut for every $x \in X$.

Consequence: Every vertex is contained in at most 4^k important sets.

Pushing argument

Lemma

If C is a good cluster of minimum size containing v, then every component of $G \setminus C$ is an important set.



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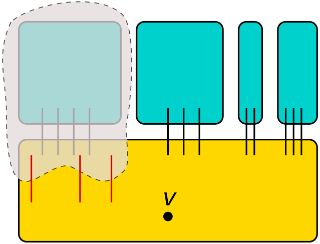

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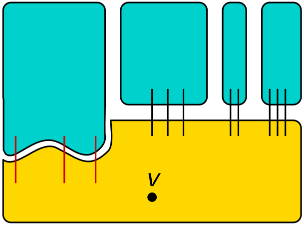

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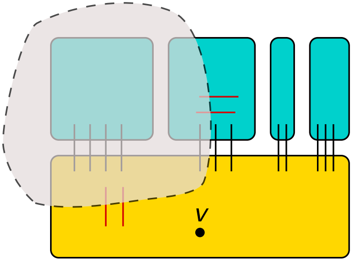

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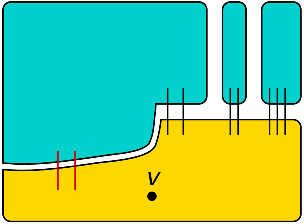

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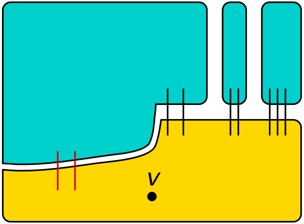


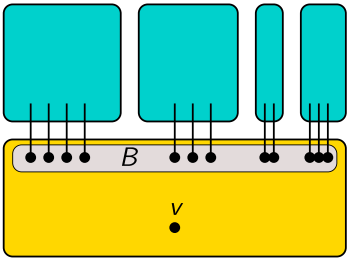
Thus C can be obtained by removing at most q important sets from $V(G)$ (but there are $n^{O(q)}$ possibilities, we cannot try all of them).

Random sampling

- Let $\mathcal{X}$ be the set of all important sets of boundary size at most q in G.
- Let $\mathcal{X}' \subseteq \mathcal{X}$ contain each set with probability $\frac{1}{2}$ independently.
- Let $Z = \bigcup_{X \in \mathcal{X}'} X$.
- Let B be the set of vertices in C with neighbors outside C.

Lemma

Let C be a good cluster of minimum size containing v. With probability $2^{-2^{O(q)}}$, Z covers $G \setminus C$ and is disjoint from B.



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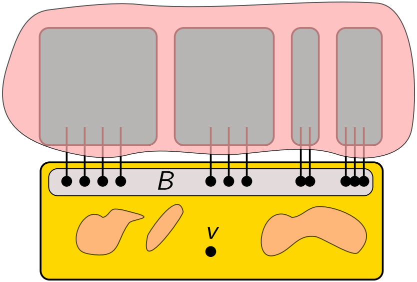

Random sampling

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Two events:

(E1) Z covers $G \setminus C$.
Each of the at most q components is an important set
$\Rightarrow$ all of them are selected by probability at least 2^{-q}.

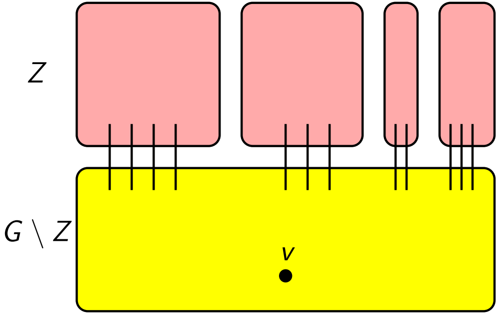
(E2) Z is disjoint from B.
Each vertex of B is in at most 4^q members of $\mathcal{X}$
$\Rightarrow$ all of them are selected by probability at least 2^{-q4^q}.

The two events are independent (involve different sets of $\mathcal{X}$), thus the claimed probability follows.

Finding good clusters

Let C be a good cluster of minimum size containing v and assume

- $G \setminus C$ is covered by Z, and
- Z is disjoint from B (hence no edge going out of C is contained in Z).



Where is the good cluster C in the figure?

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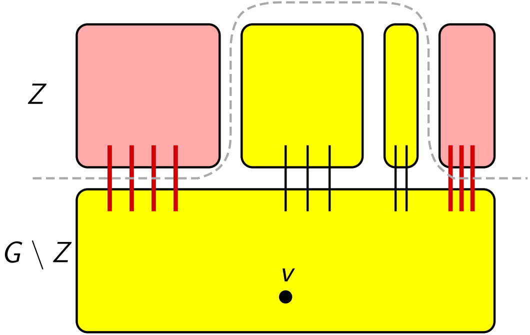

Where is the good cluster C in the figure?

Observe: Components of Z are either fully in the cluster or fully outside the cluster. What is this problem?

Finding good clusters

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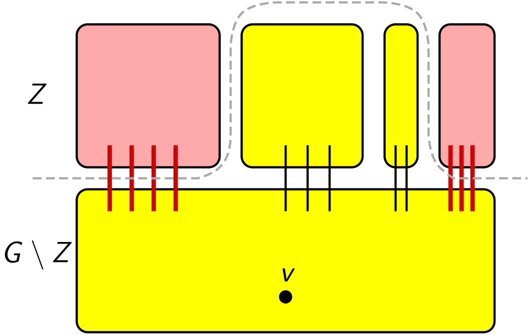
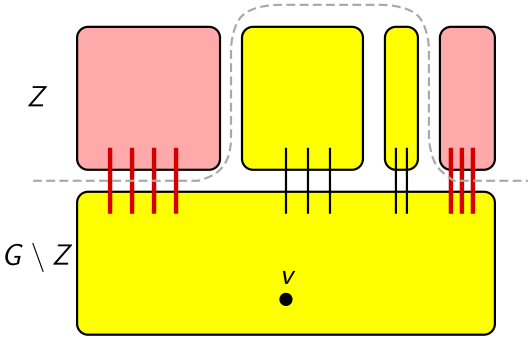


KNAPSACK!

Finding good clusters by KNAPSACK

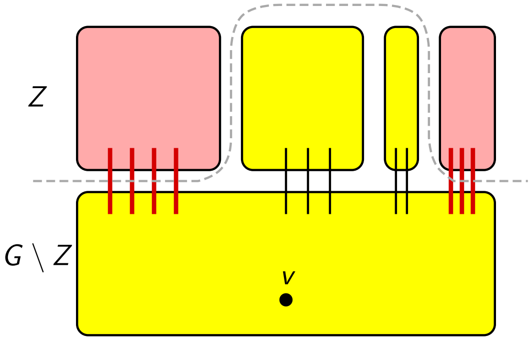


We interpret the componenents $V_1, \ldots, V_t$ of $G[Z]$ as items:

- V_i has value $\delta(V_i)$ and
- V_i has weight $|V_i|$.

The goal is to select items with total value at least $\delta(Z) - q$ and total weight at most $p - |V(G) \setminus Z|$.

Finding good clusters by KNAPSACK



Standard DP solves it in polynomial time: let $T[i,j]$ be the maximum value of a subset of the first i items having total weight at most j.

Recurrence:

$$T[i,j] = \max\{T[i-1,j], T[i-1,j-|V_i|] + \delta(V_i)\}$$

Summary of algorithm

(p, q)-CLUSTERING

Input: A graph G, integers p, q.

Find: A partition $(V_1, \ldots, V_m)$ of $V(G)$ such that for every i

- $|V_i| \leq p$ and
- $\delta(V_i) \leq q$.

- It is sufficient to check for each vertex v if it is in a good cluster.
- Enumerate all the important sets.
- Let Z be the union of random important sets.
- The solution is obtained by extending $G \setminus Z$ with some of the components of $G[Z]$.
- Knapsack.

(p, q)-CLUSTERING

- With a slightly different probability distribution, one can reduce the error probability to $2^{-O(q)}$.
- Derandomization is possible using standard techniques, but nontrivial to obtain $2^{O(q)}$ running time.
- Other variants: maximum degree in the cluster is at most p, etc.

Summary

- A simple (but essentially tight) bound on the number of important cuts.
- Algorithmic results: FPT algorithms for
  - Multiway Cut in undirected graphs,
  - Skew Multicut in directed graphs,
  - Directed Feedback Vertex/Edge Set, and
  - (p, q)-Clustering.