The square root phenomenon in planar graphs

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Main message

Are NP-hard problems easier on planar graphs?
   Yes, usually.

By how much?
   Often by exactly a square root factor.
Overview

Chapter 1:
Subexponential algorithms using treewidth.

Chapter 2:
Grid minors and bidimensionality.

Chapter 3:
Finding bounded-treewidth solutions.
Better exponential algorithms

Most NP-hard problems (e.g., \texttt{3-Coloring}, \texttt{Independent Set}, \texttt{Hamiltonian Cycle}, \texttt{Steiner Tree}, etc.) remain NP-hard on planar graphs,\(^1\) so what do we mean by “easier”?

\(^1\)Notable exception: \texttt{Max Cut} is in P for planar graphs.
Better exponential algorithms

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The running time is still exponential, but significantly smaller:

\[
2^{O(n)} \Rightarrow 2^{O(\sqrt{n})} \\
O(k) \Rightarrow O(\sqrt{k}) \\
2^{O(k)} \cdot n^{O(1)} \Rightarrow 2^{O(\sqrt{k})} \cdot n^{O(1)}
\]

\(^1\)Notable exception: \textbf{Max Cut} is in P for planar graphs.
Treewidth is a measure of “how treelike the graph is.”

We need only the following basic facts:

1. If a graph $G$ has treewidth $k$, then many classical NP-hard problems can be solved in time $2^{O(k)} \cdot n^{O(1)}$ or $2^{O(k \log k)} \cdot n^{O(1)}$ on $G$.

2. A planar graph on $n$ vertices has treewidth $O(\sqrt{n})$.

3. Excluded Grid Theorem: a planar graph of treewidth $k$ contains a $\Omega(k) \times \Omega(k)$ grid minor.
Treewidth — a measure of “tree-likeness”

Tree decomposition: Vertices are arranged in a tree structure satisfying the following properties:

1. If $u$ and $v$ are neighbors, then there is a bag containing both of them.
2. For every $v$, the bags containing $v$ form a connected subtree.
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A subtree communicates with the outside world only via the root of the subtree.
Finding tree decompositions

Various algorithms for finding optimal or approximate tree decompositions if treewidth is $w$:

- optimal decomposition in time $2^{O(w^3)} \cdot n$ [Bodlaender 1996].
- 4-approximate decomposition in time $2^{O(w)} \cdot n^2$ [Robertson and Seymour].
- 5-approximate decomposition in time $2^{O(w)} \cdot n$ [Bodlaender et al. 2013].
- $O(\sqrt{\log w})$-approximation in polynomial time [Feige, Hajiaghayi, Lee 2008].

As we are mostly interested in algorithms with running time $2^{O(w)} \cdot n^{O(1)}$, we may assume that we have a decomposition.
Theorem
Given a tree decomposition of width $w$, **3-Coloring** can be solved in time $O(3^w \cdot w^{O(1)} \cdot n)$.

$B_x$: vertices appearing in node $x$.
$V_x$: vertices appearing in the subtree rooted at $x$.

For every node $x$ and coloring $c : B_x \to \{1, 2, 3\}$, we compute the Boolean value $E[x, c]$, which is true if and only if $c$ can be extended to a proper 3-coloring of $V_x$.

**Claim:**
We can determine $E[x, c]$ if all the values are known for the children of $x$. 
Subexponential algorithm for 3-COLORING

**Theorem**

3-COLORING can be solved in time $2^{O(w)} \cdot n^{O(1)}$ on graphs of treewidth $w$.

**Theorem [Robertson and Seymour]**

A planar graph on $n$ vertices has treewidth $O(\sqrt{n})$.

**Corollary**

3-COLORING can be solved in time $2^{O(\sqrt{n})}$ on planar graphs.

```
textbook algorithm + combinatorial bound
     ↓
subexponential algorithm
```
Lower bounds

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**3-Coloring** can be solved in time $2^{O(\sqrt{n})}$ on planar graphs.

Two natural questions:

- Can we achieve this running time on general graphs?
- Can we achieve even better running time (e.g., $2^{O(\frac{3}{\sqrt{n}})}$) on planar graphs?
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$P \neq NP$ is not a sufficiently strong hypothesis: it is compatible with 3SAT being solvable in time $2^{O(n^{1/1000})}$ or even in time $n^{O(\log n)}$.

We need a stronger hypothesis!
Exponential Time Hypothesis (ETH)

Hypothesis introduced by Impagliazzo, Paturi, and Zane:

**Exponential Time Hypothesis (ETH)**

There is no $2^{o(n)}$-time algorithm for $n$-variable 3SAT.

**Note:** current best algorithm is $1.30704^n$ [Hertli 2011].

**Note:** an $n$-variable 3SAT formula can have $\Omega(n^3)$ clauses.
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**Sparsification Lemma** [Impagliazzo, Paturi, Zane 2001]

There is a $2^{o(n)}$-time algorithm for $n$-variable 3SAT.

$\uparrow$

There is a $2^{o(m)}$-time algorithm for $m$-clause 3SAT.
Lower bounds based on ETH

Exponential Time Hypothesis (ETH)

There is no $2^{o(m)}$-time algorithm for $m$-clause 3SAT.

The textbook reduction from 3SAT to 3-COLORING:

\[
\begin{align*}
\text{3SAT formula } \phi & \quad \Rightarrow \\
\text{n variables} & \quad \text{Graph } G \\
\text{m clauses} & \quad \text{O}(m) \text{ vertices} \\
& \quad \text{O}(m) \text{ edges}
\end{align*}
\]

Corollary

Assuming ETH, there is no $2^{o(n)}$ algorithm for 3-COLORING on an $n$-vertex graph $G$. 
Lower bounds based on ETH

What about $3$-Coloring on planar graphs?

The textbook reduction from $3$-Coloring to Planar $3$-Coloring uses a “crossover gadget” with 4 external connectors:

- In every 3-coloring of the gadget, opposite external connectors have the same color.
- Every coloring of the external connectors where the opposite vertices have the same color can be extended to the whole gadgets.
- If two edges cross, replace them with a crossover gadget.
Lower bounds based on ETH

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Lower bounds based on ETH

- The reduction from **3-Coloring** to **Planar 3-Coloring** introduces $O(1)$ new edge/vertices for each crossing.
- A graph with $m$ edges can be drawn with $O(m^2)$ crossings.

3SAT formula $\phi$

- $n$ variables
- $m$ clauses

$\implies$

Graph $G$

- $O(m)$ vertices
- $O(m)$ edges

$\implies$

Planar graph $G'$

- $O(m^2)$ vertices
- $O(m^2)$ edges

**Corollary**

Assuming ETH, there is a no $2^{o(\sqrt{n})}$ algorithm for **3-Coloring** on an $n$-vertex planar graph $G$.

(Essentially observed by [Cai and Juedes 2001])
Summary of Chapter 1

Streamlined way of obtaining tight upper and lower bounds for planar problems.

- **Upper bound:**
  Standard bounded-treewidth algorithm + treewidth bound on planar graphs give $2^{O(\sqrt{n})}$ time subexponential algorithms.

- **Lower bound:**
  Textbook NP-hardness proof with quadratic blow up + ETH rule out $2^{o(\sqrt{n})}$ algorithms.

Works for **Hamiltonian Cycle, Vertex Cover, Independent Set, Feedback Vertex Set, Dominating Set, Steiner Tree, ...**
More refined analysis of the running time: we express the running time as a function of input size $n$ and a parameter $k$.

**Definition**

A problem is **fixed-parameter tractable (FPT)** parameterized by $k$ if it can be solved in time $f(k) \cdot n^{O(1)}$ for some computable function $f$.

Examples of FPT problems:
- Finding a vertex cover of size $k$.
- Finding a feedback vertex set of size $k$.
- Finding a path of length $k$.
- Finding $k$ vertex-disjoint triangles.

Note: these four problems have $2^{O(k)} \cdot n^{O(1)}$ time algorithms, which is best possible on general graphs.
Bounded search tree method

Algorithm for **Vertex Cover**:

\[ e_1 = u_1 v_1 \]
Bounded search tree method

Algorithm for **Vertex Cover:**

\[ e_1 = u_1 v_1 \]

\[ u_1 \quad v_1 \]

\[ u_1 \rightarrow v_1 \]
Bounded search tree method

Algorithm for Vertex Cover:

\[ e_1 = u_1 v_1 \]

\[ e_2 = u_2 v_2 \]
Bounded search tree method

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Algorithm for **Vertex Cover**:

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\[ \vdots \]

Height of the search tree \( \leq k \) \( \Rightarrow \) at most \( 2^k \) leaves \( \Rightarrow 2^k \cdot n^{O(1)} \) time algorithm.
W[1]-hardness

Negative evidence similar to NP-completeness. If a problem is \textbf{W[1]-hard}, then the problem is not FPT unless FPT=\text{W[1]}.

Some W[1]-hard problems:
- Finding a clique/independent set of size \(k\).
- Finding a dominating set of size \(k\).
- Finding \(k\) pairwise disjoint sets.
- ... 

For these problems, the exponent of \(n\) has to depend on \(k\) (the running time is typically \(n^{O(k)}\)).
Subexponential parameterized algorithms

What kind of upper/lower bounds we have for $f(k)$?

- For most problems, we cannot expect a $2^{o(k)} \cdot n^{O(1)}$ time algorithm on **general graphs** (as this would imply a $2^{o(n)}$ algorithm).

- For most problems, we cannot expect a $2^{o(\sqrt{k})} \cdot n^{O(1)}$ time algorithm on **planar graphs** (as this would imply a $2^{o(\sqrt{n})}$ algorithm).

- However, $2^{O(\sqrt{k})} \cdot n^{O(1)}$ algorithms do exist for several problems on planar graphs, even for some W[1]-hard problems.

- Quick proofs via grid minors and bidimensionality.
  
  [Demaine, Fomin, Hajiaghayi, Thilikos 2004]
Minors

Definition

Graph $H$ is a **minor** of $G$ ($H \leq G$) if $H$ can be obtained from $G$ by deleting edges, deleting vertices, and contracting edges.

Note: minimum vertex cover size of $H$ is at most the minimum vertex cover size of $G$. 
Planar Excluded Grid Theorem

**Theorem** [Robertson, Seymour, Thomas 1994]

Every planar graph with treewidth at least $4k$ has a $k \times k$ grid minor.

**Note:** for general graphs, we need treewidth at least $k^4k^4(k+2)$ for a $k \times k$ grid minor [Diestel et al. 1999].
Bidimensionality for **Vertex Cover**

**Observation:** If the treewidth of a planar graph $G$ is at least $4\sqrt{2}k$

$\Rightarrow$ It has a $\sqrt{2k} \times \sqrt{2k}$ grid minor (Planar Excluded Grid Theorem)

$\Rightarrow$ The grid has a matching of size $k$

$\Rightarrow$ The minimum vertex cover size of the grid is at least $k$

$\Rightarrow$ The minimum vertex cover size of $G$ is at least $k$. 
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We use this observation to solve **Vertex Cover** on planar graphs:

- Set $w := 4\sqrt{2k}$.
- Find a 4-approximate tree decomposition.
  - If treewidth is at least $w$: we answer “vertex cover is $\geq k$.”
  - If we get a tree decomposition of width $4w$, then we can solve the problem in time $2^{O(w)} \cdot n^{O(1)} = 2^{O(\sqrt{k})} \cdot n^{O(1)}$. 
Bidimensionality

Definition

A graph invariant $x(G)$ is **minor-bidimensional** if

- $x(G') \leq x(G)$ for every minor $G'$ of $G$, and
- If $G_k$ is the $k \times k$ grid, then $x(G_k) \geq ck^2$
  (for some constant $c > 0$).

Examples: **minimum vertex cover**, length of the longest path, feedback vertex set are minor-bidimensional.
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Summary of Chapter 2

Tight bounds for minor-bidimensional planar problems.

- **Upper bound:** Standard bounded-treewidth algorithm + planar excluded grid theorem give $2^{O(\sqrt{k})} \cdot n^{O(1)}$ time FPT algorithms.

- **Lower bound:** Textbook NP-hardness proof with quadratic blow up + ETH rule out $2^{o(\sqrt{n})}$ time algorithms $\Rightarrow$ no $2^{o(\sqrt{k})} \cdot n^{O(1)}$ time algorithm.

Variant of theory works for *contraction-bidimensional* problems, e.g., **Independent Set**, **Dominating Set**.
Chapter 3: Finding bounded treewidth solutions

So far the way we have used treewidth is to find something (e.g., Hamiltonian cycle) in a large bounded-treewidth graph:
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Chapter 3: Finding bounded treewidth solutions

But we can also find small bounded-treewidth graphs in an arbitrary large graph.

Theorem [Alon, Yuster, Zwick 1994]

Given a graph $H$ and weighted graph $G$, we can find a minimum weight subgraph of $G$ isomorphic to $H$ in time $2^{O(|V(H)|)} \cdot n^{O(\text{tw}(H))}$. 
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If the problem can be formulated as finding a graph of treewidth $O(\sqrt{k})$, then we get an $n^{O(\sqrt{k})}$ time algorithm.
Examples

Three examples:

- **Planar $k$-Terminal Cut**
  Improvement from $n^{O(k)}$ to $2^{O(k)} \cdot n^{O(\sqrt{k})}$.

- **Planar Strongly Connected Subgraph**
  Improvement from $n^{O(k)}$ to $2^{O(k \log k)} \cdot n^{O(\sqrt{k})}$.

- **TSP** with shortest path metric of a planar graph
  Improvement from $2^{O(k)} \cdot n^{O(1)}$ to $2^{O(\sqrt{k \log k})} \cdot n^{O(1)}$. 
A classical problem

**$s - t$ Cut**

*Input:* A graph $G$, an integer $p$, vertices $s$ and $t$

*Output:* A set $S$ of at most $p$ edges such that removing $S$ separates $s$ and $t$.

---

**Theorem [Ford and Fulkerson 1956]**

A minimum $s - t$ cut can be found in polynomial time.

What about separating more than two terminals?
More than two terminals

**Multiway Cut (aka k-Terminal Cut)**

*Input:* A graph $G$, an integer $p$, and a set $T$ of $k$ terminals

*Output:* A set $S$ of at most $p$ edges such that removing $S$ separates any two vertices of $T$

**Theorem [Dalhaus et al. 1994]**

NP-hard already for $k = 3$. 
More than two terminals

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**Theorem** [Dalhaus et al. 1994] [Hartvigsen 1998] [Bentz 2012]

*Planar $k$-Terminal Cut* can be solved in time $n^{O(k)}$.

**Theorem** [Klein and M. 2012]

*Planar $k$-Terminal Cut* can be solved in time $2^{O(k)} \cdot n^{O(\sqrt{k})}$. 
The first step of the algorithms is to look at the solution in the dual graph:
Dual graph

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Recall:

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We slightly transform the problem in such a way that the terminals are represented by **vertices** in the dual graph (instead of faces).
Finding the dual solution

Main ideas of [Dalhaus et al. 1994] [Hartvigsen 1998] [Bentz 2012]:

1. The dual solution has $O(k)$ branch vertices.
2. Guess the location of branch vertices ($n^{O(k)}$ guesses).
3. Deep magic to find the paths connecting the branch vertices (shortest paths are not necessarily good!)
Finding the dual solution

Idea for $n^{O(\sqrt{k})}$ time algorithm:

- Guess the graph $H$ representing the branch vertices.
- Build a weighted complete graph $G$ representing the distances in the planar graph.
- Find in time $n^{O(tw(H))} = n^{O(\sqrt{n})}$ a minimum weight copy of $H$ in $G$.

Problem: How to ensure that the solution separates the terminals?
The Steiner tree

We find a minimum cost Steiner tree $T$ of the terminals in the dual and cut open the graph along the tree. (Steiner tree: $3^k \cdot n^{O(1)}$ time by [Dreyfus-Wagner 1972] or $2^k \cdot n^{O(1)}$ time by [Björklund 2007])
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Key idea: the paths of the dual solution between the branch points/crossing points can be assumed to be shortest paths.
Topology

Key idea: the paths of the dual solution between the branch points/crossing points can be assumed to be shortest paths.

Thus a solution can be completely described by the location of these points and which of them are connected.

A “topology” just describes the connections without the locations.
Lower bounds

**Theorem [Klein and M. 2012]**

**Planar** $k$-**Terminal Cut** can be solved in time $2^{O(k)} \cdot n^{O(\sqrt{k})}$.

Natural questions:

- Is there an $f(k) \cdot n^{o(\sqrt{k})}$ time algorithm?
- Is there an $f(k) \cdot n^{O(1)}$ time algorithm (i.e., is it fixed-parameter tractable)?
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The previous lower bound technology is of no help here: showing that there is no $2^{o(\sqrt{n})}$ time algorithm does not answer the question.

**Lower bounds:**

**Theorem [M. 2012]**

Planar $k$-Terminal Cut is W[1]-hard and has no $f(k) \cdot n^{o(\sqrt{k})}$ time algorithm (assuming ETH).
W[1]-hardness

**Definition**

A **parameterized reduction** from problem $A$ to $B$ maps an instance $(x, k)$ of $A$ to instance $(x', k')$ of $B$ such that

- $(x, k) \in A \iff (x', k') \in B$,
- $k' \leq g(k)$ for some computable function $g$,
- $(x', k')$ can be computed in time $f(k) \cdot |x|^{O(1)}$.

**Easy:** If there is a parameterized reduction from problem $A$ to problem $B$ and $B$ is FPT, then $A$ is FPT as well.

**Definition**

A problem $P$ is **W[1]-hard** if there is a parameterized reduction from $k$-CLIQUE to $P$. 
**W[1]-hardness**

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W[1]-hardness vs. NP-hardness

W[1]-hardness proofs are more delicate than NP-hardness proofs: we need to control the new parameter.

Example: \texttt{k-Independent Set} can be reduced to \texttt{k'-Vertex Cover} with \( k' := n - k \). But this is \textbf{not} a parameterized reduction.
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**Example:** \( k\)-**Independent Set** can be reduced to \( k'\)-**Vertex Cover** with \( k' := n - k \). But this is not a parameterized reduction.

### NP-hardness proof
Reduction from some graph problem. We build \( n \) vertex gadgets of constant size and \( m \) edge gadgets of constant size.

### W[1]-hardness proof
Reduction from \( k\)-**Clique**. We build \( k \) large vertex gadgets, each having \( n \) states (and/or \( \binom{k}{2} \) large edge gadgets with \( m \) states).
**Planar problems**

**Another difference:** Most problems remain NP-hard on planar graphs, but become FPT.

Algorithmic techniques for planar problems:
- Baker’s shifting technique + treewidth
- Bidimensionality
- Protrusions

Very few $W[1]$-hardness results so far for planar problems.
Tight bounds

**Theorem [Chen et al. 2004]**

Assuming ETH, there is no \( f(k) \cdot n^{o(k)} \) algorithm for \( k\text{-CLIQUE} \) for any computable function \( f \).

**Transfering to other problems:**
If there is a parameterized reduction from \( k\text{-CLIQUE} \) to problem \( A \) mapping \((x, k)\) to \((x', g(k))\), then an \( f(k) \cdot n^{o(g^{-1}(k))} \) algorithm for problem \( A \) gives an \( f(k) \cdot n^{o(k)} \) algorithm for \( k\text{-CLIQUE} \), contradicting ETH.

**Bottom line:**
To rule out \( f(k) \cdot n^{o(\sqrt{k})} \) algorithms, we need a parameterized reduction that blows up the parameter at most quadratically.
Tight bounds

**Theorem [Chen et al. 2004]**

Assuming ETH, there is no $f(k) \cdot n^{o(k)}$ algorithm for $k$-CLIQUE for any computable function $f$.

**Transfering to other problems:**

\[
\begin{array}{c|c|c}
\text{k-Clique} & \Rightarrow & \text{Problem A} \\
(x, k) & & (x', g(k)) \\

f(k) \cdot n^{o(k)} & \iff & f(k) \cdot n^{o(g^{-1}(k))}
\end{array}
\]

**Bottom line:**
To rule out $f(k) \cdot n^{o(\sqrt{k})}$ algorithms, we need a parameterized reduction that blows up the parameter at most quadratically.
Grid Tiling

**GRID TILING**

*Input:* A $k \times k$ matrix and a set of pairs $S_{i,j} \subseteq [D] \times [D]$ for each cell.

*Find:* A pair $s_{i,j} \in S_{i,j}$ for each cell such that

- Horizontal neighbors agree in the first component.
- Vertical neighbors agree in the second component.

$k = 3$, $D = 5$
**Input:** A \( k \times k \) matrix and a set of pairs \( S_{i,j} \subseteq [D] \times [D] \) for each cell.

**Find:** A pair \( s_{i,j} \in S_{i,j} \) for each cell such that

- Horizontal neighbors agree in the first component.
- Vertical neighbors agree in the second component.

### Example

\[
\begin{array}{ccc}
(1,1) & (1,5) & (1,1) \\
(1,3) & (4,1) & (4,2) \\
(4,2) & (3,5) & (3,3) \\
\hline
(2,2) & (1,3) & (2,2) \\
(4,1) & (2,1) & (3,2) \\
\hline
(3,1) & (1,1) & (3,2) \\
(3,2) & (3,1) & (3,5) \\
(3,3) & & \\
\end{array}
\]

\( k = 3, \quad D = 5 \)
Grid Tiling is W[1]-hard

**Reduction from \( k\text{-CLIQUE} \)**

**Definition of the sets:**

- For \( i = j \): \((x, y) \in S_{i,j} \iff x = y\)
- For \( i \neq j \): \((x, y) \in S_{i,j} \iff x \text{ and } y \text{ are adjacent.}\)

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Each diagonal cell defines a value \( v_i \ldots \)
Grid Tiling is W[1]-hard

Reduction from $k$\text{-\textsc{Clique}}

Definition of the sets:

- For $i = j$: $(x, y) \in S_{i,j} \iff x = y$
- For $i \neq j$: $(x, y) \in S_{i,j} \iff x$ and $y$ are adjacent.

\[\begin{array}{c|c|c|c|c}
(., v_i) & (., v_i) & (v_i, .) & (v_i, .) & (v_i, .) \\
(v_i, .) & (v_i, v_i) & (v_i, .) & (v_i, .) & (v_i, .) \\
(., v_i) & (., v_i) & (., v_i) & (., v_i) & (., v_i) \\
(., v_i) & (., v_i) & (., v_i) & (., v_i) & (., v_i) \\
\end{array}\]

\ldots which appears on a “cross”
Grid Tiling is W[1]-hard

**Reduction from $k$-CLIQUE**

**Definition of the sets:**

- For $i = j$: $(x, y) \in S_{i,j} \iff x = y$
- For $i \neq j$: $(x, y) \in S_{i,j} \iff x$ and $y$ are adjacent.

<table>
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$v_i$ and $v_j$ are adjacent for every $1 \leq i < j \leq k$. 
Grid Tiling is W[1]-hard

Reduction from $k$-CLIQUE

Definition of the sets:

- For $i = j$: $(x, y) \in S_{i,j} \iff x = y$
- For $i \neq j$: $(x, y) \in S_{i,j} \iff x$ and $y$ are adjacent.

$v_i$ and $v_j$ are adjacent for every $1 \leq i < j \leq k$. 
The gadget

For every set $S_{i,j}$, we construct a gadget such that

- for every $(x, y) \in S_{i,j}$, there is a minimum multiway cut that represents $(x, y)$.
- every minimum multiway cut represents some $(x, y) \in S_{i,j}$.

Main part of the proof: constructing these gadgets.
The gadget

For every set $S_{i,j}$, we construct a gadget such that
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Main part of the proof: constructing these gadgets.

A cut representing (2,4).
The gadget

For every set $S_{i,j}$, we construct a gadget such that

- for every $(x, y) \in S_{i,j}$, there is a minimum multiway cut that represents $(x, y)$.
- every minimum multiway cut represents some $(x, y) \in S_{i,j}$.

Main part of the proof: constructing these gadgets.

[Diagram of a gadget with labeled points and cuts representing pairs $(x, y)$ in $S_{i,j}$]

A cut not representing any pair.
Putting together the gadgets
Putting together the gadgets

Oops!
Putting together the gadgets
Planar Multiway Cut

- **Upper bound:**
  Looking at the dual + cutting open a Steiner tree + guessing a topology + finding a graph of treewidth $O(\sqrt{k})$.

- **Lower bound:**
  ETH + reduction from **Grid Tiling** + tricky gadget construction rule out $f(k) \cdot n^{o(\sqrt{k})}$ time algorithms.
Strongly Connected Subgraph

Undirected graphs:
Steiner Tree: Find a minimum weight connected subgraph that contains all $k$ terminals.

Theorem [Dreyfus-Wagner 1972]
Steiner Tree can be solved in time $2^{O(k)} \cdot n^{O(1)}$. 

Directed graphs:
Strongly Connected Subgraph:
Find a minimum weight strongly connected subgraph that contains all $k$ terminals.

Theorem [Guo, Niedermeier, Suchý 2011]
Strongly Connected Subgraph on general directed graphs is $W[1]$-hard parameterized by $k$.

Theorem [Feldman and Ruhl 2006]
Strongly Connected Subgraph can be solved in time $n^{O(k)}$ on general directed graphs.
**Strongly Connected Subgraph**

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**Strongly Connected Subgraph** can be solved in time \( n^{O(k)} \) on general directed graphs.
**Strongly Connected Subgraph** on planar graphs

**Theorem [Feldman and Ruhl 2006]**

**Strongly Connected Subgraph** can be solved in time $n^{O(k)}$ on general directed graphs.

Natural questions:

- Is there an $f(k) \cdot n^{o(k)}$ time algorithm on planar graphs?
- Is there an $f(k) \cdot n^{O(1)}$ time algorithm (i.e., is it fixed-parameter tractable) on planar graphs?
Strongly Connected Subgraph on planar graphs

Theorem [Feldman and Ruhl 2006]

Strongly Connected Subgraph can be solved in time $n^{O(k)}$ on general directed graphs.

Natural questions:
- Is there an $f(k) \cdot n^{o(k)}$ time algorithm on planar graphs?
- Is there an $f(k) \cdot n^{O(1)}$ time algorithm (i.e., is it fixed-parameter tractable) on planar graphs?

Theorem [Chitnis, Hajiaghayi, M.]

Strongly Connected Subgraph can be solved in time $2^{O(k \log k)} \cdot n^{O(\sqrt{k})}$ on planar directed graphs.

Theorem [Chitnis, Hajiaghayi, M.]

Strongly Connected Subgraph has no $f(k) \cdot n^{o(\sqrt{k})}$ time algorithm on planar directed graphs (assuming ETH).
Closely looking at the $n^{O(k)}$ algorithm of [Feldman and Ruhl 2006] shows that an optimum solution consists of directed paths and “bidirectional strips”:

With some work, we can bound the number paths/strips by $O(k)$. 
We guess the topology of the solution \(2^{O(k \log k)}\) possibilities.
Treewidth of the topology is \(O(\sqrt{k})\).
We can find the best realization of this topology (matching the location of the terminals) in time \(n^{O(\sqrt{k})}\).
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Treewidth of the topology is \(O(\sqrt{k})\).

We can find the best realization of this topology (matching the location of the terminals) in time \(n^{O(\sqrt{k})}\).
Lower bound

Theorem [Chitnis, Hajiaghayi, M.]

**Strongly Connected Subgraph** has no $f(k) \cdot n^{o(\sqrt{k})}$ time algorithm on planar directed graphs (assuming ETH).

The proof is by reduction from **Grid Tiling** and complicated construction of gadgets.
**TSP**

Input: A set $T$ of cities and a distance function $d$ on $T$

Output: A tour on $T$ with minimum total distance

---

**Theorem [Held and Karp]**

TSP with $k$ cities can be solved in time $2^k \cdot n^{O(1)}$.

**Dynamic programming:**

Let $x(v, T')$ be the minimum length of path from $v_{\text{start}}$ to $v$ visiting all the cities $T' \subseteq T$. 
TSP on planar graphs

Assume that the distance function $d$ is generated by a (weighted) planar graph and $T$ is a subset of vertices.
TSP on planar graphs

Assume that the distance function $d$ is generated by a (weighted) planar graph and $T$ is a subset of vertices.

- Can be solved in time $2^{O(\sqrt{n})}$.
- Can be solved in time $2^k \cdot n^{O(1)}$.
- Can we solve it in time $2^{O(\sqrt{k})} \cdot n^{O(1)}$?
TSP on planar graphs

Assume that the distance function $d$ is generated by a (weighted) planar graph and $T$ is a subset of vertices.

Theorem [Klein and M.]

TSP with a distance function $d$ generated by a planar graph can be solved in time $2^{O(\sqrt{k})} \cdot W^{O(1)}$, where $W$ is the maximum distance in $d$.

Note: We do not have to know the graph, only the function $d$. 
TSP and treewidth

- We wanted to formulate the problem as finding a low treewidth subgraph.
- A cycle has treewidth 2, is this of any help?

Problem:
We have to remember the subset of cities visited by the partial tour ($2^k$ possibilities).
c-change TSP

- c-change operation: removing c steps of the tour and connecting the resulting c paths in some other way.
- A solution is c-OPT if no c-change can improve it.
- We can find a c-OPT solution in $k^{O(c)} \cdot W$ time, where $W$ is maximum distance in $d$. 
\( c \)-change TSP

- \( c \)-change operation: removing \( c \) steps of the tour and connecting the resulting \( c \) paths in some other way.
- A solution is \( c \)-OPT if no \( c \)-change can improve it.
- We can find a \( c \)-OPT solution in \( k^{O(c)} \cdot W \) time, where \( W \) is maximum distance in \( d \).
\(c\)-change TSP

- \(c\)-change operation: removing \(c\) steps of the tour and connecting the resulting \(c\) paths in some other way.
- A solution is \(c\)-OPT if no \(c\)-change can improve it.
- We can find a \(c\)-OPT solution in \(k^{O(c)} \cdot W\) time, where \(W\) is maximum distance in \(d\).
The crossing graph

Consider a optimum solution and a 4-OPT solution: [assume that the two tours do not share edges, etc.]

Lemma

The crossing graph of an optimum solution and a 4-OPT solution has $O(k)$ vertices and has treewidth $O(\sqrt{k})$. 
The crossing graph

Lemma

The crossing graph of an optimum solution and a 4-OPT solution has $O(k)$ vertices and has treewidth $O(\sqrt{k})$.

- The crossing graph has separators of size $O(\sqrt{k})$.
- In each component, the set of cities visited by the optimum solution is nice: it is the same as what $O(\sqrt{k})$ segments of the 4-OPT tour visited ($k^{O(\sqrt{k})}$ possibilities).
Parameterized problems where bidimensionality does not work.

- **Upper bounds:**
  Algorithms based on finding a bounded-treewidth subgraph. Treewidth bound is problem-specific:
  - **k-Terminal Cut:** dual solution has $O(k)$ branch vertices.
  - **Planar Strongly Connected Subgraph:** solution consists of $O(k)$ paths/strips.
  - **TSP** with a planar graph metric: the crossing graph of an optimum solution and a 4-OPT solution has size $O(k)$.

- **Lower bounds:**
  To rule out $f(k) \cdot n^{o(\sqrt{k})}$ time algorithms, we have to prove $W[1]$-hardness by reduction from **Grid Tiling**.
Chapter 1: Subexponential algorithms using treewidth.
- Algorithms: standard treewidth algorithms.
- Lower bounds: textbook NP-completeness proofs + ETH.

Chapter 2: Grid minors and bidimensionality.
- Algorithms: standard treewidth algorithms + excluded grid theorem.
- Lower bounds: textbook NP-completeness proofs + ETH.

Chapter 3: Finding bounded treewidth solutions.
- Algorithms: the solution can be represented by a graph of treewidth $O(\sqrt{k})$.
- Lower bounds: grid-like W[1]-hardness proofs to rule out $f(k) \cdot n^{o(\sqrt{k})}$ algorithms.
Conclusions

- A robust understanding of why certain problems can be solved in time $2^{O(\sqrt{n})}$ etc. on planar graphs and why the square root is best possible.
Conclusions

- A robust understanding of why certain problems can be solved in time $2^{O(\sqrt{n})}$ etc. on planar graphs and why the square root is best possible.
- Going beyond the basic toolbox requires new problem-specific algorithmic techniques and hardness proofs with tricky gadget constructions.
Conclusions

- A robust understanding of why certain problems can be solved in time $2^{O(\sqrt{n})}$ etc. on planar graphs and why the square root is best possible.

- Going beyond the basic toolbox requires new problem-specific algorithmic techniques and hardness proofs with tricky gadget constructions.

- The lower bound technology on planar graphs cannot give a lower bound without a square root factor. Does this mean that there are matching algorithms for other problems as well?
  
  - $2^{O(\sqrt{k})} \cdot n^{O(1)}$ time algorithm for \textsc{Steiner Tree} with $k$ terminals in a planar graph?
  
  - $2^{O(\sqrt{k})} \cdot n^{O(1)}$ time algorithm for finding a cycle of length exactly $k$ in a planar graph?
  
  - ...