FPT suspects and tough customers: Open problems of Downey and Fellows

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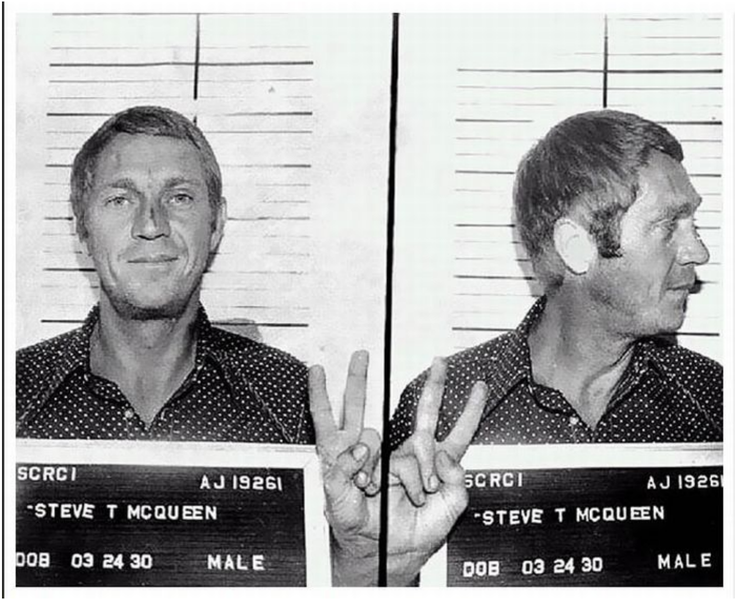
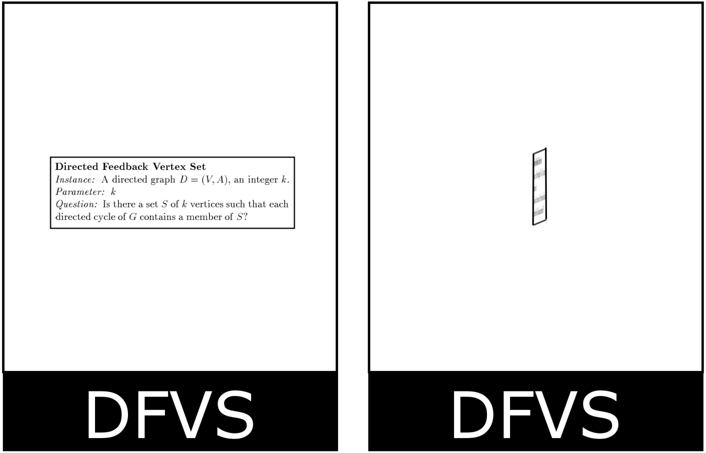
- We revisit the open problem list of the Downey-Fellows book.
- Good open problems are also significant scientific contributions.
- Were they good problems?
  - Not too easy?
  - Not impossible?
  - Any positive results?

FPT suspects



FPT suspects



FPT

Directed Feedback Vertex Set
Instance: A directed graph G
Parameter: A positive integer k
Question: Is there a set S of k vertices such that each directed cycle of G contains a member of S?

- FPT by [Chen et al. 2008]

FPT

TOPOLOGICAL CONTAINMENT
Instance: An undirected graph G
Parameter: A graph H
Question: Is H topologically contained in G?

- In XP by [Robertson and Seymour, GM13]
- FPT by [Grohe et al. 2011]

FPT

Immersion Order Test
Instance: An undirected graph G
Parameter: A graph H
Question: Does H has an immersion in G?

- FPT by reduction to Topological Containment.

W[1]-hard

Planar Directed Disjoint Paths
Instance: A directed planar graph G and k pairs $\langle r_1, s_1\rangle, \ldots, \langle r_k, s_k\rangle$ of vertices of G
Parameter: k
Question: Does G have k vertex-disjoint paths $P_1, \ldots, P_k$ with P_i running from r_i to s_i?

- In XP by [Schrijver 1994]
- FPT by [Cygan et al. 2013]

FPT

PLANAR t-NORMALIZED WEIGHTED SATISFIABILITY
Instance: A planar t-normalized formula X
Parameter: A positive integer k
Question: Does X have a satisfying assignment of weight k?

- What is exactly a planar t-normalized formula?
- FPT by standard techniques (layering + treewidth arguments or reduction to first order model checking).

W[1]-hard

PLANAR MULTIWAY CUT
Instance: A weighted undirected planar graph G with terminals $\{x_1, \ldots, x_k\}$ and a positive integer M
Parameter: k
Question: Is there a set of edges of total weight $\leq M$ whose removal disconnects each terminal from all others?

- Can be solved in time $n^{O(k)}$ by [Dahlhaus et al. 1994].
- Can be solved in time $2^{O(k)} \cdot n^{O(\sqrt{k})}$ [Klein and M. 2012]
- W[1]-hard and no $f(k) \cdot n^{o(\sqrt{k})}$ algorithm [M. 2012]

Tough customers

W[1]-hard

FIXED ALPHABET LONGEST COMMON SUBSEQUENCE (LCS)
Instance: k sequences X_i over an alphabet Σ of fixed size and a positive integer m
Parameter: k
Question: Is there a string $X \in \Sigma^*$ of length m that is a subsequence of each of the X_i?

- $O(n^{k+1})$ time by simple dynamic programming.
- W[1]-hard by [Pietrzak 2003] with binary alphabet.

FPT

Crossing Number
Instance: An undirected graph G
Parameter: A positive integer k
Question: Is the crossing number of G is at most k?

- FPT: $f(k) \cdot n^2$ algorithm by [Grohe 2001]
- $f(k) \cdot n$ algorithm by [Kawarabayashi and Reed 2007]

FPT

MINIMUM DEGREE GRAPH PARTITION
Instance: An undirected graph G
Parameter: Positive integers k and d
Question: Can $V(G)$ be partitioned into disjoint subsets $V_1, \ldots, V_m$ so that for $1 \leq i \leq m$, $|V_i| \leq k$ and at most d edges have exactly one endpoint in V_i?

- For fixed k and d, graphs with such partitions are closed under immersion [Langston and Plaut 1998].
- Immersion is wqo [Robertson and Seymour GM23].
- Immersion testing is FPT [Grohe et al. 2011].
- $\Rightarrow$ Minimum Degree Graph Partition is (nonuniform) FPT.
- $O^*(2^{O(k)})$ and $O^*(2^{O(d)})$ time by [Lokshtanov and M. 2011].

FPT

SHORT CHEAP TOUR

Instance: A graph G, integer S, and edge weighting $w\colon E(G) \to \mathbb{Z}$

Parameter: A positive integer k

Question: Is there a tour through at least k nodes of G of cost at most S?

- *"Using the methods of [PV91] or [AYZ94], it can be shown that the impoverished travelling salesman can visit at least k cities and return home for a given budget is FPT."*
  [Fellows 2001]

FPT

CHAIN MINOR
Instance: Posets P and Q
Parameter: $k = |P|$
Question: Is P a chain minor of Q?

- FPT by color coding [Błasiok and Kaminski 2017]

FPT

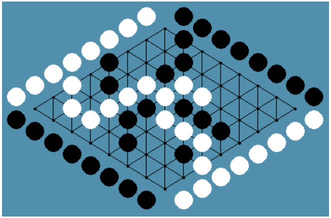
JUMP NUMBER
Instance: A poset P
Parameter: A positive integer k
Question: Is the jump number of P at most k?

- In XP by [El-Zahar and Schmerl 1984]
- FPT by [McCartin 2001]

W[1]-hard

SHORT GENERALIZED HEX
Instance: An undirected graph G with two distinguished vertices v_1 and v_2
Parameter: A positive integer k
Question: Does player one have a winning strategy of at most k moves in Generalized Hex?

- W[1]-hard [Bonnet et al. 2016]
- FPT on planar graphs [Bonnet et al. 2016]



W[1]-hard

EVEN SET

Instance: An undirected red/blue bipartite graph $G = (\mathcal{R}, \mathcal{B}, E)$

Parameter: A positive integer k

Question: Is there a non-empty set of at most k vertices $R \subseteq \mathcal{R}$, such that each member of $\mathcal{B}$ has an even number of neighbors in R?

- Hypergraph formulation.
- Minimum distance of linear codes of GF[2].
- Minimum cycle in a binary matroid.
- W[1]-hard (randomized reduction) [Bhattacharyya, Bonnet, Egri, Ghoshal, Kartik C.S., Lin, Manurangsi, Marx]

W[1]-hard

Shortest Vector
Instance: A basis $X = \{x_1, x_2, \ldots, x_t\} \subset \mathbb{Z}^n$ for a lattice $\mathcal{L}$
Parameter: A positive integer k
Question: Is there a non-zero vector $(a_1, \ldots, a_n) \in \mathcal{L}$, such that $\sum_{i=1}^{t} a_i^2 \leq k$?

- NP-hard under randomized reduction by [Ajtai 1998].
- In XP (trivial).
- W[1]-hard (randomized reduction) [Bhattacharyya, Bonnet, Egri, Ghoshal, Kartik C.S., Lin, Manurangsi, Marx] (for any L_p norm for $p > 1$)

Conclusions

- A very good list of problems.
- Only few problems turned out to be W[1]-hard (one FPT suspect and three tough customer).