The square root phenomenon in planar graphs

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Main message

Are NP-hard problems easier on planar graphs?

Yes, usually.

By how much?

Often by exactly a square root factor.

Overview

Chapter 1:
Subexponential algorithms using treewidth.

Chapter 2:
Grid minors and bidimensionality.

Chapter 3:
Finding bounded-treewidth solutions.

Better exponential algorithms

Most NP-hard problems (e.g., 3-Coloring, Independent Set, Hamiltonian Cycle, Steiner Tree, etc.) remain NP-hard on planar graphs,[1] so what do we mean by "easier"?

[1] Notable exception: Max Cut is in P for planar graphs.

Better exponential algorithms

Most NP-hard problems (e.g., 3-Coloring, Independent Set, Hamiltonian Cycle, Steiner Tree, etc.) remain NP-hard on planar graphs,[1] so what do we mean by "easier"?

The running time is still exponential, but significantly smaller:

$$
\begin{aligned}
2^{O(n)} &\quad\Rightarrow\quad 2^{O(\sqrt{n})} \\
n^{O(k)} &\quad\Rightarrow\quad n^{O(\sqrt{k})} \\
2^{O(k)} \cdot n^{O(1)} &\quad\Rightarrow\quad 2^{O(\sqrt{k})} \cdot n^{O(1)}
\end{aligned}
$$

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Chapter 1: Subexponential algorithms using treewidth

Treewidth is a measure of "how treelike the graph is."

We need only the following basic facts:

Treewidth

1. If a graph G has treewidth k, then many classical NP-hard problems can be solved in time $2^{O(k)} \cdot n^{O(1)}$ or $2^{O(k \log k)} \cdot n^{O(1)}$ on G.
2. A planar graph on n vertices has treewidth $O(\sqrt{n})$.

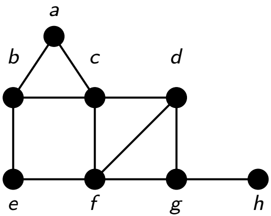
Treewidth — a measure of "tree-likeness"

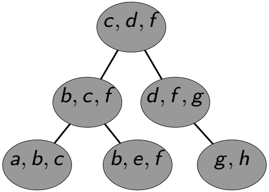
Tree decomposition: Vertices are arranged in a tree structure satisfying the following properties:

1. If u and v are neighbors, then there is a bag containing both of them.
2. For every v, the bags containing v form a connected subtree.

Width of the decomposition: largest bag size -1.

treewidth: width of the best decomposition.





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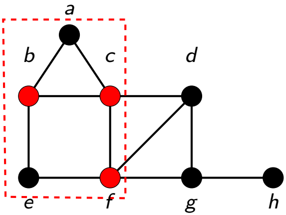
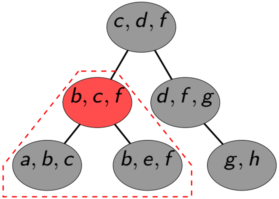




A subtree communicates with the outside world only via the root of the subtree.

Finding tree decompositions

Various algorithms for finding optimal or approximate tree decompositions if treewidth is w:

- optimal decomposition in time $2^{O(w^3)} \cdot n$ [Bodlaender 1996].
- 4-approximate decomposition in time $2^{O(w)} \cdot n^2$ [Robertson and Seymour].
- 5-approximate decomposition in time $2^{O(w)} \cdot n$ [Bodlaender et al. 2013].
- $O(\sqrt{\log w})$-approximation in polynomial time [Feige, Hajiaghayi, Lee 2008].

As we are mostly interested in algorithms with running time $2^{O(w)} \cdot n^{O(1)}$, we may assume that we have a decomposition.

3-Coloring and tree decompositions

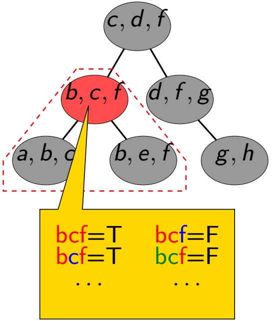
Theorem

Given a tree decomposition of width w, 3-Coloring can be solved in time $3^w \cdot w^{O(1)} \cdot n$.

B_x: vertices appearing in node x.
V_x: vertices appearing in the subtree rooted at x.

For every node x and coloring $c : B_x \to \{1,2,3\}$, we compute the Boolean value $E[x,c]$, which is true if and only if c can be extended to a proper 3-coloring of V_x.

Claim:
We can determine $E[x,c]$ if all the values are known for the children of x.



Subexponential algorithm for 3-Coloring

Theorem

3-Coloring can be solved in time $2^{O(w)} \cdot n^{O(1)}$ on graphs of treewidth w.

$$+$$

Theorem [Robertson and Seymour]

A planar graph on n vertices has treewidth $O(\sqrt{n})$.

$$\Downarrow$$

Corollary

3-Coloring can be solved in time $2^{O(\sqrt{n})}$ on planar graphs.

textbook algorithm + combinatorial bound

$\Downarrow$

subexponential algorithm

Lower bounds

Corollary

3-Coloring can be solved in time $2^{O(\sqrt{n})}$ on planar graphs.

Two natural questions:

- Can we achieve this running time on general graphs?
- Can we achieve even better running time (e.g., $2^{O(\sqrt[3]{n})}$) on planar graphs?

Lower bounds

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3-COLORING can be solved in time $2^{O(\sqrt{n})}$ on planar graphs.

Two natural questions:

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$P \neq NP$ is not a sufficiently strong hypothesis: it is compatible with 3SAT being solvable in time $2^{O(n^{1/1000})}$ or even in time $n^{O(\log n)}$.

We need a stronger hypothesis!

Exponential Time Hypothesis (ETH)

Hypothesis introduced by Impagliazzo, Paturi, and Zane:

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There is no $2^{o(n)}$-time algorithm for n-variable $\mathrm{3SAT}$.

Note: current best algorithm is 1.30704^n [Hertli 2011].

Note: an n-variable $\mathrm{3SAT}$ formula can have $\Omega(n^3)$ clauses.

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Sparsification Lemma [Impagliazzo, Paturi, Zane 2001]

There is a $2^{o(n)}$-time algorithm for n-variable $\mathrm{3SAT}$.

$$\Updownarrow$$

There is a $2^{o(m)}$-time algorithm for m-clause $\mathrm{3SAT}$.

Lower bounds based on ETH

Exponential Time Hypothesis (ETH)

There is no $2^{o(m)}$-time algorithm for m-clause 3SAT.

The textbook reduction from 3SAT to 3-COLORING:

3SAT formula ϕ		Graph G
n variables	$\Rightarrow$	$O(n+m)$ vertices
m clauses		$O(n+m)$ edges

Corollary

Assuming ETH, there is no $2^{o(n)}$ algorithm for 3-COLORING on an n-vertex graph G.

Lower bounds based on ETH

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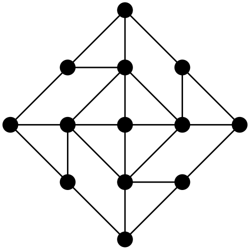
Lower bounds based on ETH

What about 3-Coloring on planar graphs?

The textbook reduction from 3-Coloring to Planar 3-Coloring uses a "crossover gadget" with 4 external connectors:



- In every 3-coloring of the gadget, opposite external connectors have the same color.
- Every coloring of the external connectors where the opposite vertices have the same color can be extended to the whole gadget.
- If two edges cross, replace them with a crossover gadget.

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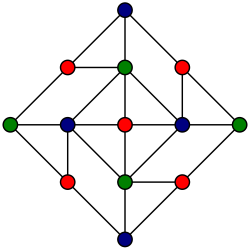

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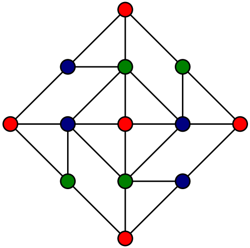

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- Every coloring of the external connectors where the opposite vertices have the same color can be extended to the whole gadget.
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Lower bounds based on ETH

- The reduction from 3-COLORING to PLANAR 3-COLORING introduces $O(1)$ new edge/vertices for each crossing.
- A graph with m edges can be drawn with $O(m^2)$ crossings.

3SAT formula ϕ n variables m clauses	$\Rightarrow$	Graph G $O(m)$ vertices $O(m)$ edges	$\Rightarrow$	Planar graph G' $O(m^2)$ vertices $O(m^2)$ edges

Corollary

Assuming ETH, there is no $2^{o(\sqrt{n})}$ algorithm for 3-COLORING on an n-vertex planar graph G.

(Essentially observed by [Cai and Juedes 2001])

Summary of Chapter 1

Streamlined way of obtaining tight upper and lower bounds for planar problems.

- **Upper bound:**
  Standard bounded-treewidth algorithm + treewidth bound on planar graphs give $2^{O(\sqrt{n})}$ time subexponential algorithms.
- **Lower bound:**
  Textbook NP-hardness proof with quadratic blow up + ETH rule out $2^{o(\sqrt{n})}$ algorithms.

Works for Hamiltonian Cycle, Vertex Cover, Independent Set, Feedback Vertex Set, Dominating Set, Steiner Tree, ...

Chapter 2: Grid minors and bidimensionality

More refined analysis of the running time: we express the running time as a function of input size n and a parameter k.

Definition

A problem is **fixed-parameter tractable (FPT)** parameterized by k if it can be solved in time $f(k) \cdot n^{O(1)}$ for some computable function f.

Examples of FPT problems:

- Finding a vertex cover of size k.
- Finding a feedback vertex set of size k.
- Finding a path of length k.
- Finding k vertex-disjoint triangles.
- $\dots$

Note: these four problems have $2^{O(k)} \cdot n^{O(1)}$ time algorithms, which is best possible on general graphs.

W[1]-hardness

Negative evidence similar to NP-completeness. If a problem is **W[1]-hard,** then the problem is not FPT unless FPT=W[1].

Some W[1]-hard problems:

- Finding a clique/independent set of size k.
- Finding a dominating set of size k.
- Finding k pairwise disjoint sets.
- $\dots$

For these problems, the exponent of n has to depend on k (the running time is typically $n^{O(k)}$).

Subexponential parameterized algorithms

What kind of upper/lower bounds we have for $f(k)$?

- For most problems, we cannot expect a $2^{o(k)} \cdot n^{O(1)}$ time algorithm on **general graphs.**
  (As this would imply a $2^{o(n)}$ algorithm.)
- For most problems, we cannot expect a $2^{o(\sqrt{k})} \cdot n^{O(1)}$ time algorithm on **planar graphs.**
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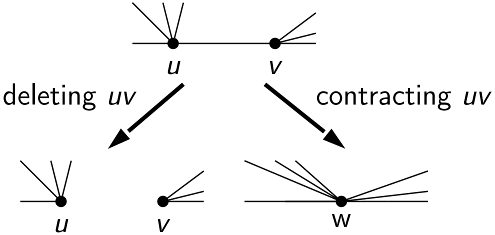
- For most problems, we cannot expect a $2^{o(\sqrt{k})} \cdot n^{O(1)}$ time algorithm on **planar graphs.**
  (As this would imply a $2^{o(\sqrt{n})}$ algorithm.)
- However, $2^{O(\sqrt{k})} \cdot n^{O(1)}$ algorithms do exist for several problems on planar graphs, even for some W[1]-hard problems.
- Quick proofs via grid minors and bidimensionality.
  [Demaine, Fomin, Hajiaghayi, Thilikos 2004]

Minors

Definition

Graph H is a **minor** of G ($H \leq G$) if H can be obtained from G by deleting edges, deleting vertices, and contracting edges.

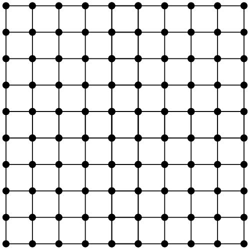


Note: length of the longest path in H is at most the length of the longest path in G.

Planar Excluded Grid Theorem

Theorem [Robertson, Seymour, Thomas 1994]

Every planar graph with treewidth at least $5k$ has a $k \times k$ grid minor.

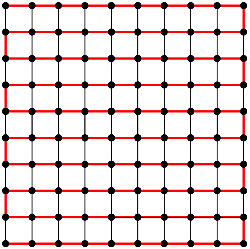


Note: for general graphs, we need treewidth at least $k^{4k^4(k+2)}$ for a $k \times k$ grid minor [Diestel et al. 1999]

(A $k^{O(1)}$ bound was just announced [Chekuri and Chuznoy 2013]!)

Bidimensionality for k-PATH

Observation: If the treewidth of a planar graph G is at least $5\sqrt{k}$
$\Rightarrow$ It has a $\sqrt{k} \times \sqrt{k}$ grid minor (Planar Excluded Grid Theorem)
$\Rightarrow$ The grid has a path of length at least k.
$\Rightarrow$ G has a path of length at least k.



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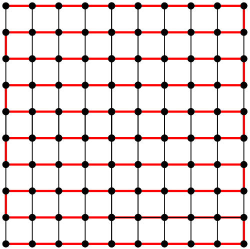
We use this observation to find a path of length at least k on planar graphs:

- Set $w := 5\sqrt{k}$.
- Find an $O(1)$-approximate tree decomposition.
  - If treewidth is at least w: we answer "there is a path of length at least k."
  - If we get a tree decomposition of width $O(w)$, then we can solve the problem in time $2^{O(w \log w)} \cdot n^{O(1)} = 2^{O(\sqrt{k} \log k)} \cdot n^{O(1)}$.

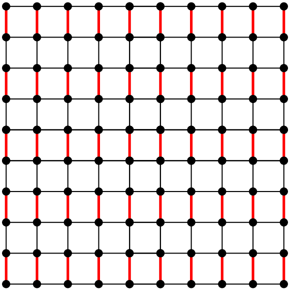


Bidimensionality

Definition

A graph invariant $x(G)$ is **minor-bidimensional** if

- $x(G') \leq x(G)$ for every minor G' of G, and
- If G_k is the $k \times k$ grid, then $x(G_k) \geq ck^2$ (for some constant $c > 0$).



Examples: minimum vertex cover, length of the longest path, feedback vertex set are minor-bidimensional.

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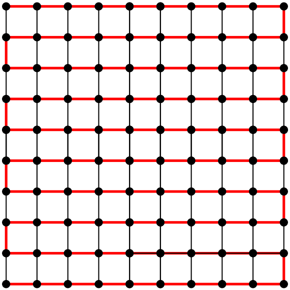

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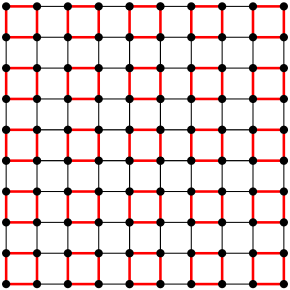


Examples: minimum vertex cover, length of the longest path, feedback vertex set are minor-bidimensional.

Summary of Chapter 2

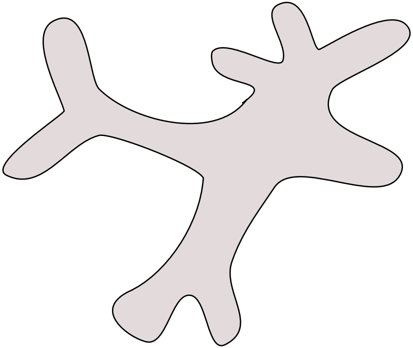
Tight bounds for minor-bidimensional planar problems.

- **Upper bound:**
  Standard bounded-treewidth algorithm + planar excluded grid theorem give $2^{O(\sqrt{k})} \cdot n^{O(1)}$ time FPT algorithms.
- **Lower bound:**
  Textbook NP-hardness proof with quadratic blow up + ETH rule out $2^{o(\sqrt{n})}$ time algorithms $\Rightarrow$ no $2^{o(\sqrt{k})} \cdot n^{O(1)}$ time algorithm.

Variant of theory works for **contraction-bidimensional** problems, e.g., Independent Set, Dominating Set.

Chapter 3: Finding bounded-treewidth solutions

So far the way we have used treewidth is to find something (e.g., Hamiltonian cycle) in a large bounded-treewidth graph:



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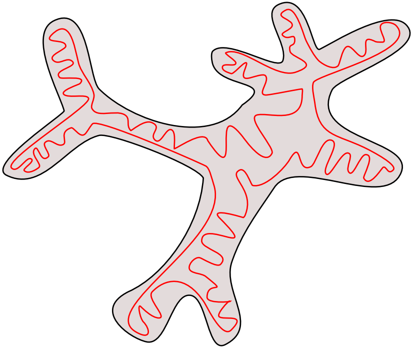
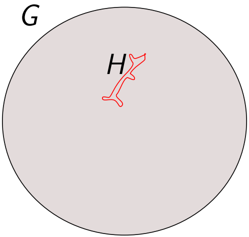


Chapter 3: Finding bounded-treewidth solutions

But we can also find small bounded-treewidth objects in an arbitrary large graph.



Theorem [Alon, Yuster, Zwick 1994]

Given a graph H and weighted graph G, we can find a minimum weight subgraph of G isomorphic to H in time $2^{O(|V(H)|)} \cdot n^{O(\mathrm{tw}(H))}$.

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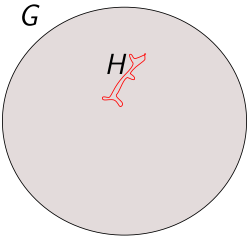

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If the problem can be formulated as finding a graph of treewidth $O(\sqrt{k})$, then we get an $n^{O(\sqrt{k})}$ time algorithm.

Examples

Three examples:

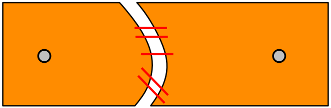
- PLANAR k-TERMINAL CUT
  Improvement from $n^{O(k)}$ to $2^{O(k)} \cdot n^{O(\sqrt{k})}$.
- PLANAR STRONGLY CONNECTED SUBGRAPH
  Improvement from $n^{O(k)}$ to $2^{O(k \log k)} \cdot n^{O(\sqrt{k})}$.
- SUBSET TSP with k cities in a planar graph
  Improvement from $2^{O(k)} \cdot n^{O(1)}$ to $2^{O(\sqrt{k} \log k)} \cdot n^{O(1)}$.

A classical problem

$s - t$ Cut

Input: A graph G, an integer p, vertices s and t

Output: A set S of at most p edges such that removing S separates s and t.



Theorem [Ford and Fulkerson 1956]

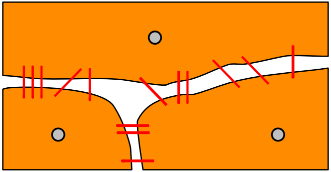
A minimum $s - t$ cut can be found in polynomial time.

What about separating more than two terminals?

More than two terminals

k-Terminal Cut (aka Multiway Cut)

Input: A graph G, an integer p, and a set T of k terminals
Output: A set S of at most p edges such that removing S separates any two vertices of T



Theorem [Dalhaus et al. 1994]

NP-hard already for $k = 3$.

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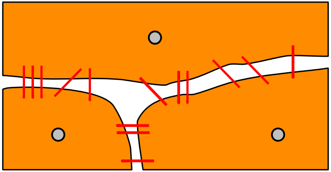


Theorem [Dalhaus et al. 1994] [Hartvigsen 1998] [Bentz 2012]

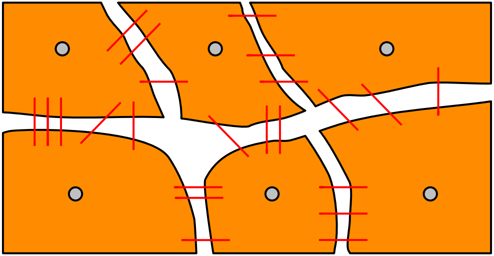
Planar k-Terminal Cut can be solved in time $n^{O(k)}$.

Theorem [Klein and M. 2012]

Planar k-Terminal Cut can be solved in time $2^{O(k)} \cdot n^{O(\sqrt{k})}$.

Dual graph

The first step of the algorithms is to look at the solution in the dual graph:



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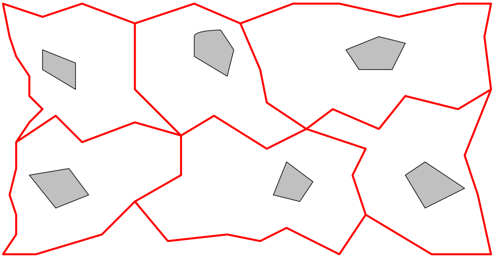

Recall:

Primal graph		Dual graph
vertices	$\Leftrightarrow$	faces
faces	$\Leftrightarrow$	vertices
edges	$\Leftrightarrow$	edges

Dual graph

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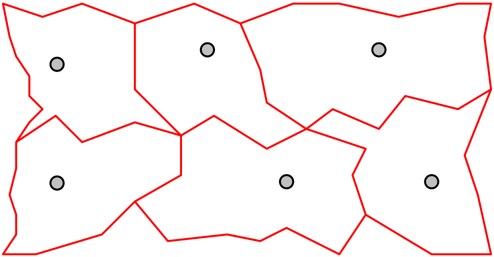

Recall:

Primal graph		Dual graph
vertices	$\Leftrightarrow$	faces
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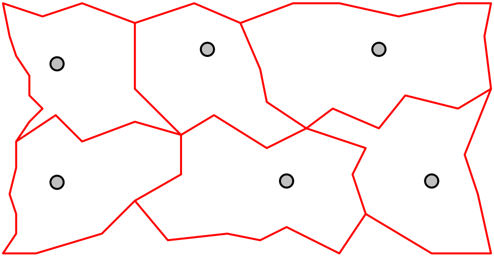
We slightly transform the problem in such a way that the terminals are represented by **vertices** in the dual graph (instead of faces).

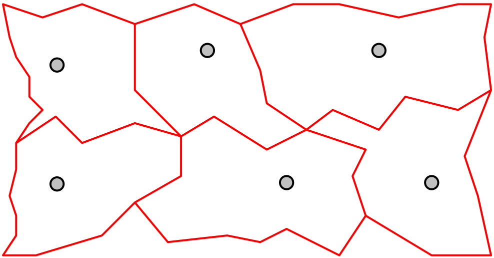
Finding the dual solution



Main ideas of [Dalhaus et al. 1994] [Hartvigsen 1998] [Bentz 2012]**:**

1. The dual solution has $O(k)$ branch vertices.
2. Guess the location of branch vertices ($n^{O(k)}$ guesses).
3. Deep magic to find the paths connecting the branch vertices (shortest paths are not necessarily good!)

Finding the dual solution



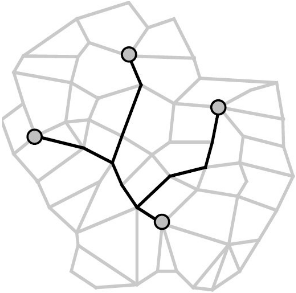
Idea for $n^{O(\sqrt{k})}$ time algorithm:

- Guess the graph H representing the branch vertices.
- Build a weighted complete graph G representing the distances in the planar graph.
- Find in time $n^{O(\mathrm{tw}(H))} = n^{O(\sqrt{k})}$ a minimum weight copy of H in G.

Problem: How to ensure that the solution separates the terminals?

The Steiner tree

We find a minimum cost Steiner tree T of the terminals in the **dual** and cut open the graph along the tree.
(Steiner tree: $3^k \cdot n^{O(1)}$ time by [Dreyfus-Wagner 1972] or $2^k \cdot n^{O(1)}$ time by [Björklund 2007])



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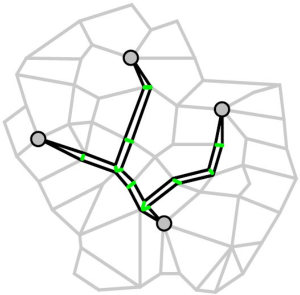

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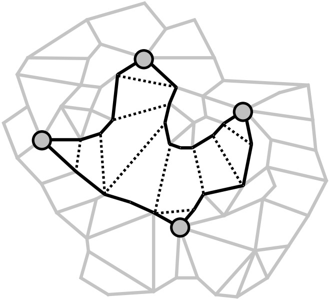

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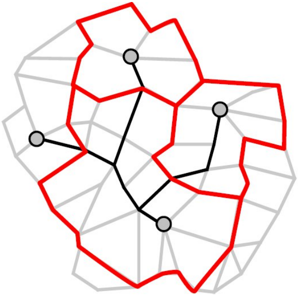

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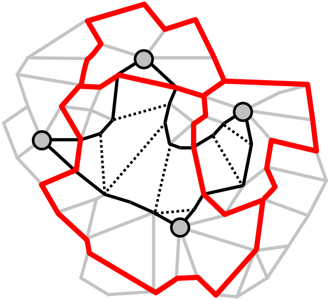

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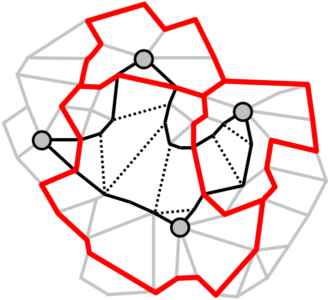

Key idea: the paths of the dual solution between the branch points/crossing points can be assumed to be shortest paths.

Topology

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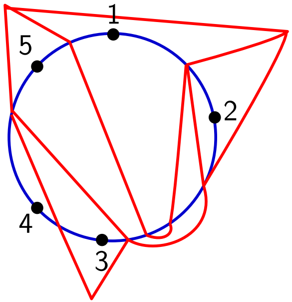

- Thus a solution can be completely described by the location of these points and which of them are connected.
- A "topology" just describes the connections without the locations.
- We can bound the size of the topology by $O(k)$ and its treewidth by $O(\sqrt{k})$.

Lower bounds

Theorem [Klein and M. 2012]

PLANAR k-TERMINAL CUT can be solved in time $2^{O(k)} \cdot n^{O(\sqrt{k})}$.

Natural questions:

- Is there an $f(k) \cdot n^{o(\sqrt{k})}$ time algorithm?
- Is there an $f(k) \cdot n^{O(1)}$ time algorithm (i.e., is it fixed-parameter tractable)?

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The previous lower bound technology is of no help here: showing that there is no $2^{o(\sqrt{n})}$ time algorithm does not answer the question.

Lower bounds:

Theorem [M. 2012]

PLANAR k-TERMINAL CUT is W[1]-hard and has no $f(k) \cdot n^{o(\sqrt{k})}$ time algorithm (assuming ETH).

W[1]-hardness

Definition

A **parameterized reduction** from problem A to B maps an instance (x, k) of A to instance (x', k') of B such that

- $(x, k) \in A \iff (x', k') \in B$,
- $k' \leq g(k)$ for some computable function g.
- (x', k') can be computed in time $f(k) \cdot |x|^{O(1)}$.

Easy: If there is a parameterized reduction from problem A to problem B and B is FPT, then A is FPT as well.

Definition

A problem P is **W[1]-hard** if there is a parameterized reduction from k-Clique to P.

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W[1]-hardness vs. NP-hardness

W[1]-hardness proofs are more delicate than NP-hardness proofs: we need to control the new parameter.

Example: k-Independent Set can be reduced to k'-Vertex Cover with $k' := n - k$. But this is **not** a parameterized reduction.

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NP-hardness proof

Reduction from some graph problem. We build n vertex gadgets of constant size and m edge gadgets of constant size.

W[1]-hardness proof

Reduction from k-Clique. We build k large vertex gadgets, each having n states (and/or $\binom{k}{2}$ large edge gadgets with m states).

Tight bounds

Theorem [Chen et al. 2004]

Assuming ETH, there is no $f(k) \cdot n^{o(k)}$ algorithm for k-CLIQUE for any computable function f.

Transfering to other problems:

k-Clique (x, k)	$\Rightarrow$	Problem A (x', k^2)
$f(k) \cdot n^{o(k)}$ algorithm	$\Leftarrow$	$f(k) \cdot n^{o(\sqrt{k})}$ algorithm

Bottom line:
To rule out $f(k) \cdot n^{o(\sqrt{k})}$ algorithms, we need a parameterized reduction that blows up the parameter at most quadratically.

Grid Tiling

GRID TILING

Input: A $k \times k$ matrix and a set of pairs $S_{i,j} \subseteq [D] \times [D]$ for each cell.

Find: A pair $s_{i,j} \in S_{i,j}$ for each cell such that

- Horizontal neighbors agree in the first component.
- Vertical neighbors agree in the second component.

(1,1) (1,3) (4,2)	(1,5) (4,1) (3,5)	(1,1) (4,2) (3,3)
(2,2) (4,1)	(1,3) (2,1)	(2,2) (3,2)
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$k = 3,\ D = 5$

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- Vertical neighbors agree in the second component.

Fact

There is a parameterized reduction from k-Clique to $k \times k$ Grid Tiling.

Grid Tiling is W[1]-hard

Reduction from k-Clique

Definition of the sets:

- For $i = j$: $\quad (x, y) \in S_{i,j} \iff x = y$
- For $i \neq j$: $\quad (x, y) \in S_{i,j} \iff x$ and y are adjacent.

	(v_i, v_i)			

Each diagonal cell defines a value v_i . . .

Grid Tiling is W[1]-hard

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	$(., v_i)$			
$(v_i, .)$	(v_i, v_i)	$(v_i, .)$	$(v_i, .)$	$(v_i, .)$
	$(., v_i)$			
	$(., v_i)$			
	$(., v_i)$			

... which appears on a "cross"

Grid Tiling is W[1]-hard

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$(v_i, .)$	(v_i, v_i)	$(v_i, .)$	$(v_i, .)$	$(v_i, .)$
	$(., v_i)$			
	$(., v_i)$		(v_j, v_j)	
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v_i and v_j are adjacent for every $1 \leq i < j \leq k$.

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$(v_i, .)$	(v_i, v_i)	$(v_i, .)$	(v_i, v_j)	$(v_i, .)$
	$(., v_i)$		$(., v_j)$	
$(v_j, .)$	(v_j, v_i)	$(v_j, .)$	(v_j, v_j)	$(v_j, .)$
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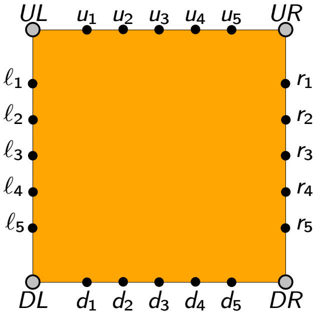
Reduction from $k \times k$ Grid Tiling to Planar k^2-Terminal Cut

For every set $S_{i,j}$, we construct a gadget with 4 terminals such that

- for every $(x, y) \in S_{i,j}$, there is a minimum multiway cut that represents (x, y).
- every minimum multiway cut represents some $(x, y) \in S_{i,j}$.

Main part of the proof: constructing these gadgets.



The gadget.

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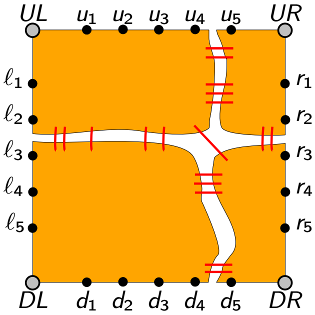

A cut representing (2,4).

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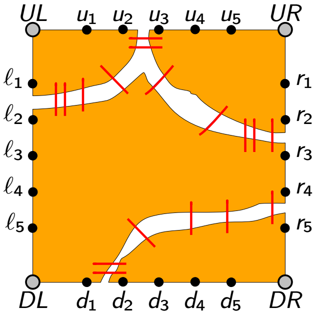
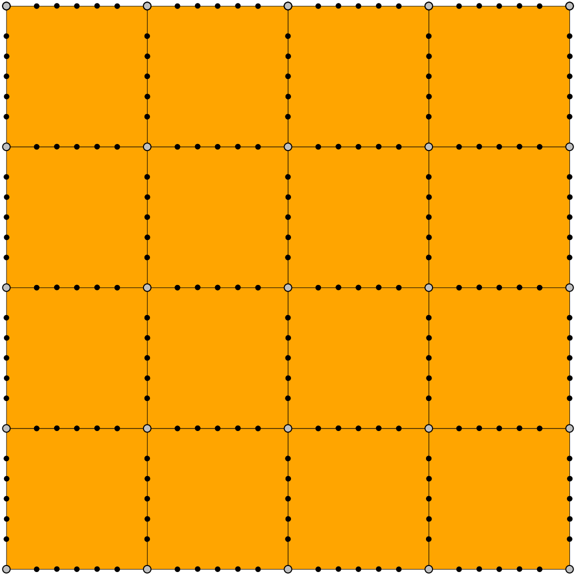


A cut not representing any pair.

Putting together the gadgets



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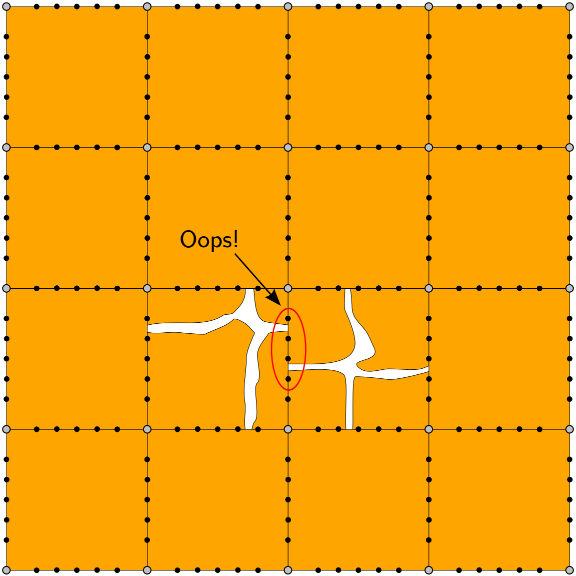

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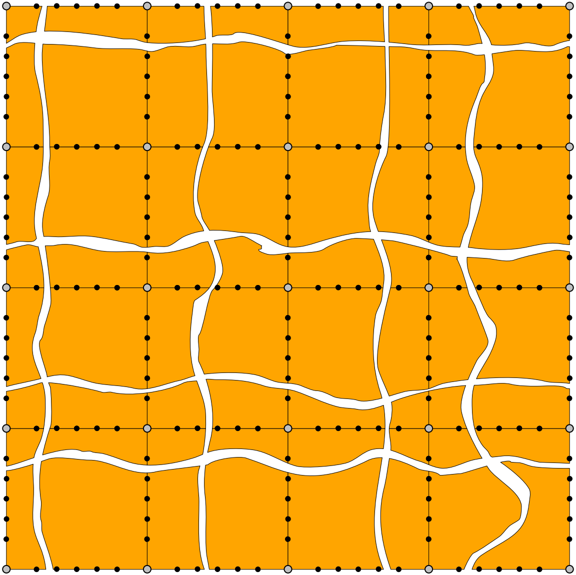

PLANAR k-TERMINAL CUT

- **Upper bound:**
  Looking at the dual + cutting open a Steiner tree + guessing a topology + finding a graph of treewidth $O(\sqrt{k})$.
- **Lower bound:**
  ETH + reduction from GRID TILING + tricky gadget construction rule out $f(k) \cdot n^{o(\sqrt{k})}$ time algorithms.

Strongly Connected Subgraph

Undirected graphs:

Steiner Tree: Find a minimum weight connected subgraph that contains all k terminals.

Theorem [Dreyfus-Wagner 1972]

Steiner Tree can be solved in time $2^{O(k)} \cdot n^{O(1)}$.

Strongly Connected Subgraph

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Directed graphs:

Strongly Connected Subgraph: Find a minimum weight strongly connected subgraph that contains all k terminals.

Theorem

Strongly Connected Subgraph on general directed graphs

- can be solved in time $n^{O(k)}$ [Feldman and Ruhl 2006],
- is W[1]-hard parameterized by k [Guo, Niedermeier, Suchý 2011].

Strongly Connected Subgraph on planar graphs

Theorem [Feldman and Ruhl 2006]

Strongly Connected Subgraph can be solved in time $n^{O(k)}$ on general directed graphs.

Natural questions:

- Is there an $f(k) \cdot n^{o(k)}$ time algorithm on planar graphs?
- Is there an $f(k) \cdot n^{O(1)}$ time algorithm (i.e., is it fixed-parameter tractable) on planar graphs?

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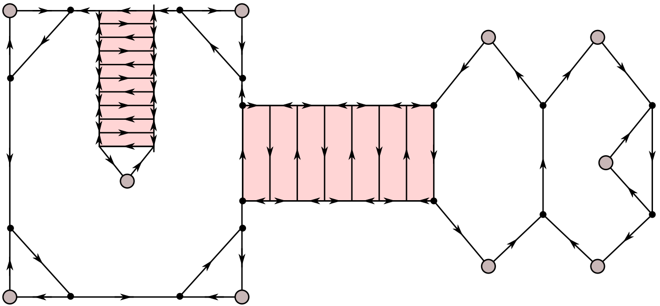
Theorem [Chitnis, Hajiaghayi, M.]

Strongly Connected Subgraph on planar directed graphs

- can be solved in time $2^{O(k \log k)} \cdot n^{O(\sqrt{k})}$,
- has no $f(k) \cdot n^{o(\sqrt{k})}$ time algorithm.

Optimum solutions

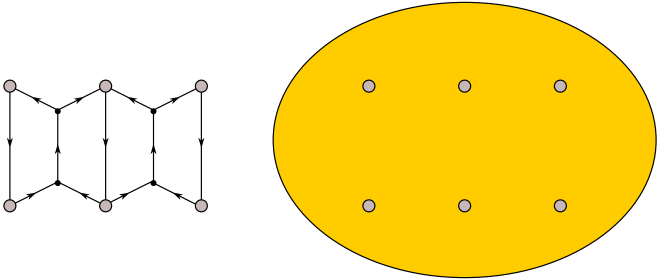
Closely looking at the $n^{O(k)}$ algorithm of [Feldman and Ruhl 2006] shows that an optimum solution consists of directed paths and "bidirectional strips":



With some work, we can bound the number paths/strips by $O(k)$.

Algorithm

[Ignore the bidirectional strips for simplicity]



- We guess the topology of the solution ($2^{O(k \log k)}$ possibilities).
- Treewidth of the topology is $O(\sqrt{k})$.
- We can find the best realization of this topology (matching the location of the terminals) in time $n^{O(\sqrt{k})}$.

Algorithm

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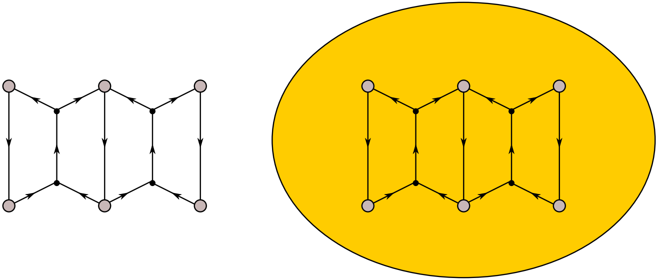

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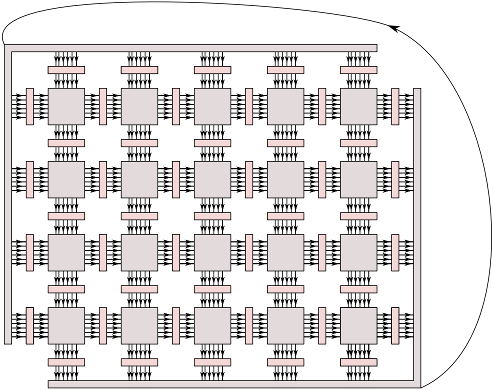
Lower bound

Theorem [Chitnis, Hajiaghayi, M.]

STRONGLY CONNECTED SUBGRAPH has no $f(k) \cdot n^{o(\sqrt{k})}$ time algorithm on planar directed graphs (assuming ETH).

The proof is by reduction from GRID TILING and complicated construction of gadgets.

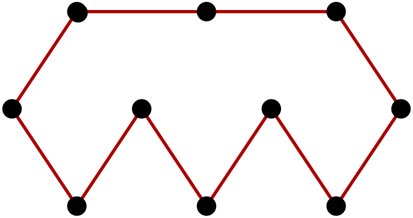


TSP

TSP

Input: A set T of cities and a distance function d on T

Output: A tour on T with minimum total distance



Theorem [Held and Karp]

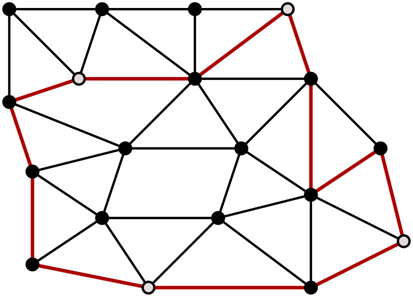
TSP with k cities can be solved in time $2^k \cdot n^{O(1)}$.

Dynamic programming:

Let $x(v, T')$ be the minimum length of path from v_{start} to v visiting all the cities $T' \subseteq T$.

Subset TSP on planar graphs

Assume that the cities correspond to a subset T of a planar graph and distance is measured in this planar graph.



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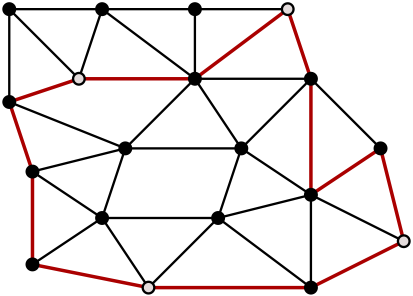

- Can be solved in time $2^{O(\sqrt{n})}$.
- Can be solved in time $2^k \cdot n^{O(1)}$.
- **Question:** Can we solve it in time $2^{O(\sqrt{k})} \cdot n^{O(1)}$?

Subset TSP on planar graphs

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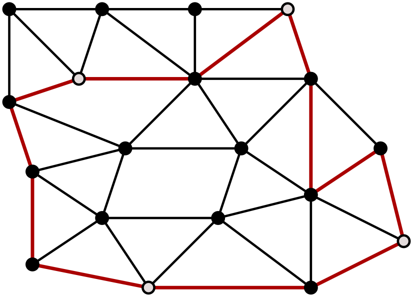
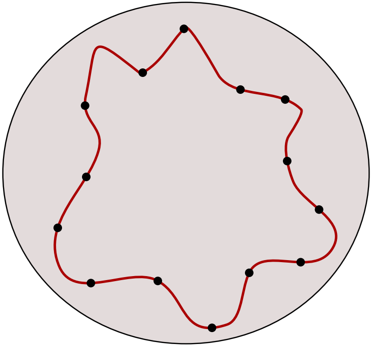


Theorem [Klein and M.]

Subset TSP for k cities in a (unit-weight) planar graph can be solved in time $2^{O(\sqrt{k}\log k)} \cdot n^{O(1)}$.

TSP and treewidth

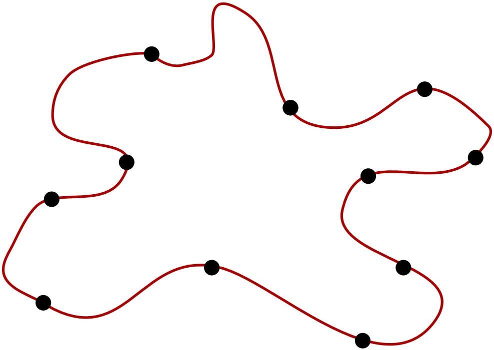
- We wanted to formulate the problem as finding a low treewidth subgraph.
- A cycle has treewidth 2, is this of any help?



Problem:
We have to remember the subset of cities visited by the partial tour (2^k possibilities).

c-change TSP

- c-change operation: removing c steps of the tour and connecting the resulting c paths in some other way.
- A solution is c-OPT if no c-change can improve it.
- We can find a c-OPT solution in $k^{O(c)} \cdot D$ time, where D is the maximum distance (if distances are integers).



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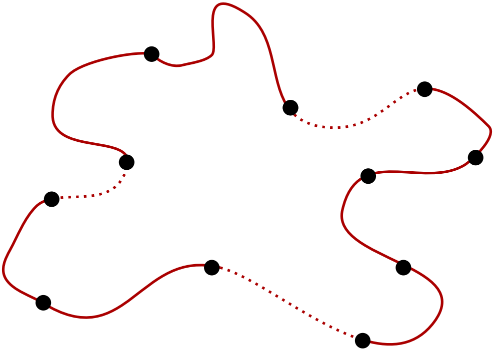

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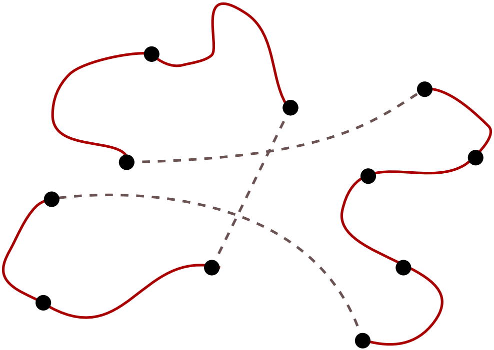
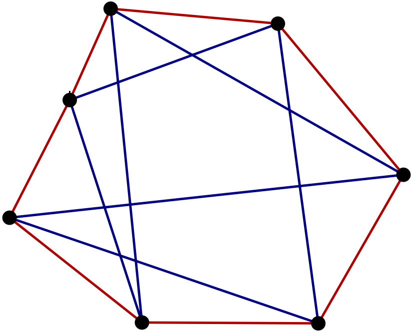


The treewidth bound

Consider the union of an optimum solution and a 4-OPT solution as a graph on k vertices:



Lemma

The union of an optimum solution and a 4-OPT solution has treewidth $O(\sqrt{k})$ [some techincal details omitted].

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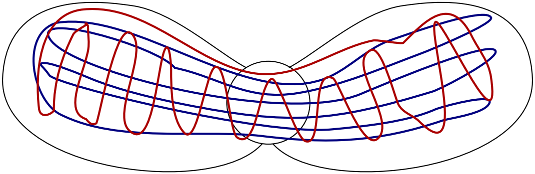
Lemma

The union of an optimum solution and a 4-OPT solution has treewidth $O(\sqrt{k})$ [some techincal details omitted].



- The union has separators of size $O(\sqrt{k})$.
- In each component, the set of cities visited by the optimum solution is nice: it is the same as what $O(\sqrt{k})$ segments of the 4-OPT tour visited ($k^{O(\sqrt{k})}$ possibilities).

Summary of Chapter 3

Parameterized problems where bidimensionality does not work.

- **Upper bounds:**
  Algorithms based on finding a bounded-treewidth subgraph.
  Treewidth bound is problem-specific:
  - k-Terminal Cut: dual solution has $O(k)$ branch vertices.
  - Planar Strongly Connected Subgraph: solution consists of $O(k)$ paths/strips.
  - Subset TSP on planar graphs: the union of an optimum solution and a 4-OPT solution has treewidth $O(k)$.
- **Lower bounds:**
  To rule out $f(k) \cdot n^{o(\sqrt{k})}$ time algorithms, we have to prove W[1]-hardness by reduction from Grid Tiling.

Conclusions

- **Chapter 1:** Subexponential algorithms using treewidth.
  - Algorithms: standard treewidth algorithms.
  - Lower bounds: textbook NP-completeness proofs + ETH.
- **Chapter 2:** Grid minors and bidimensionality.
  - Algorithms: standard treewidth algorithms + excluded grid theorem.
  - Lower bounds: textbook NP-completeness proofs + ETH.
- **Chapter 3:** Finding bounded-treewidth solutions.
  - Algorithms: the solution can be represented by a graph of treewidth $O(\sqrt{k})$.
  - Lower bounds: grid-like W[1]-hardness proofs to rule out $f(k) \cdot n^{o(\sqrt{k})}$ algorithms.

Conclusions

- A robust understanding of why certain problems can be solved in time $2^{O(\sqrt{n})}$ etc. on planar graphs and why the square root is best possible.

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Conclusions

- A robust understanding of why certain problems can be solved in time $2^{O(\sqrt{n})}$ etc. on planar graphs and why the square root is best possible.
- Going beyond the basic toolbox requires new problem-specific algorithmic techniques and hardness proofs with tricky gadget constructions.
- The lower bound technology on planar graphs cannot give a lower bound without a square root factor. Does this mean that there are matching algorithms for other problems as well?
  - $2^{O(\sqrt{k})} \cdot n^{O(1)}$ time algorithm for Steiner Tree with k terminals in a planar graph?
  - $2^{O(\sqrt{k})} \cdot n^{O(1)}$ time algorithm for finding a cycle of length **exactly** k in a planar graph?
  - ...