Complexity of clique coloring and related problems

Dániel Marx*
Institut für Informatik
Humboldt-Universität zu Berlin, Germany
dmarx@cs.bme.hu

23rd February 2011

Abstract

A k-clique-coloring of a graph G is an assignment of k colors to the vertices of G such that every maximal (i.e., not extendable) clique of G contains two vertices with different colors. We show that deciding whether a graph has a k-clique-coloring is Σ_2^p-complete for every $k \geq 2$. The complexity of two related problems are also considered. A graph is k-clique-choosable, if for every k-list-assignment on the vertices, there is a clique coloring where each vertex receives a color from its list. This problem turns out to be Π_3^p-complete for every $k \geq 2$. A graph G is hereditary k-clique-colorable if every induced subgraph of G is k-clique-colorable. We prove that deciding hereditary k-clique-colorability is also Π_3^p-complete for every $k \geq 3$. Therefore, for all the problems considered in the paper, the obvious upper bound on the complexity turns out to be the exact class where the problem belongs.

1 Introduction

Clique coloring is a variant of the classical vertex coloring. In this problem, we have to satisfy weaker requirements than in ordinary vertex coloring: instead of requiring that the two end points of each edge have two different colors, we only require that every inclusionwise maximal (not extendable) clique contains at least two different colors. It is possible that a graph is k-clique-colorable, but its chromatic number is greater than k. For example, a clique of size n is 2-clique-colorable, but its chromatic number is n. For recent results on clique coloring, see [1, 5, 2, 7, 8].

Clique coloring can be also thought of as coloring the clique hypergraph. Given a graph $G(V, E)$, the *clique hypergraph* $\mathscr{C}(G)$ of G is defined on the same vertex set V, and a subset $V' \subseteq V$ is a hyperedge of $\mathscr{C}(G)$ if and only if $|V'| > 1$ and V' induces an inclusionwise maximal clique of G. Duffus et al. [3] raised the question of k-coloring the hypergraph $\mathscr{C}(G)$, that is, assigning k colors to the vertices of the $\mathscr{C}(G)$ such that every hyperedge contains at least two colors. Clearly, a graph G is k-clique-colorable if and only if the hypergraph $\mathscr{C}(G)$ is k-colorable. Note that if the graph G is triangle-free, then the maximal cliques are the edges, hence $\mathscr{C}(G)$ is the same as G and therefore in this case G is k-clique-colorable if and only if it is k-vertex-colorable.

In general, clique coloring can be a very different problem from ordinary vertex coloring. The most notable difference is that clique coloring is not a hereditary property: it is possible that a graph is k-clique-colorable, but it has an induced subgraph that is not. The reason why this can happen is that deleting vertices can create new inclusionwise maximal cliques: it is possible that in the original

*Research supported in part by ERC Advanced grant DMMCA, the Alexander von Humboldt Foundation, and the Hungarian National Research Fund (Grant Number OTKA 67651).

graph a clique is contained in a larger clique, but after deleting some vertices this clique becomes maximal. Another difference is that a large clique is not an obstruction for clique colorability: even 2-clique-colorable graphs can contain arbitrarily large cliques. In fact, it is conjectured that every prefect graph (or perhaps every odd-hole free graph) is 3-clique-colorable (see [1]). There are no counterexamples known for this conjecture, but so far only some special cases have been proved.

In this paper we prove complexity results for clique coloring and related problems. Clique coloring is harder than ordinary vertex coloring: it is coNP-complete even to check whether a 2-clique-coloring is valid [1]. The complexity of 2-clique-colorability is investigated in [8], where they show that it is NP-hard to decide whether a perfect graph is 2-clique-colorable. However, it is not clear whether this problem belongs to NP. A valid 2-clique-coloring is not a good certificate, since we cannot verify it in polynomial time: as mentioned above, it is coNP-complete to check whether a 2-clique-coloring is valid. In Section 3 we determine the exact complexity of the problem: we show that it is Σ_2^p-complete to check whether a graph is 2-clique-colorable.

A graph is *k-clique-choosable* if whenever a list of k colors is assigned to each vertex (the lists of the different vertices do not have to be the same), then the graph has a clique coloring where the color of each vertex is taken from its list. This notion is an adaptation of choosability introduced for graphs independently by Erdős, Rubin, and Taylor [4] and by Vizing [13]. In [10] it is shown that every planar or projective planar graph is 4-clique-choosable. In Section 4 we investigate the complexity of clique-choosability. It turns out that the complexity of clique-choosability lies higher in the polynomial hierarchy than either clique-coloring or choosability: we show that for every $k \geq 2$ it is Π_3^p-complete to decide whether a graph is k-clique-choosable or not.

As mentioned above, a k-clique-colorable graph can contain an induced subgraph that is not k-clique-colorable. Therefore, it is natural to investigate graphs that are *hereditary k-clique-colorable:* graphs where every induced subgraph is k-clique-colorable. For example, Bacsó et al. [1] asked the complexity of recognizing hereditary 2-clique-colorable graphs. While we cannot answer this question for the case of 2 colors, in Section 5 we show that recognizing such graphs is Π_3^p-complete for every $k \geq 3$.

The results of the paper determine the exact complexity of certain fairly natural coloring problems. It turns out that these problems are complete for higher levels the polynomial hierarchy, which is interesting, since there are relatively few natural complete problems known for these classes (see [12]). These completeness results give us more information than knowing that the problems are NP-hard, because they also rule out the possibility that the problems are in NP or coNP (unless the polynomial hierarchy collapses). The message of these results is that the problems are "as hard as possible": they are complete for the classes they obviously belong to. If we know that a problem belongs to, say, Π_3^p, then with some clever insight or structural understanding we might be able to show that the problem actually belongs to a class on a lower level, e.g., Π_2^p or NP. However, for the problems considered in the paper, the completeness results show that there are no such insights to look for.

2 Preliminaries

In this section we introduce notation and make some preliminary observations about clique colorings. We also introduce the complexity classes that appear in our completeness results.

Clique coloring. A *clique* is a complete subgraph of at least 2 vertices. A clique is *maximal* if it cannot be extended to a larger clique. An edge is *flat* if it is not contained in any triangle. Since a flat edge is a maximal clique of size 2, the two end vertices of a flat edge receive different colors in every proper clique coloring. The *core* of G is the subgraph containing only the flat edges. Clearly,

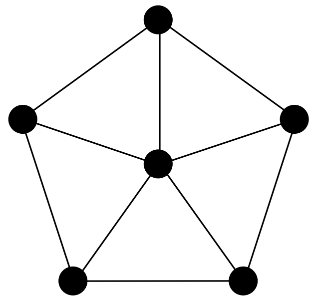


Figure 1: The graph is 2-clique-colorable, but it does not remain 2-clique-colorable after deleting the central vertex.

a proper clique coloring of G is a proper vertex coloring of the core of G. A vertex v of G is *simple* if it is not contained in any triangle, or, equivalently, all the edges incident to it are flat.

Unlike k-vertex-coloring, a k-clique-coloring of the graph G does not necessarily give a proper k-clique-coloring for the induced subgraphs of G. It is possible that deleting vertices from G makes it impossible to k-clique-color it. For example, the 5-wheel shown in Figure 1 is 2-clique-colorable, but after deleting the central vertex, the remaining C_5 is not (since it is triangle free and not 2-vertex-colorable). On the other hand, the following proposition shows that G remains k-clique-colorable if we delete only simple vertices:

Proposition 1. *Let $S \subseteq V$ be a set of simple vertices in $G(V,E)$. If ψ is a proper clique coloring of G, then ψ induces a proper clique coloring of $G \setminus S$.*

Proof. Consider the coloring ψ' of $G \setminus S$ induced by ψ. If ψ' is not a clique coloring of G, then there is a monochromatic maximal clique K in $G \setminus S$. This is not a maximal clique in G, otherwise ψ would not be a proper k-clique-coloring. Therefore, K is properly contained in a maximal clique K' of G. Since K' is not a maximal clique of $G \setminus S$, it contains at least one vertex v of S. However, K' has size at least 3, contradicting the assumption that vertex $v \in S$ is simple. □

The following two propositions will also be useful:

Proposition 2. *Let $S \subseteq V$ be an arbitrary subset of the vertices in $G(V,E)$. If ψ induces a proper clique coloring of $G \setminus S$, and every vertex in S has different color from its neighbors, then ψ is a proper clique coloring of G.*

Proof. Suppose that G has a monochromatic maximal clique K in coloring ψ. If K contains a vertex $v \in S$, then K is not monochromatic, as v has different color from its neighbors. Thus K is completely contained in $G \setminus S$ and hence it is a maximal clique of $G \setminus S$. This contradicts the assumption that ψ induces a proper clique coloring of $G \setminus S$. □

Proposition 3. *Let $S \subseteq V$ be the set of simple vertices in $G(V,E)$. If ψ is a k-clique-coloring of G, then it induces a proper k-vertex-coloring of $G[S]$, the graph induced by S.*

Proof. Observe that every edge in $G[S]$ is a flat edge and hence they are maximal cliques in G. Therefore, ψ assigns different colors to the end vertices of every edge in $G[S]$, thus it induces a proper k-vertex-coloring of $G[S]$. □

Polynomial hierarchy. We briefly recall the definitions of the complexity classes in the polynomial hierarchy; for more details and background, the reader is referred to any standard textbook on computational complexity, e.g., [11]. The complexity class $\Sigma_2^p = \mathrm{NP}^{\mathrm{NP}}$ contains those problems that can be solved by a polynomial-time nondeterministic Turing machine equipped with an NP-oracle. An oracle can be thought of as a subroutine that is capable of solving a certain problem in one step. More formally, let L be a language. A Turing machine equipped with an L oracle has a special tape called the oracle tape. Whenever the Turing machine wishes, it can ask the oracle whether the contents of the oracle tape is a word from L or not (there are special states for this purpose). Asking the oracle counts as only one step. If the language L is simple, then this oracle does not help very much. On the other hand, if L is a computationally hard language, then this oracle can increase the power of the Turing machine. We say that a Turing machine is equipped with an NP-oracle, if the language L is NP-complete. Note that here it is not really important which particular NP-complete language is L: any NP-complete language gives the same power to the Turing machine, up to a polynomial factor. Thus the class Σ_2^p contains those problems that can be solved by a polynomial-time nondeterministic Turing machine if one NP-complete problem (say, the satisfiability problem) can be solved in constant time.

Similarly to SAT, which is the canonical complete problem for NP, the problem QSAT_2 is the canonical Σ_2^p-complete problem:

2-Quantified Satisfiability ($\mathbf{QSAT_2}$)

Input:	An $n+m$ variable boolean 3DNF formula $\varphi(\mathbf{x},\mathbf{y})$ (where $\mathbf{x} = (x_1,\ldots,x_n)$, $\mathbf{y} = (y_1,\ldots,y_m)$)
Question:	Is there a vector $\mathbf{x} \in \{0,1\}^n$ such that for every $\mathbf{y} \in \{0,1\}^m$, $\varphi(\mathbf{x},\mathbf{y})$ is true? (Shorthand notation: Is it true that $\exists\mathbf{x}\forall\mathbf{y}\varphi(\mathbf{x},\mathbf{y})$?)

Recall that a 3DNF (disjunctive normal form) formula is a disjunction of terms, where each term is a conjunction of 3 literals. The complexity class Π_2^p contains those languages whose complements are in Σ_2^p.

The class Σ_3^p contains the problems solvable by a polynomial-time nondeterministic Turing machine equipped with a Σ_2^p oracle. The following problem is complete for Σ_3^p:

3-Quantified Satisfiability ($\mathbf{QSAT_3}$)

Input:	An $n+m+p$ variable boolean 3CNF formula $\varphi(\mathbf{x},\mathbf{y},\mathbf{z})$ ($\mathbf{x} = (x_1,\ldots,x_n)$, $\mathbf{y} = (y_1,\ldots,y_m)$, $\mathbf{z} = (z_1,\ldots,z_p)$)
Question:	Is there a vector $\mathbf{x} \in \{0,1\}^n$ such that for every $\mathbf{y} \in \{0,1\}^m$, there is a vector $\mathbf{z} \in \{0,1\}^p$ with $\varphi(\mathbf{x},\mathbf{y},\mathbf{z})$ true? (Shorthand notation: Is it true that $\exists\mathbf{x}\forall\mathbf{y}\exists\mathbf{z}\varphi(\mathbf{x},\mathbf{y},\mathbf{z})$?)

Similarly to NP, the classes Σ_2^p, Σ_3^p, etc. have equivalent characterizations using certificates. A problem is in NP if there is a polynomial-size certificate for each yes-instance, and verifying this certificate is a problem in P. The characterization of the class Σ_2^p is similar, but here we require only that verifying the certificate is in coNP(cf. [11] for more details). For example, to see that QSAT_2 is in Σ_2^p, observe that if the formula $\varphi(\mathbf{x},\mathbf{y})$ is a yes-instance, then an assignment $\mathbf{x}_0$ satisfying $\forall\mathbf{y}\varphi(\mathbf{x}_0,\mathbf{y})$ is a good certificate. To verify the certificate, we have to check that $\forall\mathbf{y}\varphi(\mathbf{x}_0,\mathbf{y})$ holds, or equivalently, we have to check whether there is a $\mathbf{y}$ such that $\varphi(\mathbf{x}_0,\mathbf{y})$ is false. This verification problem is in coNP, hence QSAT_2 is in Σ_2^p. For the proof that QSAT_2 is hard for Σ_2^p, see e.g., [11].

A problem is in Π_2^p if there is a polynomial-size certificate for every no-instance, and verifying this certificate is a problem in NP. The higher levels can be obtained by requiring that verifying the certificate is a problem on the previous level: for example, for Π_3^p, we require that verifying the certificates for the no-instances is a problem in Σ_2^p.

Repeating this construction, we obtain the polynomial hierarchy: let Σ_{i+1}^p contain those problems that can be solved by a polynomial-time nondeterministic Turing machine equipped with a Σ_i^p-oracle. The class Π_i^p contains a language if its complement is in Σ_i^p. The definition of these classes might seem very technical, but as the results in this paper and in the compendium [12] demonstrate, there exist fairly natural problems whose complexities are exactly characterized by these classes.

3 Complexity of clique coloring

In this section we investigate the complexity of the following problem:

***k*-Clique-Coloring**
Input: A graph $G(V, E)$
Question: Is there a *k-clique-coloring* of G, i.e., an assignment c: $V \to \{1, 2, \ldots, k\}$ such that for every maximal clique K of G, there are two vertices $u, v \in K$ with $c(u) \neq c(v)$?

Unlike ordinary vertex coloring, which is easy for two colors, this problem is hard even for $k = 2$:

Theorem 4. *2-Clique-Coloring is* Σ_2^p*-complete.*

Proof. To see that k-Clique-Coloring is in Σ_2^p, notice that the problem of verifying whether a coloring is a proper k-clique-coloring is in coNP: a monochromatic maximal clique is a polynomial-time verifiable certificate that the coloring is *not* proper. A proper k-clique-coloring is a certificate that the graph is k-clique-colorable, and this certificate can be verified in polynomial time if an NP-oracle is available. Thus clearly the problem is in $\text{NP}^{\text{NP}} = \Sigma_2^p$.

We prove that 2-Clique-Coloring is Σ_2^p-hard by a reduction from QSAT_2. For a formula $\varphi(\mathbf{x}, \mathbf{y})$, we construct a graph G that is 2-clique-colorable if and only if there is an $\mathbf{x} \in \{0, 1\}^n$ such that $\varphi(\mathbf{x}, \mathbf{y})$ is true for every $\mathbf{y} \in \{0, 1\}^m$. Graph G has $4(n + m) + 2q$ vertices, where q is the number of terms in φ. For every variable x_i ($1 \le i \le n$), the graph contains a path on 4 vertices x_i, x'_i, $\overline{x}'_i$, $\overline{x}_i$. For every variable y_j ($1 \le j \le m$), the graph contains 4 vertices y_j, y'_j, $\overline{y}_j$, $\overline{y}'_j$. Vertices y'_j and y_j are adjacent, and vertices $\overline{y}'_j$ and $\overline{y}_j$ are adjacent for every $1 \le j \le m$. Furthermore, the vertices x_i, $\overline{x}_i$, y_j, $\overline{y}_j$ form a clique of size $2(n+m)$ minus a matching: there are no edges between x_i and $\overline{x}_i$ ($1 \le i \le n$), and between y_j and $\overline{y}_j$ ($1 \le j \le m$).

For every term P_ℓ ($1 \le \ell \le q$) of the DNF formula φ, the graph contains two vertices p_ℓ and p'_ℓ. These vertices form a path p_1, p'_1, p_2, p'_2, $\ldots$, p_q, p'_q of $2q$ vertices. For every $1 \le i \le m$, vertex p'_q is connected to y'_i and $\overline{y}'_i$. Vertex p_ℓ is connected to those literals that correspond to literals not contradicting P_ℓ. That is, if x_i (resp., $\overline{x}_i$) is in P_ℓ, then p_ℓ and x_i (resp., $\overline{x}_i$) are connected. (We can assume that at most one of x_i and $\overline{x}_i$ appears in a term, otherwise this term is never satisfied and can be removed without changing the problem.) If neither x_i nor $\overline{x}_i$ appears in P_ℓ, then p_ℓ is connected to both x_i and $\overline{x}_i$. Vertices y_j and $\overline{y}_j$ are connected to p_ℓ in a similar fashion. This completes the description of the graph G. An example is shown in Figure 2. Notice that $\varphi(\mathbf{x}, \mathbf{y})$ is true for some variable assignment $\mathbf{x}$, $\mathbf{y}$ if and only if there is a vertex p_ℓ such that it is connected to all the $n + m$ vertices corresponding to the true literals of $\mathbf{x}$, $\mathbf{y}$.

Assume that $\mathbf{x} \in \{0, 1\}^n$ is such that $\varphi(\mathbf{x}, \mathbf{y})$ is true for every $\mathbf{y} \in \{0, 1\}^m$. We define a 2-clique-coloring of the graph G based on $\mathbf{x}$. Vertices p_ℓ ($1 \le \ell \le q$) and y'_j, $\overline{y}'_j$ ($1 \le j \le m$) are colored white. If x_i is true in $\mathbf{x}$, then vertices x'_i and $\overline{x}_i$ are colored white, while vertices x_i and $\overline{x}'_i$ are

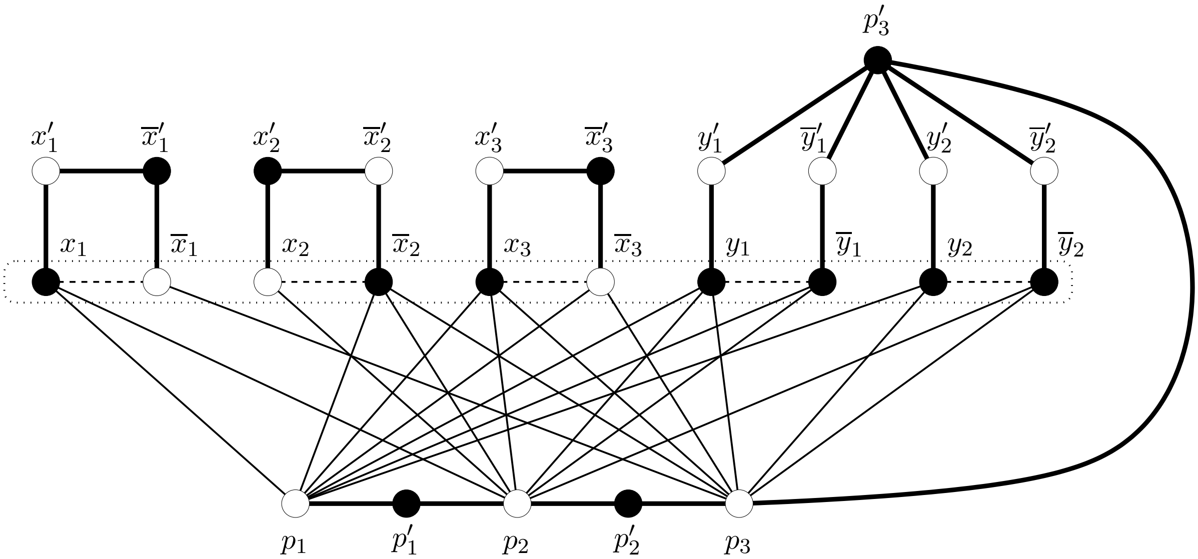


Figure 2: The construction for the formula $\varphi = (x_1 \wedge \overline{x}_2 \wedge y_2) \vee (x_1 \wedge x_3 \wedge \overline{y}_2) \vee (\overline{x}_1 \wedge \overline{x}_2 \wedge y_1)$. The vertices x_1, $\overline{x}_1$, x_2, $\dots$, y_2, $\overline{y}_2$ form a clique minus the five dashed edges. The strong edges are all flat. The coloring shown on the figure is a proper 2-clique-coloring, implying that $x_1 = 1$, $x_2 = 0$, $x_3 = 1$ satisfy φ regardless of the values of y_1 and y_2.

black; if x_i is false in $\mathbf{x}$, then vertices x'_i and $\overline{x}_i$ are colored black, and vertices x_i, $\overline{x}'_i$ are white. The remaining vertices are black.

It can be verified that the coloring defined above properly colors every flat edge of the graph. Now suppose that there is a monochromatic maximal clique K of size greater than two. Since vertices x'_i, $\overline{x}'_i$, y'_j, $\overline{y}'_j$, p'_ℓ are simple vertices, they cannot appear in K. Assume first that K is colored white, then it contains some of the $2n$ vertices x_i, $\overline{x}_i$ ($1 \le i \le n$), and at most one of the vertices p_ℓ ($1 \le \ell \le q$) (the vertices y_j, $\overline{y}_j$ are all black). However, this clique is not maximal: p_ℓ is connected to at least one of y_1 and $\overline{y}_1$, therefore K can be extended by one of these two vertices. Now suppose that K is colored black, then it can contain only vertices of the form x_i, $\overline{x}_i$, y_j, $\overline{y}_j$. Furthermore, for every $1 \le i \le n$, clique K contains exactly one of x_i and $\overline{x}_i$, and for every $1 \le j \le m$, clique K contains exactly one of y_j and $\overline{y}_j$, otherwise K is not a maximal clique. Define the vector $\mathbf{y}$ such that variable y_j is true if and only if vertex y_j is in K. By the assumption on $\mathbf{x}$, the value of $\varphi(\mathbf{x}, \mathbf{y})$ is true, therefore there is a term P_ℓ that is satisfied in $\varphi(\mathbf{x}, \mathbf{y})$. We claim that $K \cup \{p_\ell\}$ is a clique, contradicting the maximality of K. To see this, observe that $x_i \in K$ if and only if the value of variable x_i is true in $\mathbf{x}$. Therefore, K contains those vertices that correspond to true literals in the assignment $(\mathbf{x}, \mathbf{y})$. This assignment satisfies term P_ℓ, thus these literals do not contradict P_ℓ. By construction, these vertices are connected to p_ℓ, and $K \cup \{p_\ell\}$ is indeed a clique.

Now assume that G is 2-clique-colored, and suppose without loss of generality that p_1 is white. Since $\{p_\ell, p'_\ell\}$ and $\{p'_\ell, p_{\ell+1}\}$ are maximal cliques, p_ℓ is white and p'_ℓ is black for every $1 \le \ell \le q$. Because $\{p'_q, y'_j\}$ and $\{p'_q, \overline{y}'_j\}$ are maximal cliques for every $1 \le j \le m$, every y'_j and every $\overline{y}'_j$ is white. Since $\{y_j, y'_j\}$, $\{y'_j, \overline{y}'_j\}$ are maximal cliques, we also have that y_j and $\overline{y}_j$ are colored black for every $1 \le j \le m$. Finally, $\{x_i, x'_i\}$, $\{x'_i, \overline{x}'_i\}$, $\{\overline{x}'_i, \overline{x}_i\}$ are also maximal cliques, thus x_i and $\overline{x}_i$ have different color.

Define the vector $\mathbf{x}$ as variable x_i is true if and only if the color of vertex x_i is black. We show that $\varphi(\mathbf{x}, \mathbf{y})$ is true for every $\mathbf{y}$. Let K be the set of $n + m$ vertices that correspond to the true literals in the assignment $\mathbf{x}$, $\mathbf{y}$; note that K induces a clique in G. By the way $\mathbf{x}$ is defined and

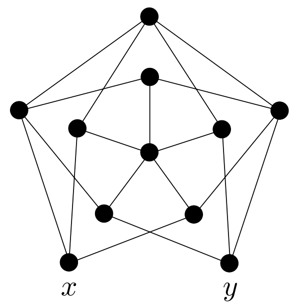


Figure 3: The graph D_4, which is the Mycielski graph M_4 (the Grötzch graph) minus the edge xy. In every 3-vertex-coloring, x and y have the same color.

from the fact that every y_j, $\overline{y}_j$ is black, we have that every vertex of K is black. Since the coloring is a proper 2-clique-coloring, clique K is not a maximal clique of G. The only way to increase it is by adding a p_ℓ, that is, some p_ℓ is adjacent with every vertex representing a true literal. By construction, this means that none of the true literals contradict the term P_ℓ, implying that $\varphi(\mathbf{x}, \mathbf{y})$ is true. □

We show that the hardness result holds for every $k > 2$ as well. The proof is by reducing k-clique-colorability to $(k+1)$-clique-colorability. The reduction uses the Mycielski graphs as gadgets.

For every $k \geq 2$, the construction of Mycielski gives a triangle-free graph M_k with chromatic number k. For completeness, we recall the construction here. For $k = 2$, the graph M_2 is a K_2, i.e., two vertices connected by an edge. To obtain the graph M_{k+1}, take a copy of M_k, let $v_1, v_2, \ldots, v_n$ be its vertices. Add $n+1$ new vertices $u_1, u_2, \ldots, u_n, w$, connect u_i to the neighbors of v_i in M_k, and connect w to every vertex u_i. It can be shown that M_{k+1} is triangle-free, and $\chi(M_{k+1}) = \chi(M_k)+1$. Moreover, M_k is edge-critical (see [9, Problem 9.18]): for every edge e of M_k, the graph $M_k \setminus e$ is $(k-1)$-colorable. Remove an arbitrary edge $e = xy$ of M_k and denote by D_k the resulting graph (see D_4 in Figure 3). It follows that in every $(k-1)$-coloring of D_k, the vertices x and y have the same color, otherwise it would be a proper $(k-1)$-coloring of M_k.

The following corollary shows that k-Clique-Coloring remains Σ_2^p-complete for every $k > 2$ (note that the problem becomes trivial for $k = 1$).

Corollary 5. *k-Clique-Coloring is Σ_2^p-complete for every $k \geq 2$.*

Proof. For every $k \geq 2$, we give a polynomial-time reduction from k-Clique-Coloring to $(k+1)$-Clique-Coloring. By Theorem 4, 2-Clique-Coloring is Σ_2^p-complete, thus the theorem follows by induction.

Let G be a graph with n vertices $v_1, v_2, \ldots, v_n$. Add $n+1$ vertices $u_1, u_2, \ldots, u_n, w$, and connect every vertex u_i with v_i. Furthermore, add n copies of the graph D_{k+2} such that vertex x of the i-th copy is identified with w, and vertex y is identified with u_i. Denote the new vertices added to G by W, observe that every vertex in W is simple. We claim that the resulting graph G' is $(k+1)$-clique-colorable if and only if G is k-clique-colorable.

Assume first that there is a $(k+1)$-clique-coloring ψ of G', we show that it induces a k-clique-coloring of G. By Prop. 3, $G'[W]$ is $(k+1)$-vertex-colored in ψ, thus the construction of the graph D_{k+2} implies that $\psi(w) = \psi(u_1) = \cdots = \psi(u_n) = \alpha$, and none of the vertices $v_1, v_2, \ldots, v_n$ has color α. Hence ψ uses at most k colors on $G = G' \setminus W$, and by Prop. 1, it is a proper k-clique-coloring.

On the other hand, if there is a proper k-clique-coloring of G, then color the vertices $u_1, \ldots, u_n, w$ with color $k+1$, and extend this coloring to the copies of the graph D_{k+2} in such a way that the coloring is a proper $(k+1)$-vertex-coloring on every copy of D_{k+2}. By Prop. 2, this results in a proper $(k+1)$-clique coloring of G', since each vertex in W has different color from its neighbors. ☐

4 Clique choosability

In this section we investigate the list coloring version of clique coloring. In a k-clique-coloring the vertices can use only the colors 1, 2, ..., k. In the list coloring version, each vertex v has a set $L(v)$ of k admissible colors, the color of the vertex has to be selected from this set. A list assignment L is a *k-list assignment* if the size of $L(v)$ is k for every vertex v. We say that a graph $G(V,E)$ is *k-clique-choosable*, if for every k-list assignment L: $V \to 2^{\mathbb{N}}$ there is a proper clique coloring ψ of G with $\psi(v) \in L(v)$. We investigate the computational complexity of the following problem:

k-Clique-Choosability
Input: A graph $G(V,E)$
Question: Is G k-clique-choosable?

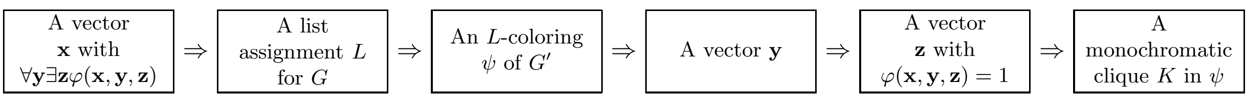
Rubin [4] characterized 2-vertex-choosable graphs. In particular, trees and cycles of even length are 2-vertex-choosable. The characterization can be turned into a polynomial-time algorithm for recognizing 2-vertex-choosable graphs. Therefore, 2-vertex-coloring and 2-vertex-choosability have the same complexity, both can be solved in polynomial time. However, 3-vertex-choosability is harder than 3-vertex-coloring: the former is Π_2^p-complete [6], whereas the latter is "only" NP-complete. The situation is different in the case of clique coloring: we show that the 2-Clique-Choosability problem is more difficult than 2-Clique-Coloring, it lies one level higher in the polynomial hierarchy.

Theorem 6. *2-Clique-Choosability is Π_3^p-complete.*

Proof. Notice first that deciding whether a graph has a proper clique coloring with the given lists is in Σ_2^p: a proper clique coloring is a certificate proving that such a coloring exists, and verifying this certificate is in coNP. Therefore, k-Clique-Choosability is in Π_3^p: if the graph is not k-clique-choosable, then an uncolorable list assignment exists, which is a Σ_2^p certificate showing that the graph is not k-clique-choosable.

We prove that the 2-Clique-Choosability problem is Π_3^p-hard by reducing QSAT$_3$ to the complement of 2-Clique-Choosability. That is, for every 3CNF formula $\varphi(\mathbf{x},\mathbf{y},\mathbf{z})$, a graph G is constructed in such a way that G is *not* 2-clique-choosable if and only if $\exists\mathbf{x}\forall\mathbf{y}\exists\mathbf{z}\varphi(\mathbf{x},\mathbf{y},\mathbf{z})$ holds.

Before describing the construction of the graph G in detail, we present the outline of the proof. Assume first that a vector $\mathbf{x}$ exists with $\forall\mathbf{y}\exists\mathbf{z}\varphi(\mathbf{x},\mathbf{y},\mathbf{z})$, it has to be shown that G is not 2-clique-choosable. Based on this vector $\mathbf{x}$, we define a 2-list assignment L of G, and claim that G is not clique colorable with this assignment. If, on the contrary, such a coloring ψ exists, then a vector $\mathbf{y}$ is defined based on this coloring. By assumption, there is a vector $\mathbf{z}$ with $\varphi(\mathbf{x},\mathbf{y},\mathbf{z})$ true. Based on vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, we construct a clique K that is monochromatic in ψ, a contradiction. This direction of the proof is summarized in the following diagram:



The other direction is to prove that if G is not 2-clique-choosable, then $\exists\mathbf{x}\forall\mathbf{y}\exists\mathbf{z}\varphi(\mathbf{x},\mathbf{y},\mathbf{z})$. The outline of this direction is the following. Given an uncolorable 2-list assignment L, we define a vector $\mathbf{x}$. Assume indirectly that there is a vector $\mathbf{y}$ with $\nexists\mathbf{z}\varphi(\mathbf{x},\mathbf{y},\mathbf{z})$. Based on this vector $\mathbf{y}$, an

L-coloring ψ of G is defined. By assumption, ψ is not a proper clique coloring, thus it contains a monochromatic maximal clique K. Based on K, a vector $\mathbf{z}$ is constructed satisfying $\varphi(\mathbf{x}, \mathbf{y}, \mathbf{z})$, a contradiction. The summary of this direction:

A list assignment L for G	$\Rightarrow$	A vector $\mathbf{x}$	$\Rightarrow$	A vector $\mathbf{y}$ with $\nexists \mathbf{z}\varphi(\mathbf{x}, \mathbf{y}, \mathbf{z})$	$\Rightarrow$	An L-coloring ψ of G	$\Rightarrow$	A monochromatic clique K in ψ	$\Rightarrow$	A vector $\mathbf{z}$ with $\varphi(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 1$

Now we define graph G. The three different types of variables are represented by vertices that have different roles. The vertices representing the x-variables can be forced by a *list assignment* to a coloring representing an assignment to the variable. The color of the vertices representing the y-variables cannot be forced to a fixed color, thus a *coloring* can freely choose a coloring that represents an assignment to a variable. Finally, the vertices corresponding to the z-variables can be forced to have the same color, thus these vertices play a role only in the selection of the monochromatic *maximal clique*.

For every variable x_i ($1 \leq i \leq n$), the graph G contains a cycle on 4 the vertices x_i, x'_i, $\overline{x}_i$, $\overline{x}'_i$ (see Figure 4). For every variable y_j ($1 \leq j \leq m$), there is a path on 4 the vertices y_j, y'_j, $\overline{y}'_j$, $\overline{y}_j$. For every variable z_k ($1 \leq k \leq p$), there are two 4-cycles z_k, z^1_k, z^2_k, z^3_k and $\overline{z}_k$, $\overline{z}^1_k$, $\overline{z}^2_k$, $\overline{z}^3_k$. For every clause C_ℓ of φ ($1 \leq \ell \leq q$), there is a 4-cycle c_ℓ, c^1_ℓ, c'_ℓ, c^2_ℓ. The edges defined so far are all flat edges in G, they form the core of G. The following edges appear in cliques greater than 2. The $2n+2m+2p+q$ vertices $H = \{x_i, \overline{x}_i, y_j, \overline{y}_j, z_k, \overline{z}_k, c'_\ell\}$ almost form a clique: the $n+m+p$ edges $x_i\overline{x}_i$, $y_i\overline{y}_i$, $z_k\overline{z}_k$ are missing from the graph. Observe that every edge in H is contained in a triangle: c'_1 is adjacent to every other vertex in H. For every $1 \leq \ell \leq q$, vertex c_ℓ is connected to every vertex that corresponds to a literal *not* satisfying clause C_ℓ. That is, if variable x_i does not appear in clause C_ℓ, then c_i is connected to x_i and $\overline{x}_i$, and if variable x_i appears in C_ℓ (but $\overline{x}_i$ does not), then c_i is connected to $\overline{x}_i$. Note that we can assume that a variable and its negation do not appear in the same clause, since in this case every assignment satisfies the clause. Thus c_ℓ is adjacent to at least one of the two literals representing each variable. As the vertices representing different variables are adjacent and there are at least two variables in ϕ, the edges connecting c_ℓ and H are not flat edges. This completes the description of the graph G.

The maximal cliques of G are of the following type. Every flat edge is a maximal clique of size 2. Among the vertices outside H, only $\{c_1, \ldots, c_q\}$ are not simple and they form an independent set. Thus if K is of size greater than 2, then K contains at most one vertex of c_ℓ and $K \setminus \{c_\ell\}$ is fully contained in H. Furthermore, $K \setminus \{c_\ell\}$ contains exactly one of x_i and $\overline{x}_i$, exactly one of y_j and $\overline{y}_j$, and exactly one of z_j and $\overline{z}_k$ for every i, j, and k: for example, c_ℓ is connected to at least one of x_i and $\overline{x}_i$, thus if neither of this two vertices is in the clique, then the clique cannot be maximal.

Assume first that there is an $\mathbf{x} \in \{0,1\}^n$ such that $\forall \mathbf{y} \exists \mathbf{z} \varphi(\mathbf{x}, \mathbf{y}, \mathbf{z})$ holds, we show that there is a list assignment L to the vertices of G such that no proper clique coloring is possible with these lists. We make the following list assignments:

- If x_i is true in $\mathbf{x}$, then set $L(x_i) = \{1,2\}$, $L(x'_i) = \{2,3\}$, $L(\overline{x}_i) = \{1,3\}$, $L(\overline{x}'_i) = \{1,2\}$. This list assignment forces x_i to color 1: giving color 2 to x_i would imply that there is color 3 on x'_i and there is color 1 on $\overline{x}'_i$, which means that there is no color left for $\overline{x}_i$.
- If x_i is false, then set $L(x_i) = \{1,3\}$, $L(x'_i) = \{2,3\}$, $L(\overline{x}_i) = \{1,2\}$, $L(\overline{x}'_i) = \{1,2\}$, forcing $\overline{x}_i$ to color 1.
- For every $1 \leq k \leq p$, we set $L(z_k) = L(\overline{z}_k) = \{1,2\}$, $L(z^1_k) = L(\overline{z}^1_k) = \{2,3\}$, $L(z^2_k) = L(\overline{z}^2_k) = \{1,3\}$, $L(z^3_k) = L(\overline{z}^3_k) = \{1,2\}$, forcing z_k and $\overline{z}_k$ to color 1.

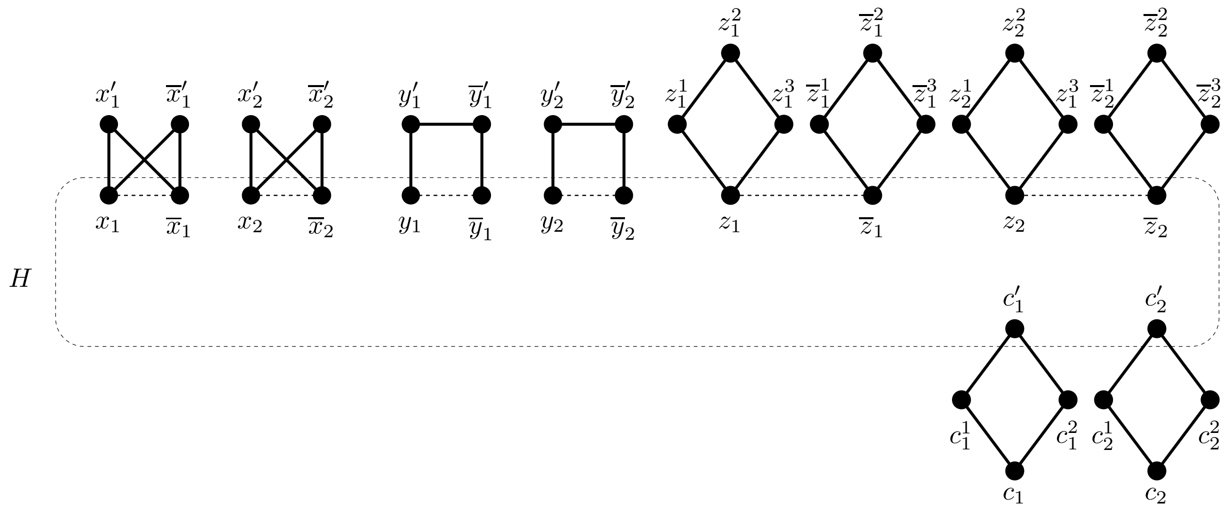


Figure 4: The structure of graph G in the proof of Theorem 6 for $n = 2$, $m = 2$, $p = 2$, $q = 3$. The set H almost forms a clique, the pairs connected by dashed lines are not neighbors. For the sake of clarity, the edges connecting the clause vertices c_1, c_2, c_3 to the vertices representing the literals are omitted.

- For every $1 \leq \ell \leq q$, we set $L(c_\ell) = \{1, 2\}$, $L(c_\ell^1) = \{1, 3\}$, $L(c_\ell') = \{1, 3\}$, $L(c_\ell^2) = \{2, 3\}$, forcing c_ℓ' to color 1.

Set $L(v) = \{1, 2\}$ for every other vertex v. We claim that there is no proper clique coloring with these list assignments.

Assume that, on the contrary, there is a proper clique coloring ψ. For every $1 \leq j \leq m$, exactly one of y_j and $\overline{y}_j$ have color 1, since edges $y_j y_j'$, $y_j'\overline{y}_j'$, $\overline{y}_j'\overline{y}_j$ are flat edges. Define the vector $\mathbf{y} \in \{0, 1\}^m$ such that variable y_j is true if and only if y_j has color 1. By assumption, there is a vector $\mathbf{z}$ such that $\varphi(\mathbf{x}, \mathbf{y}, \mathbf{z})$ holds. Based on $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, we can define a clique K of G as follows:

- $x_i \in K$ iff x_i is true,
- $\overline{x}_i \in K$ iff x_i is false,
- $y_j \in K$ iff y_j is true,
- $\overline{y}_j \in K$ iff y_j is false,
- $z_k \in K$ iff z_k is true,
- $\overline{z}_k \in K$ iff z_k is false, and
- c_ℓ' for every $1 \leq \ell \leq q$.

Notice that every vertex in clique K has color 1: if x_i is true (resp., false) then the list assignments force x_i (resp., $\overline{x}_i$) to color 1. Moreover, exactly one of y_j and $\overline{y}_j$ have color 1, and the definition of $\mathbf{y}$ and K implies that from these two vertices, the one with color 1 is selected into K. By assumption, ψ is a proper clique coloring, therefore K is not a maximal clique. It is clear that only a vertex c_ℓ can extend K to a larger clique, thus there is a c_ℓ such that $K \cup \{c_\ell\}$ is also a clique. However, by the construction, this means that in $\varphi(\mathbf{x}, \mathbf{y}, \mathbf{z})$, no variable satisfies clause C_ℓ, a contradiction.

To prove the other direction, we show that if there is a list assignment L not having a proper clique coloring, then $\exists \mathbf{x} \forall \mathbf{y} \exists \mathbf{z} \varphi(\mathbf{x}, \mathbf{y}, \mathbf{z})$ holds. The core of G is the disjoint union of trees and even cycles, hence it is 2-choosable (see [4]). We need the following observation, which can be proved by a simple case analysis:

Claim 7. *Consider a 2-list assignment L of a 4-cycle x^1, x^2, x^3, x^4 such that $c(x^1) = 1$ for every list coloring c. Then there is a list coloring c with $c(x^3) \neq 1$.*

That is, if a list assignment forces a vertex to some color, then it cannot force the opposite vertex to the same color.

Let us choose a coloring of the core of G. If $c'_1, \dots, c'_\ell$ are not all of the same color, then this is a proper clique coloring: we have seen above that every maximal clique of size greater than 2 contains $\{c'_1, \dots, c'_q\}$. Thus we can assume that the list assignment of the 4-cycles on $c'_1, \dots, c'_\ell$ force them to the same color 1. By Claim 7, we can choose a coloring where none of $c_1, \dots, c_\ell$ has color 1.

The 4-cycle formed by the vertices $x_i, x'_i, \overline{x}_i, \overline{x}'_i$ is 2-choosable, thus it can be colored with the given lists. By Claim 7, we can choose a coloring that does not assign color 1 to both $x_{i,1}$ and $\overline{x}_{i,1}$. Define the vector $\mathbf{x} \in \{0,1\}^n$ such that variable x_i is true if and only if vertex x_i has color 1.

By assumption, $\exists\mathbf{x}\forall\mathbf{y}\exists\mathbf{z}\varphi(\mathbf{x},\mathbf{y},\mathbf{z})$ does not hold, thus in particular $\forall\mathbf{y}\exists\mathbf{z}\varphi(\mathbf{x},\mathbf{y},\mathbf{z})$ is false. Therefore, there is a vector $\mathbf{y} \in \{0,1\}^m$ such that $\exists\mathbf{z}\varphi(\mathbf{x},\mathbf{y},\mathbf{z})$ does not hold. Based on the vector $\mathbf{y}$, we continue the coloring of G. The path y_j, y'_j, $\overline{y}'_j$, $\overline{y}_j$ can be colored with the lists. Moreover, this path has a coloring such that y_j does not have color 1, and it has another coloring where $\overline{y}_j$ does not have color 1. If y_j is true (resp., false), then let us color the path in such a way that $\overline{y}_j$ (resp., y_j) has a color different from 1. We claim that this coloring is a proper clique coloring. Since the coloring is a proper vertex coloring of the core of G, it is sufficient to check the maximal cliques greater than 2. Suppose that K is such a monochromatic maximal clique. As K contains the vertices $c'_1, \dots, c'_\ell$ having color 1, every vertex in K has color 1. This implies that K does not contain any of the vertices c_ℓ, since we have assigned colors different from 1 to these vertices. Therefore, K is fully contained in H. For every $1 \le k \le p$, clique K contains exactly one of z_k and $\overline{z}_k$. Define the vector $\mathbf{z} \in \{0,1\}^p$ such that variable z_k is true if and only if $z_k \in K$. Clique K contains exactly one of x_i and $\overline{x}_i$. Since K contains only vertices with color 1, and at most one of x_i and $\overline{x}_i$ has color 1, we have that $x_i \in K$ if and only if x_i is true. Similarly, K contains exactly one of y_i and $\overline{y}_i$, more precisely, $y_i \in K$ if and only if y_i is true. To arrive to a contradiction, we show that $\varphi(\mathbf{x},\mathbf{y},\mathbf{z})$ is true. Suppose that clause C_ℓ is not satisfied by this variable assignment. The vertices in K correspond to the true literals in the variable assignment $\mathbf{x},\mathbf{y},\mathbf{z}$, therefore by the construction, c_ℓ is connected to every vertex in K, contradicting the assumption that K is a maximal clique. □

The k-Clique-Choosability problem remains Π_3^p-complete for every $k > 2$. The proof is similar to the proof of Corollary 5: the case k is reduced to the case $k+1$ by attaching some special graphs. However, here we attach complete bipartite graphs instead of Mycielski graphs.

Lemma 8. *There is a k-vertex-choosable bipartite graph B_k with a distinguished vertex x such that for every color c there is a k-list assignment where every list coloring assigns color c to vertex x.*

Proof. We claim that the complete bipartite graph $B_k = K_{k,k^k-1}$ is such a graph, with $x \in V_1$ being any vertex of the smaller class V_1. To see that B_k is k-vertex-choosable, consider a k-list assignment L and assume first that $L(u) \cap L(v) \neq \emptyset$ for some $u, v \in V_1$. In this case the k vertices in V_1 can be colored such that they receive at most $k-1$ distinct colors, thus every vertex $w \in V_2$ can be given a color from $L(w)$ that is not used by the vertices in V_1. If the lists in V_1 are disjoint, then V_1 can be colored in k^k different ways, every such coloring assigns a different set of k colors to the vertices in V_1. A coloring of V_1 can be extended to V_2 unless there is a vertex $w \in V_2$ whose list contains exactly the k colors used by V_1. Since there are only $k^k - 1$ vertices in V_2, they can exclude at most $k^k - 1$ colorings of V_1, thus at least one of the k^k different colorings of V_1 can be extended to V_2.

On the other hand, let $V_1 = \{v_1, \ldots, v_k\}$ and $L(v_i) = \{c_{i,1}, c_{i,2}, \ldots, c_{i,k}\}$. There are k^k sets of the form $\{c_{1,i_1}, c_{2,i_2}, \ldots, c_{k,i_k}\}$ with $1 \leq i_1, i_2, \ldots, i_k \leq k$. Assign these sets, with the exception of $\{c_{1,1}, c_{2,1}, \ldots, c_{k,1}\}$, to the vertices in V_2. It is easy to see that with these list assignments, every coloring gives color $c_{i,1}$ to vertex v_i. Therefore, setting $x = v_1$ and $c = c_{1,1}$ satisfies the requirements. □

Corollary 9. *For every $k \geq 2$, k-Clique-Choosability is Π_3^p-complete.*

Proof. For every $k \geq 2$, we give a polynomial-time reduction from k-Clique-Choosability to $(k+1)$-Clique-Choosability. By Theorem 6, the problem 2-Clique-Choosability is Π_3^p-complete, thus the theorem follows by induction.

Let $G(V, E)$ be a graph with n vertices $v_1, v_2, \ldots, v_n$. Add n disjoint copies of the graph B_{k+1} (Lemma 8) such that vertex x_i, which is the distinguished vertex x of the i-th copy, is connected to v_i. Denote by W the new vertices added to G. Observe that every vertex in W is simple (B_{k+1} is bipartite, thus it does not contain triangles). We claim that the resulting graph $G'(V \cup W, E')$ is $(k+1)$-clique-choosable if and only if G is k-clique-choosable.

Assume first that G' is $(k+1)$-clique-choosable, we show that G is k-clique-choosable. Let L be an arbitrary k-assignment of G. Let c be a color not appearing in L. Define the $(k+1)$-assignment L' as $L'(v) = L(v) \cup \{c\}$ for every $v \in V$, and extend L' to W (i.e., to the copies of B_{k+1}) in such a way that in every list coloring of G', the vertex x_i of every copy receives the color c. By assumption, G' has a clique coloring ψ with the lists L'. By Prop. 3, ψ is a proper vertex coloring of W, therefore $\psi(x_i) = c$ for every $1 \leq i \leq n$. Thus $\psi(v_i) \neq c$ and $\psi(v_i) \in L(v_i)$ follow, hence ψ induces a list coloring of G. Moreover, by Prop. 1, ψ is a proper clique coloring of G, proving this direction of the reduction.

Now assume that G is k-clique-choosable, it has to be shown that G' is $(k+1)$-clique-choosable. Let L be a $(k+1)$-list assignment of $V \cup W$. Since B_{k+1} is $(k+1)$-choosable, every copy of B_{k+1} can be colored with these lists, let ψ be this coloring of W. Define the k-assignment L' of V as $L'(v_i) = L(v_i) \setminus \{\psi(x_i)\}$ if $\psi(x_i) \in L(v_i)$, otherwise let $L'(v_i)$ an arbitrary k element subset of $L(v_i)$. By assumption, there is a proper clique coloring of V with the lists L', extend ψ to V with these assignment of colors. By Prop. 2, ψ is also a proper clique coloring of G'. □

5 Hereditary clique coloring

Graph G is *hereditary k-clique-colorable* if every induced subgraph of G is k-clique-colorable. Since clique coloring is not a hereditary property in general, an induced subgraph of a k-clique-colorable graph G is not necessarily k-clique-colorable. Thus hereditary k-clique-colorability is not the same as k-clique-colorability. The main result of this section is showing that the decision problem Hereditary k-Clique-Coloring is Π_3^p-complete for every $k \geq 3$, that is, it lies one level higher in the polynomial hierarchy than k-clique-colorability.

Hereditary k-Clique-Coloring	
Input:	A graph $G(V, E)$
Question:	Is it true that every induced subgraph of G is k-clique-colorable?

The proof follows the same general framework as the proof of Theorem 6, but selecting an induced subgraph of G plays here the same role as selecting a list assignment in that proof. To show that $\exists \mathbf{x} \forall \mathbf{y} \exists \mathbf{z} \varphi(\mathbf{x}, \mathbf{y}, \mathbf{z})$ implies that G is *not* hereditary 3-clique-colorable, assume that a vector $\mathbf{x}$ exists with $\forall \mathbf{y} \exists \mathbf{z} \varphi(\mathbf{x}, \mathbf{y}, \mathbf{z})$. Based on this vector $\mathbf{x}$, we select an induced subgraph $G(\mathbf{x})$ of G. If subgraph

$G(\mathbf{x})$ has a 3-clique-coloring ψ, then a vector $\mathbf{y}$ can be defined based on ψ. By assumption, there is a vector $\mathbf{z}$ such that $\varphi(\mathbf{x},\mathbf{y},\mathbf{z})$ is true. We arrive to a contradiction by showing that vectors $\mathbf{x},\mathbf{y},\mathbf{z}$ can be used to find a monochromatic maximal clique K in ψ. The overview of this direction:

A vector $\mathbf{x}$ with $\forall\mathbf{y}\exists\mathbf{z}\varphi(\mathbf{x},\mathbf{y},\mathbf{z})$	$\Rightarrow$	A subgraph $G(\mathbf{x})$ of G	$\Rightarrow$	An arbitrary coloring ψ of $G(\mathbf{x})$	$\Rightarrow$	A vector $\mathbf{y}$	$\Rightarrow$	A vector $\mathbf{z}$ with $\varphi(\mathbf{x},\mathbf{y},\mathbf{z})=1$	$\Rightarrow$	A monochromatic clique K in ψ

The proof of the reverse direction is much more delicate. We have to show that if there is an induced subgraph G' of G that is not 3-clique-colorable, then $\exists\mathbf{x}\forall\mathbf{y}\exists\mathbf{z}\varphi(\mathbf{x},\mathbf{y},\mathbf{z})$ holds. If G' is a subgraph $G(\mathbf{x})$ for some vector $\mathbf{x}$ (as defined by the first direction of the proof), then we proceed as follows. Assume that $\exists\mathbf{x}\forall\mathbf{y}\exists\mathbf{z}\varphi(\mathbf{x},\mathbf{y},\mathbf{z})$ does not hold, then there is vector $\mathbf{y}$ with $\nexists\mathbf{z}\varphi(\mathbf{x},\mathbf{y},\mathbf{z})$. Based on this vector $\mathbf{y}$, one can define a 3-coloring ψ of G'. By assumption, G' is not 3-clique-colorable, thus ψ contains a monochromatic maximal clique K. Based on this maximal clique K, we can find a vector $\mathbf{z}$ satisfying $\varphi(\mathbf{x},\mathbf{y},\mathbf{z})$, a contradiction.

A subgraph G' of G	$\Rightarrow$	A vector $\mathbf{x}$	$\Rightarrow$	A vector $\mathbf{y}$ with $\nexists\mathbf{z}\varphi(\mathbf{x},\mathbf{y},\mathbf{z})$	$\Rightarrow$	A coloring ψ of G'	$\Rightarrow$	A monochromatic clique K in ψ	$\Rightarrow$	A vector $\mathbf{z}$ with $\varphi(\mathbf{x},\mathbf{y},\mathbf{z})=1$

However, it might be possible that the uncolorable induced subgraph G' is "nonstandard" in the sense that it does not correspond to a subgraph $G(\mathbf{x})$ for any vector $\mathbf{x}$. In this case the above proof does not work, we cannot define $\mathbf{x}$ based on the subgraph. In order to avoid this problem, we implement a delicate "self-destruct" mechanism, which ensures that every such nonstandard subgraph can be easily 3-clique-colored. This will be done the following way. We start with a graph G_0, and G is obtained by attaching several gadgets to G_0. Graph G_0 is easy to color, but a coloring of G_0 can be extended to the gadgets only if the coloring of G_0 satisfies certain requirements (some pairs of vertices have the same color, some pairs have different colors). If G' is a nonstandard subgraph of G (e.g., a vertex is missing from G' that cannot be missing in any subgraph $G(\mathbf{x})$), then the gadgets are "turned off," and every coloring of G_0 can be extended easily to G'. The important thing is that a single missing vertex will turn off every gadget. We define these gadgets in the following two lemmas.

Lemma 10. *There is a graph Z_1 (called the γ-*copier*), with distinguished vertices α,β,γ, satisfying the following properties:*

1. *Z_1 is triangle free.*
2. *In every 3-vertex-coloring of Z_1, vertices α and β receive the same color.*
3. *Z_1 can be 3-vertex-colored such that γ has the same color as α and β, and it can be 3-vertex-colored such that the color of γ is different from the color of α and β.*
4. *In $Z_1 \setminus \gamma$, every assignment of colors to α and β can be extended to a proper 3-vertex-coloring.*
5. *The distance between any two of α,β,γ is greater than 2 in Z_1.*

Proof. The graph Z_1' shown in Figure 5a is not triangle free, but it can be proved by inspection that Z_1' satisfies properties 2–4. The graph Z_1 is created from Z_1' as follows. Every edge $e=uv$ is replaced by a new vertex e that is adjacent to u. Furthermore, the edge between vertex e and vertex v is replaced by a copy of D_4 (see Figure 3) in such a way that the distinguished vertices x and y are identified with vertices e and v, respectively. It is clear that every 3-coloring of Z_1 induces a 3-coloring of Z_1': vertices u and v have different colors, since vertices e and v have the same color in

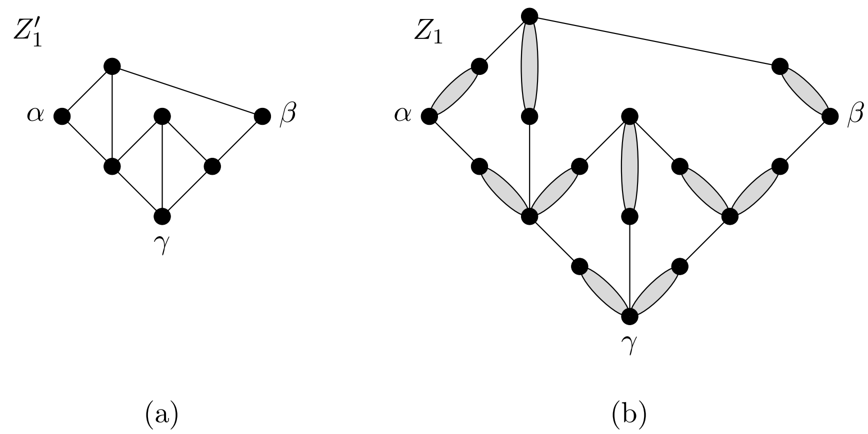


Figure 5: The γ-copier Z_1. Figure (a) shows the graph Z_1' that served as base for constructing Z_1. In Figure (b), every shaded ellipse is a copy of D_4.

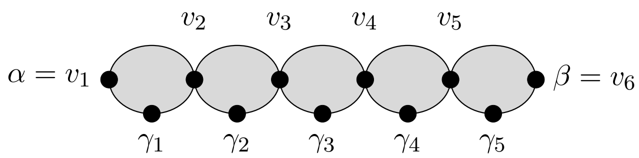


Figure 6: The n-copier Z_n. Every shaded ellipse is a copy of the γ-copier.

every 3-coloring of Z_1 (because of the properties of the graph D_4) and e, u are neighbors. Moreover, every 3-coloring of Z_1' can be extended to a coloring of Z_1. Therefore, properties 2–4 hold for Z_1 as well. It is obvious that Property 5 holds for Z_1. $\square$

Thus the γ-copier ensures that α and β have the same color, but deleting γ turns off the gadget. The gadget defined by the following lemma is similar, but the role of γ is played by several vertices $\gamma_1, \ldots, \gamma_n$, and deleting *any* of them turns off the gadget.

Lemma 11. *For every $n \geq 1$, there is a graph Z_n (called the* n-copier*), with distinguished vertices $\alpha, \beta, \gamma_1, \gamma_2, \ldots, \gamma_n$, satisfying the following properties:*

1. *Z_n is triangle free.*
2. *In every 3-vertex-coloring of Z_n, vertices α and β receive the same colors.*
3. *Every coloring of the vertices $\alpha, \beta, \gamma_1, \gamma_2, \ldots, \gamma_n$ can be extended to a 3-vertex-coloring of Z_n, if α and β have the same color.*
4. *For every $1 \leq i \leq n$, every assignment of colors to $\alpha, \beta, \gamma_1, \gamma_2, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_n$ can be extended to a proper 3-vertex-coloring of $Z_n \setminus \gamma_i$.*
5. *The distance between any two of $\alpha, \beta, \gamma_1, \gamma_2, \ldots, \gamma_n$ is greater than 2 in Z_n.*

Proof. Graph Z_n is created by concatenating n copies of the graph Z_1 defined in Lemma 10. Take $n+1$ vertices $v_1, v_2, \ldots, v_{n+1}$ and add n copies of Z_1 such that vertex α of the i-th copy is identified with vertex v_i, and vertex β is identified with vertex v_{i+1} (see Figure 6). Let $\alpha = v_1$, $\beta = v_{n+1}$, and let γ_i be vertex γ of the i-th copy.

It is clear that Z_n is triangle free. Property 2 holds, since by Property 2 of Lemma 10, vertices v_i and v_{i+1} have the same color for $1 \le i \le n$. To see that Property 3 holds, observe that if α and β have the same color c, then by Property 3 of Lemma 10, the coloring can be extended to every copy of Z_1 such that all the vertices v_i $(1 \le i \le n+1)$ are colored with c. Property 4 follows from Property 3 if the same color is assigned to α and β. Otherwise assign the same color to $v_1 = \alpha, v_2, \dots, v_i$, and the same color to $v_{i+1}, \dots, v_{n+1} = \beta$. This coloring can be extended to a 3-vertex-coloring on every copy of Z_1: for every copy but the i-th, the distinguished vertices α and β have the same color, thus there is such a coloring by Property 3 of Lemma 10. For the i-th copy, the distinguished vertex γ_i is missing, thus there is such an extension by Property 4 of Lemma 10. Property 5 follows from Property 5 of Lemma 10 and from the way Z_n is constructed. □

The *n-edge* is obtained from the n-copier by renaming vertex β to β', and connecting a new vertex β to β'. It has the same properties as the n-copier defined in Lemma 11, except that in Properties 2 and 3, vertices α and β must have *different* colors.

Now we are ready to prove the main result of the section:

Theorem 12. *Hereditary 3-Clique-Coloring is Π_3^p-complete.*

Proof. The problem is in Π_3^p: if G is not hereditary 3-clique-colorable, then it has an induced subgraph G' that is not 3-clique-colorable. This subgraph can serve as a certificate proving that G is not hereditary 3-clique-colorable. Checking 3-clique-colorability is in Σ_2^p, thus verifying this certificate is in Σ_2^p, implying that the problem is in Π_3^p.

The Π_3^p-hardness of the problem is proved by reducing the Σ_3^p-complete problem QSAT_3 to the *complement* of Hereditary 3-Clique-Choosability. That is, for every 3CNF formula $\varphi(\mathbf{x}, \mathbf{y}, \mathbf{z})$, a graph G is constructed in such a way that G is *not* hereditary 3-clique-colorable if and only if $\exists\mathbf{x}\forall\mathbf{y}\exists\mathbf{z}\varphi(\mathbf{x}, \mathbf{y}, \mathbf{z})$ holds.

The graph $G(V, E)$ is obtained from a graph $G_0(V_0, E_0)$ with some n-copiers and n-edges attached to it. G_0 contains

- 5 vertices x_i, x_i', $\overline{x}_i$, $\overline{x}_i'$, x_i^* for every variable x_i $(1 \le i \le n)$,
- 2 vertices y_j, $\overline{y}_j$ for every variable y_j $(1 \le j \le m)$,
- 2 vertices z_k, $\overline{z}_k$ for every variable z_k $(1 \le k \le p)$,
- a vertex c_ℓ for every clause C_ℓ $(1 \le \ell \le q)$,
- $2n$ vertices t_i, t_i' $(1 \le i \le n)$,
- 3 vertices f_1, f_2, f_3.

Graph G_0 has the following edges. The $4n + 2m + 2p + 1$ vertices x_i, $\overline{x}_i$, y_j, $\overline{y}_j$, z_k, $\overline{z}_k$, t_i, t_i', f_1 $(1 \le i \le n, 1 \le j \le m, 1 \le k \le p)$ almost form a clique, except that the edges $x_i\overline{x}_i$, $y_j\overline{y}_j$, $z_k\overline{z}_k$ are missing. For every $1 \le i \le n$, the 3 vertices x_i, x_i', x_i^*, and the 3 vertices $\overline{x}_i$, $\overline{x}_i'$, x_i^* form a triangle. Every vertex c_ℓ is connected to those vertices that correspond to literals *not* satisfying clause C_ℓ. Note that we can assume that a variable and its negation do not appear in the same clause, since in this case every assignment satisfies the clause. This means that c_ℓ is connected to at least one of x_i and $\overline{x}_i$. Furthermore, vertex c_ℓ is also connected to vertices f_1, t_i, t_i' $(1 \le i \le n)$.

To obtain the graph G, several n-copiers and n-edges are added to G_0. Let S contain every vertex defined above, except x_i and $\overline{x}_i$ $(1 \le i \le n)$, thus S has size $5n + 2m + 2p + q + 3$. Adding an S-copier between a and b means the following: let $S' = S \setminus \{a, b\}$, we add an $|S'|$-copier to the graph such that distinguished vertices α, β are identified with a, b, and the vertices $\gamma_1, \ldots, \gamma_{|S'|}$ are identified with the vertices in S' (in any order). Adding an S-edge is defined similarly. Adding an x_i-copier between a and b means that we add a γ-copier to the graph, and identify α, β, and γ with a, b, and x_i, respectively. The description of G is completed by adding an

- S-edge between f_1 and f_2, between f_2 and f_3, between f_1 and f_3,
- S-copier between f_1 and x_i $(1 \leq i \leq n)$,
- S-copier between f_1 and $\overline{x}_i$ $(1 \leq i \leq n)$,
- S-edge between f_3 and x'_i $(1 \leq i \leq n)$,
- S-edge between f_3 and $\overline{x}'_i$ $(1 \leq i \leq n)$,
- S-copier between x'_i and t'_i $(1 \leq i \leq n)$,
- S-copier between $\overline{x}'_i$ and t'_i $(1 \leq i \leq n)$,
- S-copier between f_2 and x^*_i $(1 \leq i \leq n)$,
- x_i-copier between f_1 and t_i $(1 \leq i \leq n)$,
- $\overline{x}_i$-copier between f_1 and t_i $(1 \leq i \leq n)$,
- S-edge between f_3 and y_j $(1 \leq j \leq m)$,
- S-edge between f_3 and $\overline{y}_j$ $(1 \leq j \leq m)$,
- S-edge between y_j and $\overline{y}_j$ $(1 \leq j \leq m)$,
- S-copier between f_1 and z_k $(1 \leq k \leq p)$,
- S-copier between f_1 and $\overline{z}_k$ $(1 \leq k \leq p)$,
- S-copier between f_3 and c_ℓ $(1 \leq \ell \leq q)$.

The graph G for $n = m = p = 2$, $q = 3$ is shown in Figure 7. It can be verified that the maximal cliques of G can be divided into the following three types:

1. The flat edges of G.
2. The *x_i-triangles* x_i, x^*_i, x'_i, and the *$\overline{x}_i$-triangles* $\overline{x}_i$, x^*_i, $\overline{x}'_i$.
3. The *assignment* cliques that contain the vertices f_1, t_i, t'_i $(1 \leq i \leq n)$. Besides these vertices, an assignment clique contains exactly one of x_i and $\overline{x}_i$, exactly one of y_j, $\overline{y}_j$, exactly one of z_k, $\overline{z}_k$, and at most one c_ℓ $(1 \leq i \leq n, 1 \leq j \leq m, 1 \leq k \leq p, 1 \leq \ell \leq q)$.

Note that the edges inside a copier or edge gadget are flat. The gadgets are triangle free and no new triangle is created even if two distinguished vertices of a gadget become connected in the above construction (for example, this is the case with the x_i-copier between f_1 and t_1): the distance of the connected vertices is greater than 2 in the gadget.

First we show that if there is an $\mathbf{x} \in \{0,1\}^n$ such that $\forall \mathbf{y} \exists \mathbf{z} \varphi(\mathbf{x}, \mathbf{y}, \mathbf{z})$, then there is an induced subgraph $G(\mathbf{x})$ of G that is not 3-clique-colorable. To obtain $G(\mathbf{x})$, delete vertex $\overline{x}_i$ from G if variable x_i is true in $\mathbf{x}$, and delete vertex x_i from G if variable x_i is false. Recall that x_i and $\overline{x}_i$ are not in S.

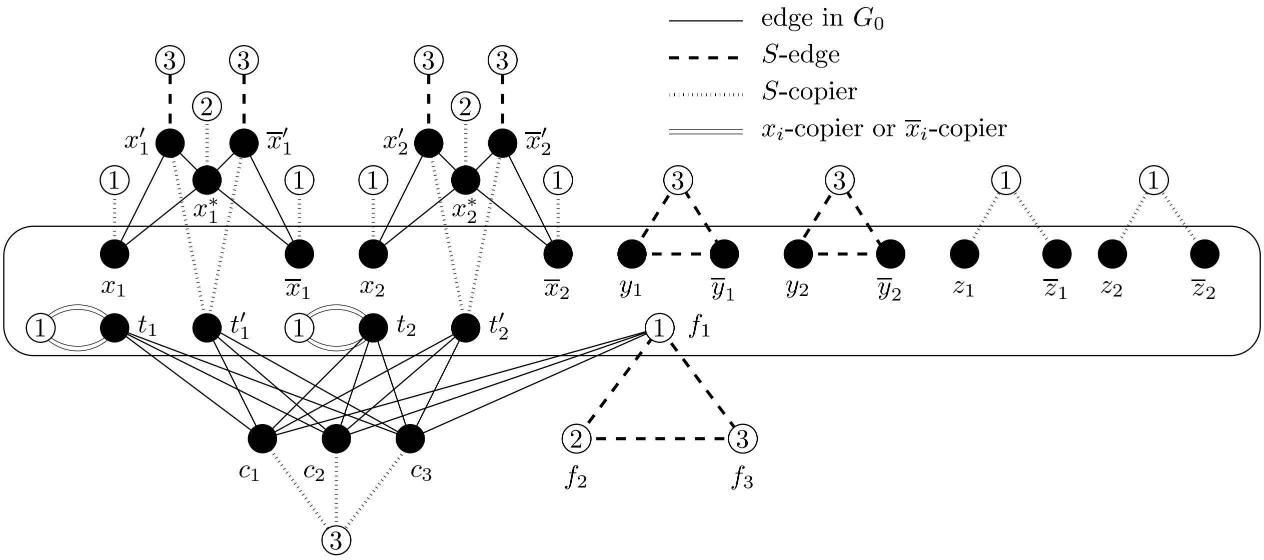


Figure 7: Sketch of the construction used in the proof of Theorem 12. The vertices $f_1,\ f_2,\ f_3$ are shown multiple times, e.g., every appearance of the white vertex 1 is identical to f_1. The two double edges between f_1 and t_1 represent the x_1-copier and the $\overline{x}_1$-copier. In the rounded box, every vertex is connected to every other vertex, except the pairs $x_i\overline{x}_i$, $y_j\overline{y}_j$, and $z_k\overline{z}_k$. Depending on the formula φ, vertex c_ℓ is connected to some vertices representing literals.

Assume that there is a 3-clique-coloring ψ of $G(\mathbf{x})$. Since every vertex of S is present in $G(\mathbf{x})$, the S-edge between f_1 and f_2 ensures that $\psi(f_1) \neq \psi(f_2)$, it can be assumed that $\psi(f_1) = 1$ and $\psi(f_2) = 2$. Because of the S-edge between f_1 and f_3, and between f_2 and f_3, we also have that $\psi(f_3) = 3$. We claim that x_i, $\overline{x}_i$ (if they are present in $G(\mathbf{x})$), t_i, t'_i, z_k, $\overline{z}_k$ all have color 1. Assume that x_i is in $G(\mathbf{x})$ (the argument is similar, if $\overline{x}_i$ is in $G(\mathbf{x})$, and x_i is not). Vertex x_i has color 1 because of the S-copier between f_1 and x_i. There is an S-copier between f_2 and x^*_i, thus $\psi(x^*_i) = 2$. Since $x_i \in G(\mathbf{x})$, the x_i-copier between f_1 and t_i ensures that $\psi(t_i) = 1$. If x_i is in $G(\mathbf{x})$, then $\overline{x}_i$ is not in $G(\mathbf{x})$ and the edge $x^*_i\overline{x}'_i$ is a maximal clique, thus $\psi(\overline{x}'_i) \neq \psi(x^*_i) = 2$. Moreover, because of the S-edge between $\overline{x}'_i$ and f_3, we have $\psi(\overline{x}'_i) \neq 3$, implying $\psi(\overline{x}'_i) = 1$. Since there is an S-copier between $\overline{x}'_i$ and t'_i, we have $\psi(t'_i) = 1$. Finally, the S-copier between f_1 and z_k, and between f_1 and $\overline{z}_k$ imply that $\psi(z_k) = \psi(\overline{z}_k) = 1$.

The S-edges between f_3 and y_j, and between f_3 and $\overline{y}_j$ ensure that y_j and $\overline{y}_j$ have color 1 or 2. Furthermore, because of the S-edge between y_j and $\overline{y}_j$, one of them has color 1, and the other has color 2. Define the vector $\mathbf{y} \in \{0,1\}^m$ such that variable y_j is true if and only if $\psi(y_j) = 1$. By assumption, there is a vector $\mathbf{z} \in \{0,1\}^p$ such that $\varphi(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is true. Let K contain all the vertices that correspond to true literals in $\mathbf{x}, \mathbf{y}, \mathbf{z}$; note that all these vertices are in $G(\mathbf{x})$. Moreover, add to K the vertices $f_1,\ t_i,\ t'_i$ $(1 \leq i \leq n)$. Clearly, K is a clique. Furthermore, because of the way K was constructed, every vertex in K has color 1. We claim that K is a monochromatic maximal clique, contradicting the assumption that ψ is a proper 3-clique-coloring of $G(\mathbf{x})$. Suppose that there is a clique $K' \supset K$ of $G(\mathbf{x})$. The clique K' is a subset of a maximal clique of G. As K' contains the vertices $f_1,\ t_i,\ t'_i$ $(1 \leq i \leq n)$, this maximal clique has to be an assignment clique. Since K contains already $1 + 2n + m + p + q$ vertices, the only possibility is that $K' \setminus K$ contains a vertex c_ℓ corresponding to a clause. However, in this case the assignment $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ does not satisfy φ since clause C_ℓ is not satisfied: otherwise there is a vertex in K that corresponds to a literal satisfying C_ℓ, and by the construction c_ℓ is not connected to this vertex.

To prove the other direction of the reduction, assume that there is an induced subgraph G' of G that is not 3-clique-colorable, we have to show that $\exists \mathbf{x} \forall \mathbf{y} \exists \mathbf{z} \varphi(\mathbf{x}, \mathbf{y}, \mathbf{z})$ holds. By Prop. 1, it can be assumed that G' contains all the simple vertices of G: adding simple vertices to G' does not make it 3-clique-colorable. In particular, G' contains the internal vertices of all the gadgets.

Call an induced subgraph of G *standard*, if for every $1 \le i \le n$, it contains exactly one of x_i and $\overline{x}_i$, and it contains every other vertex of G (in particular, it contains every vertex of S). First we show that every nonstandard subgraph of G is 3-clique-colorable, thus G' must be standard. Next we show that if there is a standard subgraph G' of G that is not 3-clique-colorable, then there is an $\mathbf{x} \in \{0,1\}^n$ satisfying $\forall \mathbf{y} \exists \mathbf{z} \varphi(\mathbf{x}, \mathbf{y}, \mathbf{z})$. These two lemmas complete the proof of this direction of the reduction.

Lemma 13. *If G' is a nonstandard induced subgraph of G, then G' is 3-clique-colorable.*

Proof. Let G' be a nonstandard subgraph of G. By Prop. 1 it can be assumed that G' contains every simple vertex of G. We show that G' contains every vertex of S. Assume that a vertex $v \in S$ is missing from G'. The absence of v turns off the S-copiers and the S-edges, which makes the coloring very easy. However, the x_i-copiers might still be working, thus we have to pay attention that the coloring can be extended to the internal vertices of these gadgets. Let G'_0 be the induced subgraph of G' containing only those vertices that are in G_0. We show that there is a 3-clique-coloring of G'_0 with the following property:

> If both f_1 and t_i are in G'_0 for some $1 \le i \le n$, and at least one of $x_i, \overline{x}_i$ is in G'_0, then f_1 and t_i have the same color.

If this is true, then this coloring can be extended to a 3-clique-coloring of G': by Property 4 of Lemma 11 and since by assumption $v \in S$ is missing from G', the coloring can be extended to the internal vertices of every S-copier and S-edge. Here we use Proposition 2: if we extend the coloring of G'_0 such that every gadget is 3-vertex-colored, then it gives a 3-clique-coloring of G. Moreover, the internal vertices of the x_i-copier and the $\overline{x}_i$-copier between f_1 and t_i can be colored as well, since either both x_i and $\overline{x}_i$ are missing (Property 4 of Lemma 10), or f_1 and t_i have the same color (Property 3 of Lemma 10).

We consider the following 3 cases:

Case 1: $x_i, \overline{x}_i, y_j, \overline{y}_j, z_k, \overline{z}_k \notin G'_0$ ($1 \le i \le n$, $1 \le j \le m$, $1 \le k \le p$). In this case, we color G'_0 as follows:

- Vertices x'_i, $\overline{x}'_i$ have color 1 ($1 \le i \le n$).
- Vertex f_2 has color 2.
- One of f_1, t_i, t'_i ($1 \le i \le n$) has color 1, the rest has color 2.
- Vertices f_3, x^*_i, c_ℓ have color 3 ($1 \le i \le n$, $1 \le \ell \le q$).

It is clear that there is no monochromatic clique of color 1 or 3, since these color classes are independent sets. A monochromatic clique K with color 2 cannot contain f_2 (since it is not adjacent to any other vertex with color 2), thus K can be extended to the clique f_1, t_i, t'_i ($1 \le i \le n$), which contains a vertex of color 1.

Case 2: f_1, t_i, $t'_i \notin G'_0$ ($1 \le i \le n$). Consider the following coloring G'_0:

- If $x^*_i \in G'_0$, then vertices x'_i, $\overline{x}'_i$ have color 1; otherwise they have color 3 ($1 \le i \le n$).

- Vertices $f_2,\ x_i,\ \overline{x}_i,\ y_j,\ \overline{y}_j,\ z_k,\ \overline{z}_k$ have color 2 ($1 \le i \le n$, $1 \le j \le m$, $1 \le k \le p$).
- Vertices $f_3,\ x_i^*,\ c_\ell$ have color 3 ($1 \le i \le n$, $1 \le \ell \le q$).

We change the coloring defined above on at most 2 vertices. Because we are not in Case 1, there is a pair $(x_i, \overline{x}_i)$, or $(y_j, \overline{y}_j)$, $(z_k, \overline{z}_k)$ such that at least one vertex of the pair is in G'_0. Let us fix such a pair and recolor the vertices of the pair with color 1.

Color class 3 induces an independent set. The only possibility of two adjacent vertices having color 1 is that the pair $(x_i, \overline{x}_i)$ was recolored to color 1 and x'_i, $\overline{x}'_i$ also have color 1. However, in this case $x_i^* \in G'_0$ and has color 3, thus $\{x_i, x'_i\}$ and $\{\overline{x}_i, \overline{x}'_i\}$ are not maximal cliques: they can be extended with x_i^*.

Finally, a monochromatic clique with color 2 cannot contain f_2, since it is not adjacent to any other vertex with color 2. Thus such a clique cannot be maximal, as it can be extended with a vertex of the recolored pair.

Case 3: G'_0 contains a vertex $w_1 \in \{f_1,\ t_i,\ t'_i \mid 1 \le i \le n\}$ and a vertex $w_2 \in \{x_i, \overline{x}_i, y_j, \overline{y}_j, z_k, \overline{z}_k \mid 1 \le i \le n, 1 \le j \le m, 1 \le k \le p\}$. In this case, consider the following assignment of colors:

- Vertices $f_1,\ x'_i,\ \overline{x}'_i,\ t_i,\ t'_i$ have color 1 ($1 \le i \le n$).
- Vertices $f_2,\ x_i,\ \overline{x}_i,\ y_j,\ \overline{y}_j,\ z_k,\ \overline{z}_k$ have color 2 ($1 \le i \le n$, $1 \le j \le m$, $1 \le k \le p$).
- Vertices $f_3,\ x_i^*,\ c_\ell$ have color 3 ($1 \le i \le n$, $1 \le \ell \le q$).

A monochromatic clique of color 1 has to be a subset of $\{f_1,\ t_i,\ t'_i \mid 1 \le i \le n\}$, hence it can be extended by w_2 having color 2. A monochromatic clique of color 2 cannot contain f_2 (since it is not adjacent to any other vertex of color 2), thus it can be extended with vertex w_1 having color 1. Color class 3 induces an independent set. This completes Case 3.

We have shown that every nonstandard induced subgraph G' is 3-clique-colorable if a vertex of S is missing from G'. Now assume that G' is a nonstandard subgraph of G and every vertex of S is in G'. Since the graph is nonstandard, there is an $1 \le i_0 \le n$ such that either G' contains both x_{i_0} and $\overline{x}_{i_0}$, or G' contains neither x_{i_0} nor $\overline{x}_{i_0}$. The following coloring of G'_0 can be extended to a proper 3-clique-coloring of G':

- Vertices $f_1, x_i, \overline{x}_i, x'_i, \overline{x}'_i, t_i, t'_i$ have color 1, where $1 \le i \le n$ and $i \ne i_0$.
- Vertices $y_j, \overline{y}_j, z_k, \overline{z}_k$ have color 1 ($1 \le j \le m$, $1 \le k \le p$)
- Vertices f_2, x_i^* have color 2 ($1 \le i \le n$).
- Vertices $f_3,\ c_\ell$ have color 3 ($1 \le \ell \le q$).
- If both x_{i_0} and $\overline{x}_{i_0}$ are in G', then x'_{i_0}, $\overline{x}'_{i_0}$, t'_{i_0} have color 2 and x_{i_0}, $\overline{x}_{i_0}$, t_{i_0} have color 1.
- If neither x_{i_0} nor $\overline{x}_{i_0}$ is in G', then x'_{i_0}, $\overline{x}'_{i_0}$, t'_{i_0} have color 1 and t_{i_0} has color 2.

This coloring can be extended to G' in such a way that the flat edges are properly colored (that is, it can be extended to the internal vertices of the copier and edge gadgets). Indeed, it can be verified by inspection that the two distinguished vertices of the S-copiers (resp., S-edges) have the same (resp., different) colors, respectively. Moreover, for $i \ne i_0$, both f_1 and t_i have color 1, thus the coloring can be extended to the x_i-copier and $\overline{x}_i$-copier between f_1 and t_i. However, if both x_{i_0} and $\overline{x}_{i_0}$ are missing from G', then f_1 has color 1 and t_{i_0} has color 2. But in this case the absence of

x_{i_0} and $\overline{x}_{i_0}$ ensures that the two copiers between f_1 and t_{i_0} can be colored, regardless of the color of f_1 and t_{i_0} (Property 4 of Lemma 10).

The triangles x_i, x_i^*, x_i' and $\overline{x}_i$, x_i^*, $\overline{x}_i'$ contain both color 1 and 2. Therefore, only the assignment cliques can be monochromatic in this coloring. However, every assignment clique contains t_{i_0} and t'_{i_0}, and these two vertices have different colors. □

Therefore, we can assume that G' is a standard subgraph. We show that based on G' we can define an assignment $\mathbf{x}$ such that $\forall \mathbf{y} \exists \mathbf{z} \varphi(\mathbf{x}, \mathbf{y}, \mathbf{z})$. The proof is similar to the proof of the first direction.

Lemma 14. *If there is a standard subgraph G' of G that is not 3-clique-colorable, then there is an $\boldsymbol{x} \in \{0,1\}^n$ satisfying $\forall \boldsymbol{y} \exists \boldsymbol{z} \varphi(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$.*

Proof. Define vector $\mathbf{x} \in \{0,1\}^n$ by setting variable x_i to true if $x_i \in G'$, and to false if $\overline{x}_i \in G'$. We claim that $\forall \mathbf{y} \exists \mathbf{z} \varphi(\mathbf{x}, \mathbf{y}, \mathbf{z})$. Suppose that, on the contrary, there is a vector $\mathbf{y} \in \{0,1\}^m$ such that $\varphi(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is false for every $\mathbf{z} \in \{0,1\}^p$.

Consider the following coloring of G':

- Vertices $f_1,\ x_i,\ \overline{x}_i,\ x_i',\ \overline{x}_i',\ t_i',\ t_i,\ z_k,\ \overline{z}_k$ have color 1 ($1 \le i \le n$, $1 \le k \le p$).
- Vertices $f_2,\ x_i^*$ have color 2 ($1 \le i \le n$).
- Vertices $f_3,\ c_\ell$ have color 3 ($1 \le \ell \le q$).
- If variable y_j is true in $\mathbf{y}$, then vertex y_j has color 1 and vertex $\overline{y}_j$ has color 2 ($1 \le j \le m$).
- If variable y_j is false in $\mathbf{y}$, then vertex y_j has color 2 and vertex $\overline{y}_j$ has color 1 ($1 \le j \le m$).

As in the proof of Lemma 13, this coloring can be extended to the whole G' in such a way that every flat edge and every x_i-triangle is properly colored. By assumption, this coloring is not a proper 3-clique-coloring, thus there is a monochromatic maximal clique K, which must be an assignment clique. Since every assignment clique contains f_1, therefore every vertex of K has color 1. By the definition of the coloring, this means that K contains y_j if and only if y_j is true in $\mathbf{y}$. For every $1 \le k \le p$, an assignment clique contains exactly one of z_k and $\overline{z}_k$, define the vector $\mathbf{z} \in \{0,1\}^p$ by setting variable z_k to true if and only if $z_k \in K$. Notice that apart from $f_1,\ t_i,\ t_i'$, clique K contains those vertices that correspond to true literals in the assignment $\mathbf{x}, \mathbf{y}, \mathbf{z}$.

We claim that $\varphi(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is true. To see this, assume that clause C_ℓ is not satisfied by this assignment. Vertex c_ℓ is not in K, since c_ℓ has color 3. Now clique K contains the vertices $f_1,\ t_i,\ t_i'$, and vertices corresponding to literals not satisfying C_ℓ, therefore $K \cup \{c_\ell\}$ is also a clique, contradicting the maximality of K. □

Putting together these two lemmas completes the proof of the theorem. □

Hereditary k-clique-coloring remains Π_3^p-complete for every $k > 3$. The proof is analogous to the proof of Corollary 5, the same construction can be used to reduce the case of k colors to $k+1$ colors.

Corollary 15. *For every $k \ge 3$, Hereditary k-Clique-Coloring is Π_3^p-complete.* □

The complexity of the case $k = 2$ remains an open question. The problem seems to be very different if there are only 2 colors. The proofs of this section used gadgets having only certain kind of 3-clique-colorings; more precisely, the gadget were triangle free, thus 3-clique-coloring and 3-vertex-coloring coincides, and we can control the possible 3-clique-colorings by controlling the possible 3-vertex-colorings. However, in the case of 2 colors, such an approach is unlikely to work, since there are only two possible ways of 2-vertex-coloring a connected graph, hence we cannot build such versatile gadgets this way.

Acknowledgments

I'm grateful to David Défossez and to an anonymous reviewer for pointing out an error in the reductions and for suggesting a simplification of the proof of Theorem 6.

References

[1] G. Bacsó, S. Gravier, A. Gyárfás, M. Preissmann, and A. Sebő. Coloring the maximal cliques of graphs. *SIAM J. Discrete Math.*, 17(3):361–376, 2004.

[2] D. Défossez. Clique-coloring some classes of odd-hole-free graphs. *Journal of Graph Theory*, 53(3):233–249, 2006.

[3] D. Duffus, B. Sands, N. Sauer, and R. E. Woodrow. Two-colouring all two-element maximal antichains. *J. Combin. Theory Ser. A*, 57(1):109–116, 1991.

[4] P. Erdős, A. L. Rubin, and H. Taylor. Choosability in graphs. In *Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing (Humboldt State Univ., Arcata, Calif., 1979)*, pages 125–157, Winnipeg, Man., 1980. Utilitas Math.

[5] S. Gravier, C. T. Hoàng, and F. Maffray. Coloring the hypergraph of maximal cliques of a graph with no long path. *Discrete Math.*, 272(2-3):285–290, 2003.

[6] S. Gutner. The complexity of planar graph choosability. *Discrete Math.*, 159(1-3):119–130, 1996.

[7] C. T. Hoàng and C. McDiarmid. On the divisibility of graphs. *Discrete Math.*, 242(1-3):145–156, 2002.

[8] J. Kratochvíl and Zs. Tuza. On the complexity of bicoloring clique hypergraphs of graphs. *J. Algorithms*, 45(1):40–54, 2002.

[9] L. Lovász. *Combinatorial problems and exercises.* North-Holland Publishing Co., Amsterdam, second edition, 1993.

[10] B. Mohar and R. Škrekovski. The Grötzsch theorem for the hypergraph of maximal cliques. *Electron. J. Combin.*, 6(1):Research Paper 26, 13 pp., 1999.

[11] C. H. Papadimitriou. *Computational complexity.* Addison-Wesley Publishing Company, Reading, MA, 1994.

[12] M. Schaefer and C. Umans. Completeness in the polynomial-time hierarchy: a compendium. *SIGACT News*, 33(3):32–49, 2002.

[13] V. G. Vizing. Coloring the vertices of a graph in prescribed colors. *Diskret. Analiz*, (29 Metody Diskret. Anal. v Teorii Kodov i Shem):3–10, 101, 1976.