Parameterized coloring problems on chordal graphs

Dániel Marx

Budapest University of Technology and Economics

`dmarx@cs.bme.hu`

International Workshop on Parameterized and Exact Computation

September 15, 2004

Bergen, Norway

Two coloring problems

- Let $\mathcal{G}$ be a class of graphs that is easy to color.

  How to color a graph from $\mathcal{G}$ if some of the vertices already have a color? (Precoloring extension)

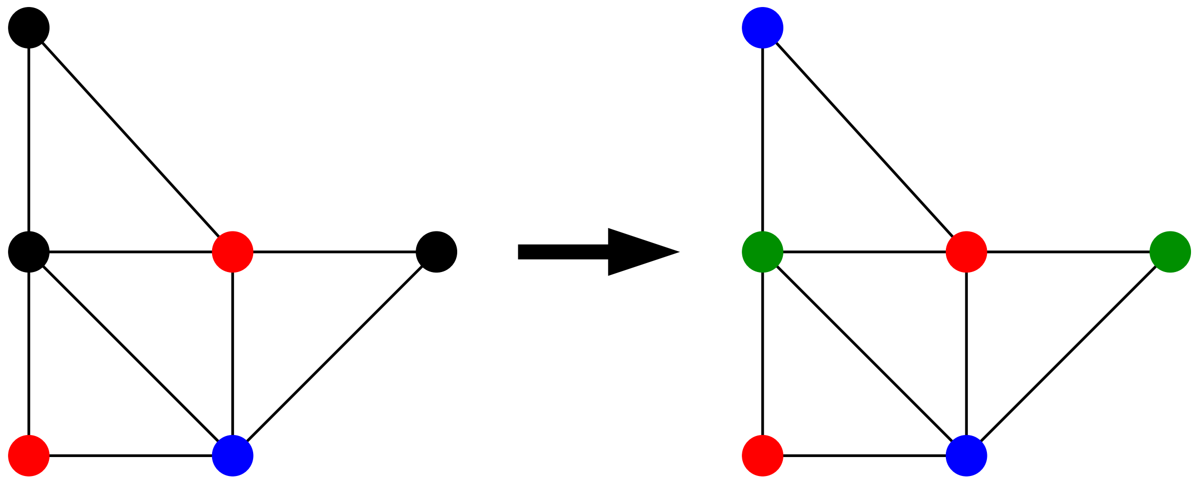
- Let $\mathcal{G}$ be a class of graphs that is easy to color.

  How to color a graph that is “almost” in $\mathcal{G}$?

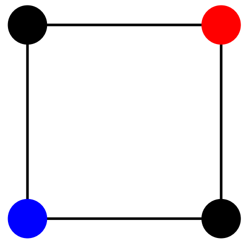
Precoloring extension

Generalization of vertex coloring: given a partial coloring, extend it to the whole graph using k colors.

Example for $k = 3$:



Example for $k = 2$:



Cannot be extended!

Precoloring extension (cont.)

Vertex coloring is a special case of precoloring extension (PREXT).

PREXT is polynomial time solvable for

- complements of bipartite graphs
- cographs
- split graphs
- trees
- partial k-trees

PREXT is NP-complete for

- bipartite graphs
- line graphs of bipartite graphs
- line graphs of planar graphs
- interval graphs

Two parameterizations

Precoloring Extension (PREXT)

- **Input:** a graph $G(V, E)$ with a partial coloring c on $W \subseteq V$, an integer k.
- **Parameter 1:** the size of W.
- **Parameter 2:** the number of colors used on W.
- **Question:** Is there a k-coloring of G that extends c?

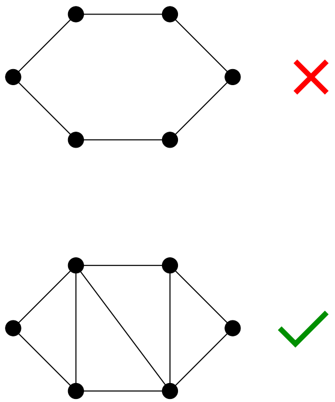
PREXT with parameter 1 is easier than PREXT with parameter 2.

We will investigate both problems on chordal and interval graphs.

Chordal graphs

A graph is **chordal** if it does not contain induced cycles longer than 3.

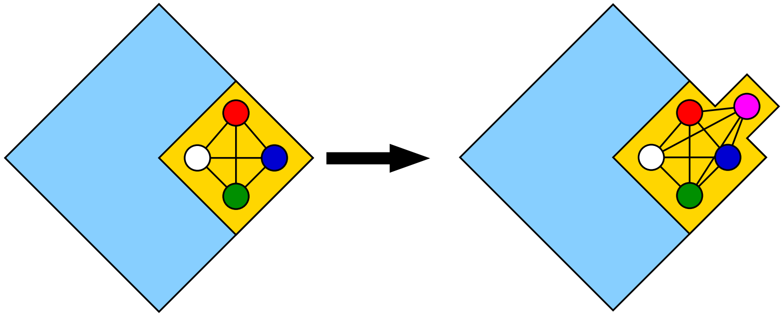
- Interval graphs are chordal.
- Intersection graphs of intervals on a line ⇔ interval graphs.
- Intersection graphs of subtrees in a tree ⇔ chordal graphs.
- Chordal graphs are perfect.



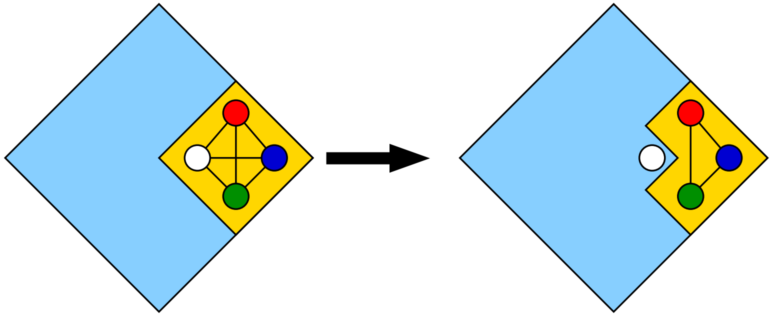
Tree decomposition

Every chordal graph can be built using the following operations. We consider chordal graphs with a distinguished clique.

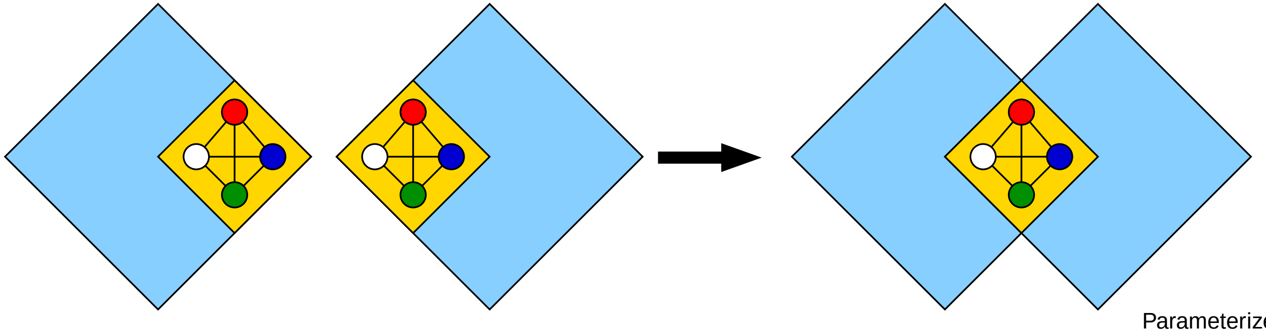
Add: attach a new vertex to the clique.



Forget: remove a vertex from the clique.



Join: identify the cliques of two chordal graphs.

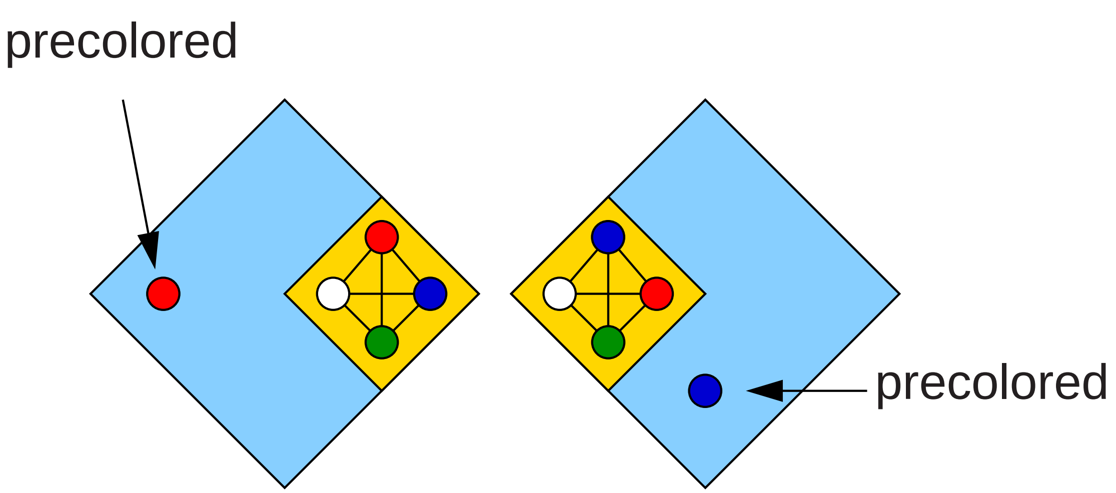


Coloring chordal graphs

Tree decomposition gives a method of coloring chordal graphs.

Main idea: before the **join** operation we can permute the colors such that the clique has the same coloring in both graphs.

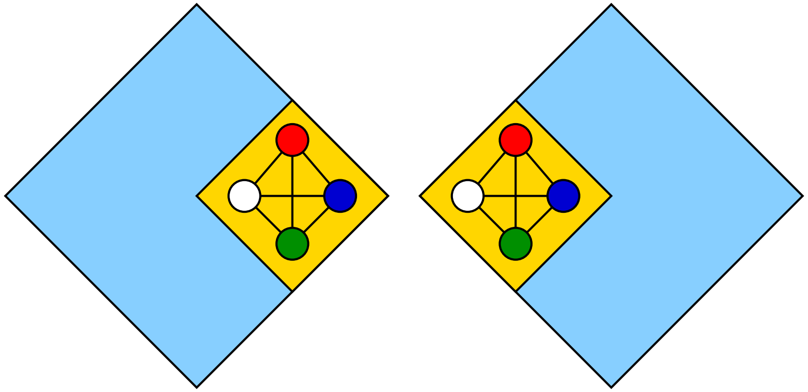
This approach does not work if there are precolored vertices:



The two colorings cannot be joined!

Precoloring extension

Idea 1: For each subgraph appearing in the construction of the chordal graph, list all possible colorings that can appear on the distinguished clique in a precoloring extension.



The graphs can be joined if they have colorings that agree on the clique. The colors outside the clique are not important.

Problem: There can be too many (exponentially many) colorings.

Colorings of the clique

Idea 2: Only those colors have to be distinguished that appear on the precolored vertices. The rest of the colors can be freely permuted.

⇒If only k colors are used in the precoloring, then there are only $O(n^k)$ different colorings that have to be considered for each clique.

Theorem: PREXT can be solved in $O(n^{k+2})$ time if only k colors are used in the precoloring.

Hardness for interval graphs

Theorem: PREXT is W[1]-hard for interval graphs if the parameter is the number of precolored vertices.

The reduction is from the disjoint paths problem:

DISJOINT PATHS

- **Input:** a graph $G(V, E)$ with pairs of vertices (s_1, t_2), (s_2, t_2), $\dots$, (s_k, t_k).
- **Parameter:** k
- **Find:** edge disjoint paths P_1, P_2, $\dots$, P_k such that path P_i connects s_i and t_i.

Theorem: [Slivkins, ESA 2003] The DISJOINT PATHS problem with parameter k is W[1]-hard for directed acyclic graphs.

The “almost” graph classes

Let $\mathcal{G}$ be a class of graphs. Leizhen Cai [2003] introduced the following parameterized classes:

- $\mathcal{G} + ke$: a graph from $\mathcal{G}$ with k extra edges.
- $\mathcal{G} - ke$: a graph from $\mathcal{G}$ with k edges deleted.
- $\mathcal{G} + kv$: graphs that can be made to be in $\mathcal{G}$ by deleting k vertices.
- $\mathcal{G} - kv$: a graph from $\mathcal{G}$ with k vertices deleted.

If $\mathcal{G}$ is hereditary, then $\mathcal{G}$ and $\mathcal{G} - kv$ are the same $\Rightarrow$ $\mathcal{G} - kv$ is not a very interesting class.

Coloring the parameterized classes

Consider the problem of coloring, say, $\mathcal{G} + ke$ graphs (parameter: k).
We expect three possible outcomes:

- Coloring is NP-hard for some k.
- Coloring is polynomial-time solvable for every k, but W[1]-hard.
- Coloring is fixed-parameter tractable.

Theorem: [Cai 2003] Coloring is linear time solvable for bipartite$+\mathbf{1}v$ and bipartite$+\mathbf{2}e$ graphs, but NP-hard for bipartite$+\mathbf{2}v$ and bipartite$+\mathbf{3}e$ graphs.

Theorem: [Cai 2003] If coloring is polynomial-time solvable in $\mathcal{G}$, and $\mathcal{G}$ is closed for contraction, then coloring is in FPT for $\mathcal{G} - ke$.

Almost chordal and interval graphs

What can we say about classes related to interval and chordal graphs?

Theorem: Under mild assumptions on $\mathcal{G}$, we have the following reductions:

Coloring $\mathcal{G} + ke \preceq$ PREXT on $\mathcal{G}$ (param: no. of precolored vertices) $\preceq$ Coloring $\mathcal{G} + kv$

Theorem: If $\mathcal{G}$ is closed for attaching degree 1 vertices, then

Coloring $\mathcal{G} + kv \preceq$ PREXT on $\mathcal{G}$ (param: no. of colors in precoloring)

Theorem: Coloring interval+kv and chordal+kv graphs is W[1]-hard, but polynomial-time solvable for fixed k.

Coloring chordal+ke graphs

Theorem: Coloring chordal+$\boldsymbol{ke}$ graphs is fixed-parameter tractable.

Method: Remove the extra edges. Find a coloring for the remaining chordal graph that respects the extra edges.

Bollobás' Inequality is used to reduce the number of subproblems that have to be considered for each subgraph in the tree decomposition.

Conclusions

- PREXT is polynomial-time solvable for chordal graphs if the number of colors in the precoloring is small.
- PREXT for interval graphs is W[1]-hard if the parameter is the number of precolored vertices.
- Connections between PREXT on $\mathcal{G}$ and the coloring of $\mathcal{G} + kv$, $\mathcal{G} + ke$ graphs.
- Coloring chordal$+kv$ graphs is W[1]-hard, but coloring chordal$+ke$ graphs is FPT.