CSPs and fixed-parameter tractability

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Algorithms Seminar
University of Bergen, Norway
June 7, 2013

Reactions to FPT

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Hmm... Is my favorite graph problem FPT parameterized by the size of the solution/number of objects/etc. ?

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Typical CSP researcher:

SAT is trivially FPT parameterized by the number of variables. So why should I care?

Parameterizing SAT

Trivial: 3SAT is FPT parameterized by the number of variables ($2^k \cdot n^{O(1)}$ time algorithm).

Trivial: 3SAT is FPT parameterized by the number of clauses ($2^{3k} \cdot n^{O(1)}$ time algorithm).

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What about SAT parameterized by the number k of clauses?

Algorithm 1: Problem kernel

- If a clause has more than k literals: can be ignored, removing it does not make the problem any easier.
- If every clause has at most k literals: there are at most k^2 variables, use brute force.

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What about SAT parameterized by the number k of clauses?

Algorithm 2: Bounded search tree

- Pick a variable occuring both positively and negatively, branch on setting it to 0 or 1.
- In both branches, the number of clauses strictly decreases $\Rightarrow$ search tree of size 2^k.

Max Sat

- Max Sat: Given a formula, satisfy at least k clauses.
- Polynomial for fixed k: guess the k clauses, use the previous algorithm to check if they are satisfiable.
- Is the problem FPT?

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- Is the problem FPT?
- YES: If there are at least $2k$ clauses, a random assignment satisfies k clauses on average. Otherwise, use the previous algorithm.

This is not very insightful, can we say anything more interesting?

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- Above average MAX SAT (satisfy $m/2 + k$ clauses) is FPT [Mahajan and Raman 1999]
- Above average MAX r-SAT (satisfy $(1 - 1/2^r)m + k$ clauses) is FPT [Alon et al. 2010]
- Satisfying $\sum_{i=1}^{m}(1 - 1/2^{r_i}) + k$ clauses is NP-hard for $k = 2$ [Crowston et al. 2012]
- Above average MAX r-LIN-2 (satisfy $m/2 + k$ linear equations) is FPT [Gutin et al. 2010]
- Permutation CSPs such as MAXIMUM ACYCLIC SUBGRAPH and BETWEENNESS [Gutin et al. 2010].
- ...

Boolean constraint satisfaction problems

Let Γ be a set of **Boolean** relations. An Γ-formula is a conjunction of relations in Γ:

$$R_1(x_1, x_4, x_5) \wedge R_2(x_2, x_1) \wedge R_1(x_3, x_3, x_3) \wedge R_3(x_5, x_1, x_4, x_1)$$

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- Given: an Γ-formula φ
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$\Gamma = \{a \neq b\} \Rightarrow \text{SAT}(\Gamma) = 2$-coloring of a graph
$\Gamma = \{a \vee b,\ a \vee \bar{b},\ \bar{a} \vee \bar{b}\} \Rightarrow \text{SAT}(\Gamma) = \text{2SAT}$
$\Gamma = \{a \vee b \vee c, a \vee b \vee \bar{c}, a \vee \bar{b} \vee \bar{c}, \bar{a} \vee \bar{b} \vee \bar{c}\} \Rightarrow \text{SAT}(\Gamma) = \text{3SAT}$

Question: SAT(Γ) is polynomial time solvable for which Γ?
It is NP-complete for which Γ?

Schaefer's Dichotomy Theorem (1978)

Theorem [Schaefer 1978]

For every Γ, the $\mathrm{SAT}(\Gamma)$ problem is polynomial-time solvable if one of the following holds, and NP-complete otherwise:

- Every relation is satisfied by the all 0 assignment
- Every relation is satisfied by the all 1 assignment
- Every relation can be expressed by a 2SAT formula
- Every relation can be expressed by a Horn formula
- Every relation can be expressed by an anti-Horn formula
- Every relation is an affine subspace over GF(2)

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This is surprising for two reasons:

- this family does not contain NP-intermediate problems and
- the boundary of polynomial-time and NP-hard problems can be cleanly characterized.

Other dichotomy results

- Approximability of MAX-SAT, MIN-UNSAT [Khanna et al. 2001]
- Approximability of MAXONES-SAT, MINONES-SAT [Khanna et al. 2001]
- Generalization to 3-valued variables [Bulatov 2002]
- Inverse satisfiability [Kavvadias and Sideri, 1999]
- etc.

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- etc.

Celebrated open question: generalize Schaefer's result to relations over variables with non-Boolean, but fixed domain.

CSP(Γ): similar to SAT(Γ), but with non-Boolean domain.

Conjecture [Feder and Vardi 1998]

Let Γ be a finite set of relations over an arbitrary fixed domain. Then CSP(Γ) is either polynomial-time solvable or NP-complete.

Weighted problems

Parameterizing by the weight (= number of 1s) of the solution.

- MinOnes-Sat(Γ) :
  Find a satisfying assignment with weight at most k
- ExactOnes-Sat(Γ) :
  Find a satisfying assignment with weight exactly k
- MaxOnes-Sat(Γ) :
  Find a satisfying assignment with weight at least k

The first two problems can be always solved in $n^{O(k)}$ time, and the third one as well if Sat(Γ) is in P.

Goal: Characterize which languages Γ make these problems FPT.

ExactOnes-Sat(Γ)

Theorem [Marx 2004]

ExactOnes-Sat(Γ) is FPT if Γ is weakly separable and W[1]-hard otherwise.

Examples of weakly separable constraints:

- affine constraints
- "0 or 5 out of 8"

Examples of not weakly separable constraints:

- $(\neg x \vee \neg y)$
- $x \rightarrow y$
- "0 or 4 out of 8"

Larger domains

What is the generalization of $\textsc{ExactOnes-Sat}(\Gamma)$ to larger domains?

1. Find a solution with exactly k nonzero values (zeros constraint).
2. Find a solution where nonzero value i appears exactly k_i times (cardinality constraint).

Theorem [Bulatov and M. 2011]

For every Γ closed under substituting constants, $\mathrm{CSP}(\Gamma)$ with zeros constraint is FPT or W[1]-hard.

Larger domains

The following two problems are equivalent:

- CSP(Γ) with cardinality constraint, where Γ contains only the relation $R = \{00, 10, 02\}$.
- BICLIQUE: Find a complete bipartite graph with k vertices on each side. The fixed-parameter tractability of BICLIQUE is a notorious open problem (conjectured to be hard).

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- CSP(Γ) with cardinality constraint, where Γ contains only the relation $R = \{00, 10, 02\}$.
- Biclique: Find a complete bipartite graph with k vertices on each side. The fixed-parameter tractability of Biclique is a notorious open problem (conjectured to be hard).

So the best we can get at this point:

Theorem [Bulatov and M. 2011]

For every Γ closed under substituting constants, CSP(Γ) with cardinality constraint is FPT or Biclique-hard.

MinOnes-Sat(Γ)

The bounded-search tree algorithm for Vertex Cover can be generalized to MinOnes-Sat.

Observation

MinOnes-Sat(Γ) is FPT for every finite Γ.

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But can we solve the problem simply by preprocessing?

Definition

A polynomial kernel is a polynomial-time reduction creating an equivalent instance whose size is polynomial in k.

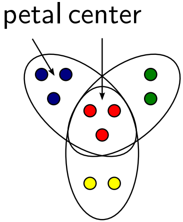
Goal: Characterize the languages Γ for which MinOnes-Sat(Γ) has a polynomial kernel.

Example: the special case d-Hitting Set (where Γ contains only $R = x_1 \vee \cdots \vee x_d$) has a polynomial kernel.

Sunflower lemma

Definition

Sets S_1, S_2, $\ldots$, S_k form a **sunflower** if the sets $S_i \setminus (S_1 \cap S_2 \cap \cdots \cap S_k)$ are disjoint.



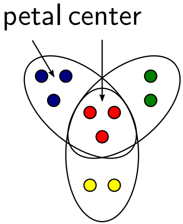
Lemma [Erdős and Rado, 1960]

If the size of a set system is greater than $(p-1)^d \cdot d!$ and it contains only sets of size at most d, then the system contains a sunflower with p petals.

Sunflowers and d-Hitting Set

d-Hitting Set

Given a collection $\mathcal{S}$ of sets of size at most d and an integer k, find a set S of k elements that intersects every set of $\mathcal{S}$.



Reduction Rule

Suppose more than $k+1$ sets form a sunflower.

- If the sets are disjoint $\Rightarrow$ No solution.
- Otherwise, keep only $k+1$ of the sets.

Dichotomy for kernelization

Kernelization for general MinOnes-Sat(Γ) generalizes the sunflower reduction, and requires that Γ is "mergeable."

Theorem [Kratsch and Wahlström 2010]

(1) If MinOnes-Sat(Γ) is polynomial-time solvable or Γ is mergeable, then MinOnes-Sat(Γ) has a polynomial kernelization.

(2) If MinOnes-Sat(Γ) is NP-hard and Γ is not mergebable, then MinOnes-Sat(Γ) does not have a polynomial kernel, unless the polynomial hierarchy collapses.

Dichotomy for kernelization

Similar results for other problems:

Theorem [Kratsch, M., Wahlström 2010]

- If Γ has property X, then MaxOnes-Sat(Γ) has a polynomial kernel, and otherwise no (unless the polynomial hierarchy collapses).
- If Γ has property Y, then ExactOnes-Sat(Γ) has a polynomial kernel, and otherwise no (unless the polynomial hierarchy collapses).

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Local search

Walk in the solution space by iteratively replacing the current solution with a better solution in the local neighborhood.

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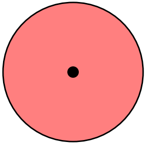

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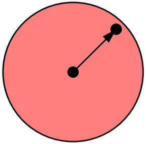

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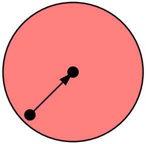

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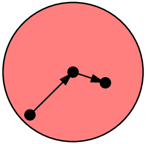

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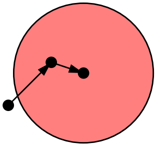

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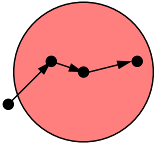

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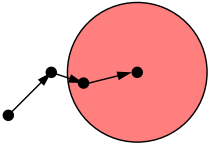

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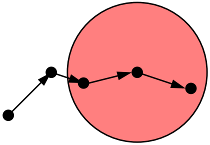

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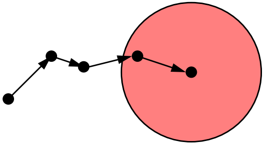

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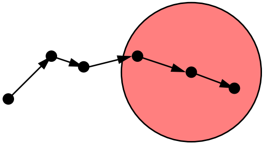

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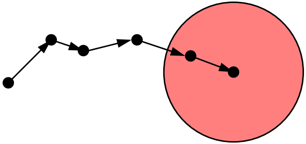

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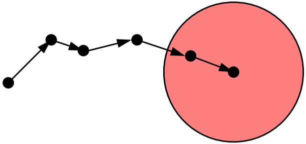

Problem: local search can stop at a local optimum (no better solution in the local neighborhood).

More sophisticated variants: simulated annealing, tabu search, etc.

Local neighborhood

The local neighborhood is defined in a problem-specific way:

- For TSP, the neighbors are obtained by swapping 2 cities or replacing 2 edges.
- For a problem with 0-1 variables, the neighbors are obtained by flipping a single variable.
- For subgraph problems, the neighbors are obtained by adding/removing one edge.

More generally: reordering k cities, flipping k variables, etc.

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More generally: reordering k cities, flipping k variables, etc.

Larger neighborhood (larger k):

- algorithm is less likely to get stuck in a local optimum,
- it is more difficult to check if there is a better solution in the neighborhood.

Searching the neighborhood

Question: Is there an efficient way of finding a better solution in the k-neighborhood?

We study the complexity of the following problem:

k-step Local Search

Input:	instance I, solution x, integer k
Find:	A solution x' with $\mathrm{dist}(x, x') \leq k$ that is "better" than x.

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Remark 1: If the optimization problem is hard, then it is unlikely that this local search problem is polynomial-time solvable: otherwise we would be able to find an optimum solution.

Remark 2: Size of the k-neighborhood is usually $n^{O(k)} \Rightarrow$ local search is polynomial-time solvable for every fixed k, but this is not practical for larger k.

k-step Local Search

The question that we want to investigate:

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Is k-step Local Search FPT for a particular problem?

If yes, then local search algorithms can consider larger neighborhoods, improving their efficiency.

Important: k is the number of allowed changes and **not** the size of the solution. Relevant even if solution size is large.

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Examples:

- Local search is easy: it is FPT to find a larger independent set in a planar graph with at most k exchanges [Fellows et al. 2008].
- Local search is hard: it is W[1]-hard to check if it is possible to obtain a shorter TSP tour by replacing at most k arcs [M. 2008].

Local search for SAT

Simple satisfiability:

Theorem [Dantsin et al. 2002]

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A family of problems:

Theorem [Krokhin and M. 2008]

Dichotomy results for MINONES-SAT(Γ).

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Something strange: for some problems (e.g., VERTEX COVER on bipartite graphs), local search is hard, even though the problem is polynomial-time solvable.

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Strict k-step Local Search

Input: instance I, solution x, integer k

Find: A solution x' with $\mathrm{dist}(x, x') \leq k$ that is "better" than x.

Permissive k-step Local Search

Input: instance I, solution x, integer k

Find: Any solution x' "better" than x, if there is such a solution at distance at most k.

Constraint Satisfaction Problems (CSP)

A CSP instance is given by describing the

- variables,
- domain of the variables,
- constraints on the variables.

Task: Find an assignment that satisfies every constraint.

$$I = C_1(x_1, x_2, x_3) \wedge C_2(x_2, x_4) \wedge C_3(x_1, x_3, x_4)$$

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Examples:

- 3SAT: 2-element domain, every constraint is ternary
- VERTEX COLORING: domain is the set of colors, binary constraints
- k-CLIQUE (in graph G): k variables, domain is the vertices of G, $\binom{k}{2}$ binary constraints

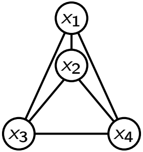
Graphs and hypergraphs related to CSP

Gaifman/primal graph: vertices are the variables, two variables are adjacent if they appear in a common constraint.

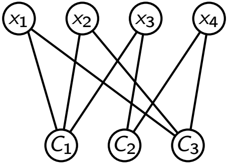
Incidence graph: bipartite graph, vertices are the variables and constraints.

Hypergraph: vertices are the variables, constraints are the hyperedges.

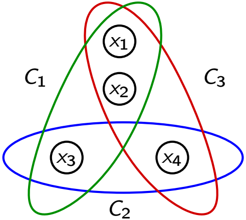
$$I = C_1(x_2, x_1, x_3) \wedge C_2(x_4, x_3) \wedge C_3(x_1, x_4, x_2)$$



Primal graph



Incidence graph



Hypergraph

Treewidth and CSP

Theorem [Freuder 1990]

For every fixed k, CSP can be solved in polynomial time if the primal graph of the instance has treewidth at most k.

Note: The running time is $|D|^{O(k)}$, which is not FPT parameterized by treewidth.

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Note: The running time is $|D|^{O(k)}$, which is not FPT parameterized by treewidth.

We know that binary CSP($\mathcal{G}$) is polynomial-time solvable for every class $\mathcal{G}$ of graphs with bounded treewidth. Are there other polynomial cases?

Tractable structures

Question: Which graph properties lead to polynomial-time solvable CSP instances?

Systematic study:

- Binary CSP: Every constraint is of arity 2.
- CSP($\mathcal{G}$): problem restricted to binary CSP instances with primal graph in $\mathcal{G}$.
- Which classes $\mathcal{G}$ make CSP($\mathcal{G}$) FPT?
- E.g., if $\mathcal{G}$ is the set of trees, then it is easy, if $\mathcal{G}$ is the set of 3-regular graphs, then it is W[1]-hard.

Dichotomy for binary CSP

Complete answer for **every** class $\mathcal{G}$:

Theorem [Grohe-Schwentick-Segoufin 2001]

Let $\mathcal{G}$ be a computable class of graphs.

(1) If $\mathcal{G}$ has bounded treewidth, then CSP($\mathcal{G}$) is FPT parameterized by number of variables (in fact, polynomial-time solvable).

(2) If $\mathcal{G}$ has unbounded treewidth, then CSP($\mathcal{G}$) is W[1]-hard parameterized by number of variables.

Note: In (2), CSP($\mathcal{G}$) is not necessarily NP-hard.

Dichotomy for binary CSP

Complete answer for **every** class $\mathcal{G}$:

Theorem [Grohe-Schwentick-Segoufin 2001]

Let $\mathcal{G}$ be a recursively enumerable class of graphs. Assuming FPT $\neq$ W[1], the following are equivalent:

- Binary CSP($\mathcal{G}$) is polynomial-time solvable.
- Binary CSP($\mathcal{G}$) is FPT.
- $\mathcal{G}$ has bounded treewidth.

Note: Fixed-parameter tractability does not give us more power here than polynomial-time solvability!

Combination of parameters

CSP can be parameterized by many (combination of) parameters.

Examples:

- CSP is W[1]-hard parameterized by the treewidth of the primal graph.
- CSP is FPT parameterized by the treewidth of the primal graph and the domain size.

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- CSP is FPT parameterized by the treewidth of the primal graph and the domain size.

[Samer and Szeider 2010] considered 11 parameters and determined the complexity of CSP by any subset of these parameters.

tw:	treewidth of primal graph	arity:	maximum arity
tw^d:	tw of dual graph	dep:	largest relation size
tw^*:	tw of incidence graph	deg:	largest variable occurrence
vars:	number of variables	ovl:	largest overlap between scopes
dom:	domain size	diff:	largest difference between scopes
cons:	number of constraints		

Summary

- Fixed-parameter tractability results for SAT and CSPs do exist.
- Choice of parameter is not obvious.
- Above average parameterization.
- Local search.
- Parameters related to the graph of the constraints.