

# Constant-Factor Approximations for Asymmetric TSP on Nearly-Embeddable Graphs\*

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## Abstract

In the Asymmetric Traveling Salesperson Problem (ATSP) the goal is to find a closed walk of minimum cost in a directed graph visiting every vertex. We consider the approximability of ATSP on topologically restricted graphs. It has been shown by Oveis Gharan and Saberi [13] that there exists polynomial-time constant-factor approximations on planar graphs and more generally graphs of constant orientable genus. This result was extended to non-orientable genus by Erickson and Sidiropoulos [8].

We show that for any class of *nearly-embeddable* graphs, ATSP admits a polynomial-time constant-factor approximation. More precisely, we show that for any fixed  $k \geq 0$ , there exist  $\alpha, \beta > 0$ , such that ATSP on  $n$ -vertex  $k$ -nearly-embeddable graphs admits an  $\alpha$ -approximation in time  $O(n^\beta)$ . The class of  $k$ -nearly-embeddable graphs contains graphs with at most  $k$  apices,  $k$  vortices of width at most  $k$ , and an underlying surface of either orientable or non-orientable genus at most  $k$ . Prior to our work, even the case of graphs with a single apex was open. Our algorithm combines tools from rounding the Held-Karp LP via thin trees with dynamic programming.

We complement our upper bounds by showing that solving ATSP exactly on graphs of pathwidth  $k$  (and hence on  $k$ -nearly embeddable graphs) requires time  $n^{\Omega(k)}$ , assuming the Exponential-Time Hypothesis (ETH). This is surprising in light of the fact that both TSP on undirected graphs and Minimum Cost Hamiltonian Cycle on directed graphs are FPT parameterized by treewidth.

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## 1 Introduction

An instance of the Asymmetric Traveling Salesman Problem (ATSP) consists of a directed graph  $\vec{G}$  and a (not necessarily symmetric) cost function  $c : E(\vec{G}) \rightarrow \mathbb{R}^+$ . The goal is to find a spanning closed walk of  $\vec{G}$  with minimum total cost. This is one of the most well-studied NP-hard problems.

Asadpour *et al.* [2] obtained a polynomial-time  $O(\log n / \log \log n)$ -approximation algorithm for ATSP, which was the first asymptotic improvement in almost 30 years [12, 3, 9, 18]. Building on their techniques, Oveis Gharan and Saberi [13] described a polynomial-time  $O(\sqrt{g} \log g)$ -approximation algorithm when the input includes an embedding of the input graph into an orientable surface of genus  $g$ . Erickson and Sidiropoulos [8] improved the dependence on the genus by obtaining a  $O(\log g / \log \log g)$ -approximation.

Anari and Oveis Gharan [1] have recently shown that the integrality gap of the natural linear programming relaxation of ATSP proposed by Held and Karp [16] is  $\log \log^{O(1)} n$ . This implies a polynomial-time  $\log \log^{O(1)} n$ -approximation algorithm for the *value* of ATSP. We remark that the best known lower bound on the integrality gap of the Held-Karp LP is 2 [4]. Obtaining a polynomial-time constant-factor approximation algorithm for ATSP is a central open problem in optimization.

### 1.1 Our contribution

We study the approximability of ATSP on topologically restricted graphs. Prior to our work, a constant-factor approximation algorithm was known only for graphs of bounded genus. We significantly extend this result by showing that there exists a polynomial-time constant-factor approximation algorithm for ATSP on nearly embeddable graphs. These graphs include graphs with bounded genus, with a bounded number of apices and a bounded number of vortices of bounded pathwidth. We remark that prior to our work, even the case of planar graphs with a single apex was open<sup>1</sup>. For any  $a, g, k, p \geq 0$ , we say that a graph is  $(a, g, k, p)$ -nearly embeddable if it is obtained from a graph of Euler genus  $g$  by adding  $a$  apices and  $k$  vortices of pathwidth  $p$  (see [20, 19, 7] for more precise definitions). The following summarizes our result.

► **Theorem 1.** *Let  $a, g, k \geq 0, p \geq 1$ . There is a  $O(a(g+k+1)+p^2)$ -approximation algorithm for ATSP on  $(a, g, k, p)$ -nearly embeddable digraphs, with running time  $n^{O((a+p)(g+k+1)^4)}$ .*

The above algorithm is obtained via a new technique that combines the Held-Karp LP with a dynamic program that solves the problem on vortices. We remark that it is not known whether the integrality gap of the LP is constant for graphs of constant pathwidth.

We complement this result by showing that solving ATSP exactly on graphs of pathwidth  $p$  (and hence on  $p$ -nearly embeddable graphs) requires time  $n^{\Omega(p)}$ , assuming the Exponential-Time Hypothesis (ETH). This is surprising in light of the fact that both TSP on undirected graphs and Minimum Cost Hamiltonian Cycle on directed graphs are FPT parameterized by treewidth. The following summarizes our lower bound.

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<sup>1</sup> Previous algorithms for constructing thin trees [13] and forests [8] on surface-embedded graphs depend critically on the relation between cuts and cycles in the dual graph, and thus are not directly applicable to the case of graphs even with a single apex. We also remark that the optimal walk might traverse the apex arbitrarily many times; thus, any approach that attempts to first solve the problem on the subgraph obtained by removing the apex, cannot yield a constant-factor approximation.

► **Theorem 2.** *Assuming ETH, there is no  $f(p)n^{o(p)}$  time algorithm for ATSP on graphs of pathwidth at most  $p$  for any computable function  $f$ .*

## 1.2 Overview of the algorithm

We now give a high level overview of the main steps of the algorithm and highlight some of the main challenges.

**Step 1: Reducing the number of vortices.** We first reduce the problem to the case of nearly embeddable graphs with a single vortex. This is done by iteratively merging pairs of vortices. We can merge two vortices by adding a new handle on the underlying surface-embedded graph. For the remainder we will focus on the case of graphs with a single vortex.

**Step 2: Traversing a vortex.** We obtain an exact polynomial-time algorithm for computing a closed walk that visits all the vertices in the vortex. We remark that this subsumes as a special case the problem of visiting all the vertices in a single face of a planar graph, which was open prior to our work.

Let us first consider the case of a vortex in a planar graph. Let  $\vec{W}$  be an optimal walk that visits all the vertices in the vortex. Let  $F$  be the face on which the vortex is attached. We give a dynamic program that maintains a set of partial solutions for each subpath of  $F$ . We prove correctness of the algorithm by establishing structural properties of  $\vec{W}$ . The main technical difficulty is that  $\vec{W}$  might be self-crossing. We first decompose  $\vec{W}$  into a collection  $\mathcal{W}$  of non-crossing walks. We form a conflict graph  $\mathcal{I}$  of  $\mathcal{W}$  and consider a spanning forest  $\mathcal{F}$  of  $\mathcal{I}$ . This allows us to prove correctness via induction on the trees of  $\mathcal{F}$ .

The above algorithm can be extended to graphs of bounded genus. The main difference is that the dynamic program now computes a set of partial solutions for each bounded collection of subpaths of  $F$ .

Finally, the algorithm is extended to the case of nearly-embeddable graphs by adding the apices to the vortex without changing the cost of the optimum walk.

**Step 3: Finding a thin forest in the absence of vortices.** The constant-factor approximation for graphs of bounded genus was obtained by constructing thin forests with a bounded number of components in these graphs [13, 8]. We extend this result by constructing thin forests with a bounded number of components in graphs of bounded genus and with a bounded number of additional apices. Prior to our work even the case of planar graphs with a single apex was open; in fact, no constant-factor approximation algorithm was known for these graphs.

**Step 4: Combining the Held-Karp LP with the dynamic program.** We next combine the dynamic program with the thin forest construction. We first compute an optimal walk  $\vec{W}$  visiting all the vertices in the vortex, and we contract the vortex into a single vertex. A natural approach would be to compute a thin forest in the contracted graph. Unfortunately this fails because such a forest might not be thin in the original graph. In order to overcome this obstacle we change the feasible solution of the Held-Karp LP by taking into account  $\vec{W}$ , and we modify the forest construction so that it outputs a subgraph that is thin with respect to this new feasible solution.

**Step 5: Rounding the forest into a walk.** Once we have a thin spanning subgraph of  $G$  we can compute a solution to ATSP via circulations, as in previous work.

### 1.3 Organization

The rest of the paper is organized as follows. Section 2 introduces some basic notation. Section 3 defines the Held-Karp LP for ATSP. Section 4 presents the main algorithm, using the dynamic program and the thin forest construction as a black box. Section 5 presents the technique for combining the dynamic program with the Held-Karp LP. Section 6 gives the algorithm for computing a thin tree in a 1-apex graph. This algorithm is generalized to graphs with a bounded number of apices in Section 7, and to graphs of bounded genus and with a bounded number of apices in Section 8. In Section 9 we show how to modify the thin forest construction so that we can compute a spanning thin subgraph in a nearly-embeddable graph, using the solution of the dynamic program.

The dynamic program is given in Sections 10, 11, 12, 13, and 14. More precisely, Section 10 introduces a certain preprocessing step. Section 11 establishes a structural property of the optimal solution. Section 12 presents the dynamic program for a vortex in a planar graph. Sections 13 and 14 generalize this dynamic program to graphs of bounded genus and with a bounded number of apices respectively.

Finally, Section 15 presents the lower bound.

## 2 Notation

In this section we introduce some basic notation that will be used throughout the paper.

**Graphs.** Unless otherwise specified, we will assume that for every pair of vertices in a graph there exists a unique shortest path; this property can always be achieved by breaking ties between different shortest paths in a consistent manner (e.g. lexicographically). Moreover for every edge of a graph (either directed or undirected) we will assume that its length is equal to the shortest path distance between its endpoints. Let  $\vec{G}$  be some digraph. Let  $G$  be the undirected graph obtained from  $\vec{G}$  by ignoring the directions of the edges, that is  $V(G) = V(\vec{G})$  and  $E(G) = \{\{u, v\} : (u, v) \in E(\vec{G}) \text{ or } (v, u) \in E(\vec{G})\}$ . We say that  $G$  is the *symmetrization* of  $\vec{G}$ . For some  $x : E(\vec{G}) \rightarrow \mathbb{R}$  we define  $\text{cost}_{\vec{G}}(x) = \sum_{(u,v) \in E(\vec{G})} x((u,v)) \cdot d_{\vec{G}}(u,v)$ . For a subgraph  $S \subseteq G$  we define  $\text{cost}_G(S) = \sum_{e \in E(S)} c(e)$ . Let  $z$  be a weight function on the edges of  $G$ . For any  $A, B \subseteq V(G)$  we define  $z(A, B) = \sum_{a \in A, b \in B} z(\{a, b\})$ .

**Asymmetric TSP.** Let  $\vec{G}$  be a directed graph with non-negative arc costs. For each arc  $(u, v) \in E(\vec{G})$  we denote the cost of  $(u, v)$  by  $c(u, v)$ . A *tour* in  $\vec{G}$  is a closed walk in  $\vec{G}$ . The *cost* of a tour  $\tau = v_1, v_2, \dots, v_k, v_1$  is defined to be  $\text{cost}_{\vec{G}}(\tau) = d_{\vec{G}}(v_k, v_1) + \sum_{i=1}^{k-1} d_{\vec{G}}(v_i, v_{i+1})$ . Similarly the cost of an open walk  $W = v_1, \dots, v_k$  is defined to be  $\text{cost}_{\vec{G}}(W) = \sum_{i=1}^{k-1} d_{\vec{G}}(v_i, v_{i+1})$ . The cost of a collection  $\mathcal{W}$  of walks is defined to be  $\text{cost}_{\vec{G}}(\mathcal{W}) = \sum_{W \in \mathcal{W}} \text{cost}_{\vec{G}}(W)$ . We denote by  $\text{OPT}_{\vec{G}}$  the minimum cost of a tour traversing all vertices in  $\vec{G}$ . For some  $U \subseteq V(\vec{G})$  we denote by  $\text{OPT}_{\vec{G}}(U)$  the minimum cost of a tour in  $\vec{G}$  that visits all vertices in  $U$ .

## 3 The Held-Karp LP

We recall the Held-Karp LP for ATSP [15]. Fix a directed graph  $\vec{G}$  and a cost function  $c : E(\vec{G}) \rightarrow \mathbb{R}^+$ . For any subset  $U \subseteq V$ , we define  $\delta_{\vec{G}}^+(U) := \{(u, v) \in E(\vec{G}) : u \in U \text{ and } v \notin U\}$  and  $\delta_{\vec{G}}^-(U) := \delta_{\vec{G}}^+(V \setminus U)$ . We omit the subscript  $\vec{G}$  when the underlying graph is clear from context. We also write  $\delta^+(v) = \delta^+(\{v\})$  and  $\delta^-(v) = \delta^-(\{v\})$  for any single vertex  $v$ .

Let  $G$  be the symmetrization of  $\vec{G}$ . For any  $U \subseteq V(G)$ , we define  $\delta_G(U) := \{\{u, v\} \in E(G) : u \in U \text{ and } v \notin U\}$ . Again, we omit the subscript  $G$  when the underlying graph is clear from context. We also extend the cost function  $c$  to undirected edges by defining  $c(\{u, v\}) := \min\{c(u, v), c(v, u)\}$ . For any function  $x: E(\vec{G}) \rightarrow \mathbb{R}$  and any subset  $W \subseteq E(\vec{G})$ , we write  $x(W) = \sum_{a \in W} x(a)$ . With this notation, the Held-Karp LP relaxation is defined as follows.

$$\begin{array}{ll} \text{minimize} & \sum_{a \in E(\vec{G})} c(a) \cdot x(a) \\ \text{subject to} & x(\delta^+(U)) \geq 1 \quad \text{for all nonempty } U \subsetneq V(\vec{G}) \\ & x(\delta^+(v)) = x(\delta^-(v)) \quad \text{for all } v \in V(\vec{G}) \\ & x(a) \geq 0 \quad \text{for all } a \in E(\vec{G}) \end{array}$$

We define the *symmetrization* of  $x$  as the function  $z: E(G) \rightarrow \mathbb{R}$  where  $z(\{u, v\}) := x(u, v) + x(v, u)$  for every edge  $\{u, v\} \in E(G)$ . For any subset  $W \subseteq E(G)$  of edges, we write  $z(W) := \sum_{e \in W} z(e)$ . Let  $\vec{W} \subseteq \vec{G}$ . Let  $\alpha, s > 0$ . We say that  $\vec{W}$  is  $\alpha$ -thin (w.r.t.  $z$ ) if for all  $U \subseteq V$  we have  $|E(\vec{W}) \cap \delta(U)| \leq \alpha \cdot z(\delta(U))$ . We also say that  $\vec{W}$  is  $(\alpha, s)$ -thin (w.r.t.  $x$ ) if  $\vec{W}$  is  $\alpha$ -thin (w.r.t.  $z$ ) and  $c(E(\vec{W})) \leq s \cdot \sum_{e \in E(\vec{G})} c(e) \cdot x(e)$ . We say that  $z$  is  $\vec{W}$ -dense if for all  $(u, v) \in E(\vec{W})$  we have  $z(\{u, v\}) \geq 1$ . We say that  $z$  is  $\varepsilon$ -thick if for all  $U \subsetneq V(G)$  with  $U \neq \emptyset$  we have  $z(\delta(U)) \geq \varepsilon$ .

#### 4 An approximation algorithm for nearly-embeddable graphs

The following Lemma is implicit in the work of Erickson and Sidiropoulos [8] (see also [2]).

► **Lemma 3.** *Let  $\vec{G}$  be a digraph and let  $x$  be a feasible solution for the Held-Karp LP for  $\vec{G}$ . Let  $\alpha, s > 0$ , and let  $S$  be a  $(\alpha, s)$ -thin spanning subgraph of  $G$  (w.r.t.  $x$ ), with at most  $k$  connected components. Then, there exists a polynomial-time algorithm which computes a collection of closed walks  $C_1, \dots, C_{k'}$ , for some  $k' \leq k$ , such that their union visits all the vertices in  $V(\vec{G})$ , and such that  $\sum_{i=1}^{k'} \text{cost}_{\vec{G}}(C_i) \leq (2\alpha + s) \sum_{e \in E(\vec{G})} c(e) \cdot x(e)$ .*

The following is the main technical Lemma that combines a solution to the Held-Karp LP with a walk traversing the vortex that is computed via the dynamic program. The proof of Lemma 4 is deferred to Section 5. A similar result, for the case of graphs of orientable genus, was first obtained in [13].

► **Lemma 4.** *Let  $a, g, p > 0$ , let  $\vec{G}$  be a  $(a, g, 1, p)$ -nearly embeddable graph, and let  $G$  be its symmetrization. There exists an algorithm with running time  $n^{O((a+p)g^4)}$  which computes a feasible solution  $x$  for the Held-Karp LP for  $\vec{G}$  with cost  $O(\text{OPT}_{\vec{G}})$  and a spanning subgraph  $S$  of  $G$  with at most  $O(a + g)$  connected components, such that  $S$  is  $(O(a \cdot g + p^2), O(1))$ -thin w.r.t.  $x$ .*

Using Lemma 4 we are now ready to obtain an approximation algorithm for nearly-embeddable graphs with a single vortex.

► **Theorem 5.** *Let  $a, g \geq 0$ ,  $p \geq 1$ . There exists a  $O(a \cdot g + p^2)$ -approximation algorithm for ATSP on  $(a, g, 1, p)$ -nearly embeddable digraphs, with running time  $n^{O((a+p)(g+1)^4)}$ .*

**Proof.** We follow a similar approach to [8]. The only difference is that in [8] the algorithm uses an optimal solution to the Held-Karp LP. In contrast, here we use a feasible solution that is obtained by Lemma 4, together with an appropriate thin subgraph.

Let  $\vec{G}$  be  $(a, g, 1, p)$ -nearly embeddable digraph. By using Lemma 4, we find in time  $n^{O((a+p)g^4)}$  a feasible solution  $x$  for the Held-Karp LP for  $\vec{G}$  with cost  $O(\text{OPT}_{\vec{G}})$  and a spanning subgraph  $S$  of  $G$  with at most  $O(a + g)$  connected components, such that  $S$  is  $(O(a \cdot g + p^2), O(1))$ -thin w.r.t.  $x$ . Now we compute in polynomial time a collection of closed walks  $C_1, \dots, C_{k'}$ , for some  $k' \in O(a + g)$ , that visit all the vertices in  $V(\vec{G})$ , and such that the total cost of all walks is at most  $O((a \cdot g + p^2) \cdot \text{OPT}_{\vec{G}})$ , using Lemma 3. For every  $i \in \{1, \dots, k'\}$ , let  $v_i \in V(\vec{G})$  be an arbitrary vertex visited by  $C_i$ . We construct a new instance  $(\vec{G}', c')$  of ATSP as follows. Let  $V(\vec{G}') = \{v_1, \dots, v_{k'}\}$ . For any  $u, v \in V(\vec{G}')$ , we have an edge  $(u, v)$  in  $E(\vec{G}')$ , with  $c'(u, v)$  being the shortest-path distance between  $u$  and  $v$  in  $G$  with edge weights given by  $c$ . By construction we have  $\text{OPT}_{\vec{G}'} \leq \text{OPT}_{\vec{G}}$ . We find a closed tour  $C$  in  $\vec{G}'$  with  $\text{cost}_{\vec{G}'}(C) = \text{OPT}_{\vec{G}'}$  in time  $2^{O(|V(\vec{G}')|)} \cdot n^{O(1)} = 2^{O(a+g)} \cdot n^{O(1)}$ . By composing  $C$  with the  $k'$  closed walks  $C_1, \dots, C_{k'}$ , and shortcutting as in [11], we obtain a solution for the original instance, of total cost  $O(a \cdot g + p^2) \cdot \text{OPT}_{\vec{G}}$ . ◀

We are now ready to prove the main algorithmic result of this paper.

**Proof of Theorem 1.** We may assume  $k \geq 2$  since otherwise the assertion follows by Theorem 5. We may also assume w.l.o.g. that  $p \geq 2$ . Let  $\vec{G}$  be a  $(a, g, k, p)$ -nearly embeddable digraph. It suffices to show that there exists a polynomial time computable  $(a, g + k - 1, 1, 2p)$ -nearly embeddable digraph  $\vec{G}'$  with  $V(\vec{G}') = V(\vec{G})$  such that for all  $u, v \in V(\vec{G})$  we have  $d_{\vec{G}}(u, v) = d_{\vec{G}'}(u, v)$ . We compute  $\vec{G}'$  as follows. Let  $\vec{H}_1, \dots, \vec{H}_k$  be the vortices of  $\vec{G}$  and let  $\vec{F}_1, \dots, \vec{F}_k$  be the faces on which they are attached. For each  $i \in \{1, \dots, k\}$  pick distinct  $e_i, f_i \in E(\vec{F}_i)$ , with  $e_i = \{w_i, w'_i\}$ ,  $f_i = \{z_i, z'_i\}$ . There exists a path decomposition  $B_{i,1}, \dots, B_{i,\ell_i}$  of  $\vec{H}_i$ , of width at most  $2p$ , and such that  $B_{i,1} = e_i$ , and  $B_{i,\ell_i} = f_i$ . For each  $i \in \{1, \dots, k-1\}$ , we add edges  $(w_{i+1}, z_i)$ ,  $(z_i, w_{i+1})$ ,  $(w'_{i+1}, z'_i)$ , and  $(z'_i, w'_{i+1})$  to  $\vec{G}'$ , and we set their length to be equal to the shortest path distance between their endpoints in  $\vec{G}$ . We also add a handle connecting punctures in the disks bounded by  $\vec{F}_i$  and  $\vec{F}_{i+1}$  respectively, and we route the four new edges along this handle. Since we add  $k-1$  handles in total the Euler genus of the underlying surface increases by at most  $k-1$ . We let  $\vec{H}$  be the single vortex in  $\vec{G}'$  with  $V(\vec{H}) = \bigcup_{i=1}^k V(\vec{H}_i)$  and  $E(\vec{H}) = \left( \bigcup_{i=1}^k E(\vec{H}_i) \right) \cup \left( \bigcup_{i=1}^{k-1} \{(w_{i+1}, z_i), (z_i, w_{i+1}), (w'_{i+1}, z'_i), (z'_i, w'_{i+1})\} \right)$ . It is immediate that

$$B_{1,1}, \dots, B_{1,\ell_1}, \{f_1, e_2\}, B_{2,1}, \dots, B_{2,\ell_2}, \{f_2, e_3\}, \dots, B_{k,1}, \dots, B_{k,\ell_k}$$

is a path decomposition of  $\vec{H}$  of width at most  $2p$ . Thus  $\vec{G}'$  is  $(a, g + k - 1, 1, 2p)$ -nearly embeddable, which concludes the proof. ◀

## 5 Combining the Held-Karp LP with the dynamic program

In this Section we show how to combine the dynamic program that finds an optimal closed walk traversing all the vertices in a vortex, with the Held-Karp LP. The following summarizes our exact algorithm for traversing the vortex in a nearly-embeddable graph. The proof of Theorem 6 is deferred to Section 14.

► **Theorem 6.** *Let  $\vec{G}$  be an  $n$ -vertex  $(a, g, 1, p)$ -nearly embeddable graph and let  $\vec{H}$  be the single vortex of  $\vec{G}$ . Then there exists an algorithm which computes a walk  $\vec{W}$  visiting all vertices in  $V(\vec{H})$  of total length at most  $\text{OPT}_{\vec{G}}(V(\vec{H}))$  in time  $n^{O((a+p)g^4)}$ .*

► **Definition 7** ( $\vec{W}$ -augmentation). Let  $\vec{G}$  be a directed graph. Let  $x : E(\vec{G}) \rightarrow \mathbb{R}$  and let  $\vec{W} \subseteq \vec{G}$ . We define the  $\vec{W}$ -augmentation of  $x$  to be the function  $x' : E(\vec{G}) \rightarrow \mathbb{R}$  such that for all  $e \in E(\vec{G})$  we have

$$x'(e) = \begin{cases} x(e) + 1 & \text{if } e \in E(\vec{W}) \\ x(e) & \text{otherwise} \end{cases}$$

The following summarizes the main technical result for computing a thin spanning subgraph in a nearly embeddable graph. The proof of Lemma 8 is deferred to Section 9.

► **Lemma 8.** *Let  $\vec{G}$  be a  $(a, g, 1, p)$ -nearly embeddable digraph, let  $\vec{H}$  be its vortex, and let  $\vec{W}$  be a walk in  $\vec{G}$  visiting all vertices in  $V(\vec{H})$ . Let  $G, H,$  and  $W$  be the symmetrizations of  $\vec{G}, \vec{H},$  and  $\vec{W}$  respectively. Let  $z : E(G) \rightarrow \mathbb{R}_{\geq 0}$  be  $\alpha$ -thick for some  $\alpha \geq 2$ , and  $\vec{W}$ -dense. Then there exists a polynomial time algorithm which given  $\vec{G}, \vec{H}, A, \vec{W}, z,$  and an embedding of  $\vec{G} \setminus (A \cup \vec{H})$  into a surface of genus  $g$ , outputs a subgraph  $S \subseteq G \setminus H$ , satisfying the following conditions:*

1.  $W \cup S$  is a spanning subgraph of  $G$  and has  $O(a + g)$  connected components.
2.  $W \cup S$  is  $O(a \cdot g + p^2)$ -thin w.r.t.  $z$ .

We are now ready to prove the main result of this section.

**Proof of Lemma 4.** Let  $\vec{H}$  be the single vortex of  $\vec{G}$ . We compute an optimal solution  $y : E(\vec{G}) \rightarrow \mathbb{R}$  for the Held-Karp LP for  $\vec{G}$ . We find a tour  $\vec{W}$  in  $\vec{G}$  visiting all vertices in  $V(\vec{H})$ , with  $\text{cost}_{\vec{G}}(\vec{W}) = O(\text{OPT}_{\vec{G}})$  using Theorem 6. Let  $x : E(\vec{G}) \rightarrow \mathbb{R}$  be the  $\vec{W}$ -augmentation of  $y$ . Since for all  $e \in E(\vec{G})$  we have  $x(e) \geq y(e)$ , it follows that  $x$  is a feasible solution for the Held-Karp LP. Moreover since  $\text{cost}_{\vec{G}}(\vec{W}) = O(\text{OPT}_{\vec{G}})$ , we obtain that  $\text{cost}_{\vec{G}}(x) = \text{cost}_{\vec{G}}(y) + \text{cost}_G(\vec{W}) = O(\text{OPT}_{\vec{G}})$ . Let  $z$  be the symmetrization of  $x$ .

Note that  $z$  is 2-thick and  $\vec{W}$ -dense. Therefore, by Lemma 8 we can find a subgraph  $S \subseteq G \setminus H$  such that  $T = W \cup S$  is a  $O(a \cdot g + p^2)$ -thin spanning subgraph of  $G$  (w.r.t.  $z$ ), with at most  $O(g + a)$  connected components. Therefore, there exists a constant  $\alpha$  such that for every  $U \subseteq V(G)$  we have  $|T \cap \delta(U)| \leq \alpha \cdot (a \cdot g + p^2) \cdot z(\delta(U))$ . We can assume that  $\alpha \geq 1$ . Now we follow a similar approach to [8].

Let  $m = \lfloor n^2/\alpha \rfloor$ . We define a sequence of functions  $z_0, \dots, z_m$ , and a sequence of spanning forests  $T_1, \dots, T_m$  satisfying the following conditions.

1. For any  $i \in \{0, \dots, m\}$ ,  $z_i$  is non-negative, 2-thick and  $\vec{W}$ -dense.
2. For any  $i \in \{1, \dots, m\}$ ,  $T_i$  has at most  $O(a + g)$  connected components.
3. For every  $U \subseteq V(G)$  we have  $|T_{i+1} \cap \delta(U)| \leq \alpha \cdot (a \cdot g + p^2) \cdot z_i(\delta(U))$ .

We set  $z_0 = 3 \lfloor zn^2 \rfloor / n^2$ . Now suppose for  $i \in \{0, \dots, m-1\}$  we have defined  $z_i$ . We define  $z_{i+1}$  and  $T_{i+1}$  as follows. We apply Lemma 8 and we obtain a subgraph  $T_{i+1}$  of  $G$  with at most  $O(a + g)$  connected components such that for every  $U \subseteq V(G)$  we have  $|T_{i+1} \cap \delta(U)| \leq \alpha \cdot (a \cdot g + p^2) \cdot z_i(\delta(U))$ . Also, for every  $e \in E(G)$  we set  $z_{i+1}(e) = z_i(e)$  if  $e \notin T_{i+1}$ , and  $z_{i+1}(e) = z_i(e) - 1/n^2$  if  $e \in T_{i+1}$ . Now by using the same argument as in [8], we obtain that  $z_{i+1}$  is non-negative and 2-thick. By the construction, we know that  $z$  is  $\vec{W}$ -dense and thus for all  $(u, v) \in \vec{W}$  we have  $z_0(\{u, v\}) \geq 3$ . Note that for all  $e \in E(G)$  we have  $z_{i+1}(e) \geq z_i(e) - 1/n^2$ . Thus for all  $e \in E(G)$  and for all  $i \in \{0, \dots, m\}$  we have  $z_i(e) \geq 2$ . Therefore for all  $i \in \{0, \dots, m\}$  we have that  $z_i$  is  $\vec{W}$ -dense.

Now, similar to [8] we set the desired  $S$  to be the subgraph  $T_i$  that minimizes  $\text{cost}_G(T_i)$ , which implies that  $S$  is a  $(O(a \cdot g + p^2), O(1))$ -thin spanning subgraph with at most  $O(a + g)$  connected components. ◀

## 6 Thin trees in 1-apex graphs

In this section we show how to compute thin trees in 1-apex graphs. The following is implicit in the work of Oveis Gharan and Saberi [13].

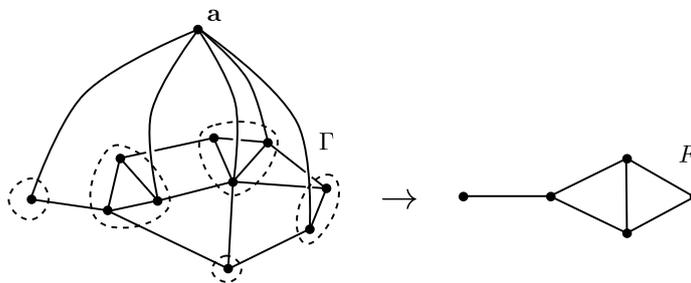
► **Theorem 9.** *If  $G$  is a planar graph and  $z$  is an  $\alpha$ -thick weight function on the edges of  $G$  for some  $\alpha > 0$ , then there exists a  $10/\alpha$ -thin spanning tree in  $G$  w.r.t.  $z$ .*

For the remainder of this section, let  $G$  be an 1-apex graph with planar part  $\Gamma$  and apex  $\mathbf{a}$ . Let  $z$  be a 2-thick weight function on the edges of  $G$ . We will find a  $O(1)$ -thin spanning tree in  $G$  (w.r.t.  $z$ ). We describe an algorithm for finding such a tree in polynomial time. The algorithm proceeds in five phases.

**Phase 1.** We say that a cut  $U$  is *tiny* (w.r.t.  $z$ ) if  $z(\delta(U)) < 0.1$ . We start with  $\Gamma$  and we proceed to partition it via tiny cuts. Each time we find a tiny cut  $U$ , we partition the remaining graph by deleting all edges crossing  $U$ . This process will stop in at most  $n$  steps. Let  $\Gamma'$  be the resulting subgraph of  $\Gamma$  where  $V(\Gamma') = V(\Gamma)$  and  $E(\Gamma') \subset E(\Gamma)$ .

**Phase 2.** By the construction, we know that there is no tiny cuts in each connected component of  $\Gamma'$ . Therefore, following [13], in each connected component  $C$  of  $\Gamma'$ , we can find a  $O(1)$ -thin spanning tree  $T_C$  (w.r.t.  $z$ ). More specifically, we will find a 100-thin spanning tree in each of them.

**Phase 3.** We define a graph  $F$  with  $V(F)$  being the set of connected components of  $\Gamma'$  and  $\{C, C'\} \in E(F)$  iff there exists an edge between some vertex in  $C$  and some vertex in  $C'$  in  $\Gamma$ . We set the weight of  $\{C, C'\}$  to be  $z(C, C')$ . We call  $F$  the *graph of components*.



We define a graph  $G'$  obtained from  $G$  by contracting every connected component of  $\Gamma'$  into a single vertex. We remark that we may get parallel edges in  $G'$ .

**Phase 4.** In this phase, we construct a tree  $T'$  in  $G'$ . We say that a vertex in  $F$  is *originally heavy*, if it has degree of at most 15 in  $F$ . Since  $F$  is planar, the minimum degree of  $F$  is at most 5. We contract all vertices in  $V(G') \setminus \{\mathbf{a}\}$  into the apex sequentially. In each step, we find a vertex in  $F$  with degree at most 5, we contract it to the apex in  $G'$ , and we delete it from  $F$ . Since the remaining graph  $F$  is always planar, there is always a vertex of degree at most 5 in it, and thus we can continue this process until all vertices of  $V(G') \setminus \{\mathbf{a}\}$  are contracted into the apex.

Initially we consider all vertices of  $F$  having no parent. In each step when we contract a vertex  $C$  with degree at most 5 in  $F$  to the apex in  $G'$ , for each neighbor  $C'$  of  $C$  in  $F$ , we make  $C$  the *parent* of  $C'$  if  $C'$  does not have any parents so far. Note that each vertex in  $F$  can be the parent of at most 5 other vertices, and can have at most one parent.

Every time we contract a vertex  $C$  to the apex, we add an edge  $e$  to  $T'$ . If  $C$  is originally heavy, we add an arbitrary edge  $e$  from  $C$  to the apex; we will show in Lemma 11 that  $z(C, \{a\}) \geq 0.5$ , which implies that such an edge always exists in  $G$ . Otherwise, we add an arbitrary edge from  $C$  to its parent (which is a neighbor vertex, therefore such an edge exists in  $G$ ). We will show in the next section that each vertex in  $F$  is originally heavy or it has a parent (or both). Therefore,  $T'$  is a tree on  $G'$ .

**Phase 5.** In this last phase we compute a tree  $T$  in  $G$ . We set  $E(T) = E(T') \cup \bigcup_{C \in V(F)} E(T_C)$ . We prove in the next subsection that  $T$  is a  $O(1)$ -thin spanning tree in  $G$ .

## 6.1 Analysis

We next show that  $T$  is a  $O(1)$ -thin spanning tree in  $G$ .

► **Lemma 10.** *The weight of every edge in  $F$  is less than 0.1.*

**Proof.** Let  $\{C, C'\} \in E(F)$ . By construction, each component of  $\Gamma'$  is formed by finding a tiny cut in some other component. Suppose  $C$  was formed either simultaneously with  $C'$  or later than  $C'$  by finding a tiny cut in some  $C''$ . If  $C' \subseteq C''$  then  $z(C, C') < z(\delta(C)) < 0.1$ . Otherwise, the total weight of edges from  $C$  to  $C'$  is a part of a tiny cut which means that  $z(C, C') < 0.1$ . ◀

► **Lemma 11.** *Let  $C$  be an originally heavy vertex in  $F$ . Then  $z(C, \{a\}) \geq 0.5$ .*

**Proof.** For every neighbor  $C'$  of  $C$  in  $F$ , by Lemma 10 we have that the weight of  $\{C, C'\}$  is less than 0.1. By the assumption on  $z$ , we have that  $z(\delta(C)) \geq 2$ . Now since  $C$  has degree of at most 15, we have that  $z(C, \{a\}) \geq 0.5$ , as desired. ◀

► **Lemma 12.** *Each vertex  $C \in V(F)$  is originally heavy or it has a parent (both cases might happen for some vertices).*

**Proof.** Let  $C \in V(F)$ . If it is originally heavy, we are done. Otherwise, it has degree of at least 15. We know that all vertices in  $F$  are going to be contracted to  $\mathbf{a}$  at some point, and we only contract vertices with degree at most 5 in each iteration. This means that at least 10 other neighbors of  $C$  were contracted to the apex before we decided to contract  $C$  to the apex. Therefore one of them is the parent of  $C$ . ◀

► **Lemma 13.**  *$T$  is a spanning tree in  $G$ .*

**Proof.** Suppose  $G$  has  $n$  vertices and  $F$  has  $m$  vertices. After the second phase of our algorithm, we obtain a spanning forest on  $\Gamma$  with  $m$  components and  $n - m - 1$  edges. Each time we contract a vertex of  $F$  to the apex, we add a single edge to  $T$ . Therefore,  $T$  has  $n - 1$  edges. It is now sufficient to show that  $T$  is connected.

We will show that for every vertex  $u$  in  $\Gamma$ , there is path between  $u$  and  $\mathbf{a}$  in  $T$ . Let  $u$  be a vertex of  $\Gamma$ . Suppose  $u$  is in some component  $C_u$  which is a vertex of  $F$ . If  $C_u$  is originally heavy, then there is an edge  $e$  in  $T$  between a vertex  $v \in C_u$  and the apex. Since we have a spanning tree in  $C_u$ , there is a path between  $u$  and  $v$  in  $T$ . Therefore, there is path between  $u$  and the apex  $a$  in  $T$ .

Otherwise,  $C_u$  must have some parent  $C_{u_1}$  and there is an edge between these two components. Therefore, there is a path between  $u$  and each vertex of these two components. Now, the same argument applies for  $C_{u_1}$ . Either it is originally heavy or it has a parent  $C_{u_2}$ .

If it is originally heavy, we are done. Otherwise, we use the same argument for  $C_{u_2}$ . Note that by construction and the definition of a parent, we do not reach the same component in this sequence. Therefore, at some point, we reach a component  $C_{u_k}$  which is originally heavy and we are done. ◀

Now we are ready to show that  $T$  is a  $O(1)$ -thin tree in  $G$  (w.r.t.  $z$ ). We have to show that there exists some constant  $\alpha$  such that for every cut  $U$ ,  $|E(T) \cap \delta(U)| \leq \alpha \cdot z(\delta(U))$ . Let  $U$  be a cut in  $G$ . We can assume w.l.o.g. that  $\mathbf{a} \notin U$ , since otherwise we can consider the cut  $V(G) \setminus U$ . We partition  $E(T) \cap \delta(U)$  into three subsets:

1.  $T_1 = \{\{\mathbf{a}, v\} \in E(T) \cap \delta(U) : v \in V(\Gamma)\}$ .
2.  $T_2 = \{\{u, v\} \in E(T) \cap \delta(U) : u \text{ and } v \text{ are in the same component of } \Gamma'\}$ .
3.  $T_3 = \{\{u, v\} \in E(T) \cap \delta(U) : u \text{ and } v \text{ are in different components of } \Gamma'\}$ .

► **Lemma 14.** *There exists some constant  $\alpha_1$  such that  $|T_1| \leq \alpha_1 \cdot z(\delta(U))$ .*

**Proof.** Let  $e = \{\mathbf{a}, v\} \in T_1$  where  $v \in V(\Gamma)$ . Let  $C_v \in V(F)$  such that  $v \in C_v$ . By the construction of  $T$ ,  $C_v$  is originally heavy. If  $C_v \subseteq U$ , we can charge  $e$  to  $z(C_v, \{\mathbf{a}\})$ , which we know is at least 0.5. Otherwise suppose  $C_v$  is not a subset of  $U$ . By the assumption we have  $\mathbf{a} \notin U$  and thus  $v \in U$  which implies that  $U \cap C_v \neq \emptyset$ . By the construction, we know that there is no tiny cuts in  $C_v$ . Therefore,  $z(\delta(U) \cap E(G[C_v])) \geq 0.1$ . Thus we can charge  $e$  to the total weight of the edges in  $\delta(U) \cap E(G[C_v])$ . Note that for each  $C_v \in V(F)$ , there is at most one edge in  $T_1$  between  $\mathbf{a}$  and  $C_v$ . Therefore we have that  $|T_1| \leq 10 \cdot z(\delta(U))$ . ◀

► **Lemma 15.** *There exists some constant  $\alpha_2$  such that  $|T_2| \leq \alpha_2 \cdot z(\delta(U))$ .*

**Proof.** We have

$$\begin{aligned} |T_2| &= \left| \bigcup_{C \in V(F)} (E(C) \cap T_2) \right| = \left| \bigcup_{C \in V(F)} (E(C) \cap E(T) \cap \delta(U)) \right| \\ &= \left| \bigcup_{C \in V(F)} (E(T_C) \cap \delta(U)) \right| \leq \sum_{C \in V(F)} 100 \cdot z(\delta(U) \cap E(C)) \leq 100 \cdot z(\delta(U)), \end{aligned}$$

concluding the proof. ◀

► **Lemma 16.** *There exists some constant  $\alpha_3$  such that  $|T_3| \leq \alpha_3 \cdot z(\delta(U))$ .*

**Proof.** We partition  $T_3$  into three subsets:

1.  $T_{31} = \{\{u, v\} \in T_3 : u \in U, v \notin U, \exists C_u, C_v \in V(F) \text{ s.t. } u \in C_u, v \in C_v, U \cap C_v \neq \emptyset\}$ .
2.  $T_{32} = \{\{u, v\} \in T_3 : u \in U, v \notin U, \exists C_u, C_v \in V(F) \text{ s.t. } u \in C_u, v \in C_v, U \cap C_u \neq \emptyset\}$ .
3.  $T_{33} = \{\{u, v\} \in T_3 : u \in U, v \notin U, \exists C_u, C_v \in V(F) \text{ s.t. } u \in C_u, v \in C_v, U \cap C_v = \emptyset, C_u \subseteq U\}$ .

First for each  $e = \{u, v\} \in T_{31}$  where  $v \in C_v$  for some  $C_v \in V(F)$ , we have that  $U_v = U \cap C_v$  is a cut in  $C_v$  which is not tiny. By the construction,  $C_v$  can be the parent of at most five other vertices in  $F$  and it can have at most one parent. Therefore, there are at most six edges in  $T_3$  with a vertex in  $C_v$ . So we can charge  $e$  and at most five other edges to  $z(\delta(U_v))$ . Since  $z(\delta(U_v)) \geq 0.1$  we get  $|T_{31}| \leq 60 \cdot z(\delta(U))$ .

Second for each  $e = \{u, v\} \in T_{32}$  where  $u \in C_u$  for some  $C_u \in V(F)$ , we have that  $U_u = U \cap C_u$  is a cut in  $C_u$  which is not tiny. Therefore, the same argument for  $C_v$  as in the first case, applies here for  $C_u$  and we get  $|T_{32}| \leq 60 \cdot z(\delta(U))$ .

Finally, for  $T_{33}$  we need to find a constant  $\alpha_{33}$  such that  $|T_{33}| \leq \alpha_{33} \cdot z(\delta(U))$ . First, we define a new cut  $U_1$  as follows. For every  $C \in V(F)$  with  $C \cap U \neq \emptyset$ , if  $C \cap U \neq C$ , we add all the other vertices of  $C$  to  $U$  and we say that  $C$  is *important*. This process leads to a new cut  $U_1$  such that for every  $C \in F$ , either  $C \cap U_1 = \emptyset$  or  $C \subseteq U_1$ . Let  $U_2 = \{C \in V(F) : C \subseteq U_1\}$ . Let  $X = \{C \in V(F) : C \not\subseteq U_2\}$  and  $Y = X \cup \{\mathbf{a}\}$ .

Let

$$T_{331} = \{\{u, v\} \in T_{33} : u \in U, u \in C \text{ for some } C \in V(F) \text{ with } \deg_{F[U_2]}(C) < 19\}.$$

Let also  $T_{332} = T_{33} \setminus T_{331}$ .

For each edge  $e = \{u, v\} \in T_{331}$  where  $u \in U$  and  $u \in C_u$  for some  $C_u \in V(F)$ , we have  $\deg_{F[U_2]}(C_u) < 19$ . By Lemma 10, we know that for any  $C, C' \in V(F)$ ,  $z(C, C') \leq 0.1$ . Therefore, we get  $z(C_u, Y) \geq 0.2$ . Note that there are at most six edges in  $T_{331}$  with a vertex in  $C_u$ . So we can charge  $e$  and at most five other edges to  $z(C_u, Y)$ . Therefore,  $|T_{331}| \leq 30 \cdot z(\delta(U))$ .

Let  $V_1 = \{C \in U_2 : \deg_{F[U_2]}(C) \geq 19\}$  and  $V_2 = \{C \in U_2 : \deg_{F[U_2]}(C) \leq 5\}$ . By Euler's formula, we know that the average degree of a planar graph is at most 6. Since  $F[U_2]$  is planar, we get  $|V_1| \leq |V_2|$ . For any  $C \in V_2$ , if  $C$  is important, then  $C \cap U$  is a cut for  $C$  and we have  $z(C \cap U, Y) \geq 0.1$ . If  $C$  is not important, then we have  $z(C, Y) \geq 1.5$ . Note that for any  $C' \in V_1$ , there are at most six edges in  $T_{332}$  with a vertex in  $C'$ . Therefore, we have  $|T_{332}| \leq 60 \cdot z(\delta(U))$ .

Now since  $T_3 = T_{31} \cup T_{32} \cup T_{331} \cup T_{332}$ , we have  $|T_3| \leq 210 \cdot z(\delta(U))$  completing the proof.  $\blacktriangleleft$

► **Lemma 17.**  *$T$  is a  $O(1)$ -thin spanning tree in  $G$ .*

**Proof.** By Lemma 13 we know that  $T$  is a spanning tree. For any  $U \subseteq V(G)$ , by Lemmas 14, 15 and 16 we get  $|T| \leq 320 \cdot z(\delta(U))$ . This completes the proof.  $\blacktriangleleft$

We are now ready to prove the main result of this Section.

► **Theorem 18.** *Let  $G$  be a 1-apex graph and let  $z : E(G) \rightarrow \mathbb{R}_{\geq 0}$  be  $\beta$ -thick for some  $\beta > 0$ . Then there exists a polynomial time algorithm which given  $G$  and  $z$  outputs a  $O(1/\beta)$ -thin spanning tree in  $G$  (w.r.t.  $z$ ).*

**Proof.** For  $\beta \geq 2$ , by Lemma 17 we know that we can find a 320-thin spanning tree in  $G$ . For any  $\beta$  with  $0 < \beta < 2$ , the assertion follows by scaling  $z$  by a factor of  $2/\beta$ .  $\blacktriangleleft$

## 7 Thin forests in graphs with many apices

Let  $a \geq 1$ . In this section, we describe an algorithm for finding thin-forests in an  $a$ -apex graph. The high level approach is analogous to the case of 1-apex graphs. We construct a similar graph  $F$  of components and contract each vertex of  $F$  to some apex.

Let  $e_0 = \{u_0, v_0\} \in E(G)$ . Let  $G'$  be obtained from  $G$  by contracting  $e_0$ . We define a new weight function  $z'$  on the edges of  $G'$  as follows. For any  $\{u, v_0\} \in E(G)$ , we set  $z'(\{u, v_0\}) = z(\{u, v_0\}) + z(\{u, u_0\})$ . For any other edge  $e$  we set  $z'(e) = z(e)$ . We say that  $z'$  is *induced* by  $z$ . Similarly, when  $G'$  is obtained by contracting a subset  $X$  of edges in  $G$ , we define  $z'$  by inductively contracting the edges in  $X$  in some arbitrary order.

For the remainder of this section let  $G$  be a  $a$ -apex graph with the set of apices  $A = \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_a\}$ . Let  $\Gamma$  be the planar part of  $G$ . Let  $z$  be a 2-thick weight function on the edges of  $G$ . The algorithm proceeds in 5 phases.

**Phase 1.** We say that a cut  $U$  is *tiny* (w.r.t.  $z$ ) if  $z(\delta(U)) < 1/(100 \cdot a)$ . Similar to the case of 1-apex graphs, we start with  $\Gamma$  and repeatedly partition it via tiny cuts until there are no more such cuts and we let  $\Gamma'$  be the resulting graph.

**Phase 2.** For each connected component  $C$  of  $\Gamma'$  we find a  $O(a)$ -thin tree  $T_C$  using Theorem 9.

**Phase 3.** We define  $F$  and  $G'$  exactly the same way as in the case of 1-apex graphs.

► **Lemma 19.** *For every  $\{C, C'\} \in E(F)$ , we have  $z(C, C') \leq 1/(100 \cdot a)$ .*

**Proof.** The same argument as in Lemma 10 applies here. The only difference here is that a cut  $U$  is tiny if  $z(\delta(U)) < 1/(100 \cdot a)$ . ◀

**Phase 4.** We construct a forest  $T'$  on  $G'$ . Let  $m = |V(F)|$ . We define a sequence of planar graphs  $F_0, F_1, \dots, F_m$ , a sequence of graphs  $G'_0, G'_1, \dots, G'_m$  and a sequence of weight functions  $z_0, z_1, \dots, z_m$  as follows. Let  $F_0 = F$ ,  $G'_0 = G'$  and  $z_0 = z$ . We also define a sequence of forests  $P_0, \dots, P_m$  where each  $P_j$  contains a tree rooted at each  $\mathbf{a}_i \in A$ . We set  $P_0$  to be the forest that contains a tree for each  $\mathbf{a}_i \in A$  and with no other vertices.

Let  $C \in V(F_j)$  for some  $j$ . For any  $\mathbf{a}_i \in A$ , we say that  $C$  is  $\mathbf{a}_i$ -heavy in  $F_j$  if  $z_j(C, \{\mathbf{a}_i\}) \geq 1/a$ . Let  $C' \in V(F)$ . For any  $\mathbf{a}_i \in A$ , we say that  $C'$  is *originally  $\mathbf{a}_i$ -heavy* if  $C'$  is  $\mathbf{a}_i$ -heavy in  $F$ .

We maintain the following inductive invariant:

(I1) For any  $j \in \{0, \dots, m-1\}$ , let  $v \in V(F_j)$  be a vertex of minimum degree. Then either there exists some  $\mathbf{a}_i \in A$  such that  $v$  is originally  $\mathbf{a}_i$ -heavy or  $v \in V(P_j)$ .

Consider some  $i \in \{0, \dots, m-1\}$ . Let  $v_i \in V(F_i)$  be a vertex with minimum degree. If  $v_i$  is originally  $\mathbf{a}_j$ -heavy for some  $\mathbf{a}_j \in A$ , then we contract  $v_i$  to  $\mathbf{a}_j$ . Otherwise, by the inductive invariant (I1), we have that  $v_i \in V(P_i)$ . Thus there exists a tree in  $P_i$  containing  $v_i$  that is rooted in some  $\mathbf{a}_j \in A$ ; we contract  $v_i$  to  $\mathbf{a}_j$ . In either case, by contracting  $v_i$  to  $\mathbf{a}_j$  we obtained  $G'_{i+1}$  from  $G'_i$ . We also delete  $v_i$  from  $F_i$  to obtain  $F_{i+1}$ . We let  $z_{i+1}$  be the weight function on  $G'_{i+1}$  induced by  $z_i$ .

Finally, we need to define  $P_{i+1}$ . If  $v_i$  was originally  $\mathbf{a}_j$ -heavy then we add  $v_i$  to  $P_i$  via an edge  $\{v_i, \mathbf{a}_j\}$ . For each  $u \in V(F_i)$  that is a neighbor of  $v_i$ , and is not in  $V(P_i)$ , we add  $u$  to  $P_{i+1}$  by adding the edge  $\{v_i, u\}$  iff the following conditions hold:

- (i) For all  $\mathbf{a}_r \in A$ , we have that  $u$  is not  $\mathbf{a}_r$ -heavy in  $F_i$ .
- (ii)  $u$  is  $\mathbf{a}_j$ -heavy in  $F_{i+1}$ .

In this case we say that  $v$  is the *parent* of  $u$ . This completes the description of the process that contracts each vertex in  $V(G')$  into some apex.

► **Lemma 20.** *Let  $j \in \{0, 1, \dots, m-1\}$ . Let  $C \in V(F_j)$  be a vertex with minimum degree. Then there exists  $\mathbf{a}_i \in A$  such that  $C$  is  $\mathbf{a}_i$ -heavy in  $F_j$ .*

**Proof.** Since  $C$  has at most 5 neighbors in  $F_j$ , by Lemma 19 we have  $z_j(C, A) \geq 2 - 5/(100 \cdot a) \geq 1$ . Therefore by averaging, there exists an apex  $\mathbf{a}_i$  such that  $z_j(C, \{\mathbf{a}_i\}) \geq 1/a$ . ◀

► **Lemma 21.** *For any  $j \in \{0, \dots, m\}$  and for any  $v \in V(\Gamma) \cap V(P_j)$ , we have  $\deg_{P_j}(v) \leq 6$ .*

**Proof.** By the construction, in each step we pick a vertex of minimum degree and contract it into some apex. Since  $F_i$  is planar, its minimum degree is at most 5. This means that for any  $v \in V(\Gamma) \cap V(P_j)$ ,  $v$  can be the parent of at most five other vertices and can have at most one parent. This completes the proof. ◀

► **Lemma 22.** *The inductive invariant (I1) is maintained.*

**Proof.** For any  $j \in \{0, \dots, m-1\}$ , let  $v \in V(F_j)$  be a vertex of minimum degree. If there exists some  $\mathbf{a}_i \in A$  such that  $v$  is originally  $\mathbf{a}_i$ -heavy, then we are done. Suppose for all  $\mathbf{a}_i \in A$ ,  $v$  is not originally  $\mathbf{a}_i$ -heavy. By Lemma 20 we know that there exists some  $\mathbf{a}_l \in A$  such that  $v$  is  $\mathbf{a}_l$ -heavy in  $F_j$ . Let  $j^* \in \{1, \dots, j\}$  be minimum such that  $v$  is not  $\mathbf{a}_l$ -heavy in  $F_{j^*-1}$  for all  $\mathbf{a}_l \in A$ , and  $v$  is  $\mathbf{a}_{l'}$ -heavy in  $F_{j^*}$  for some  $\mathbf{a}_{l'} \in A$ . Let  $u \in V(F_{j^*-1})$  be vertex that is contracted to some apex in step  $j^*$ . It follows by construction that  $u$  is the parent of  $v$  in  $P_{j^*}$ . Since  $j \geq j^*$  it follows that  $v \in V(P_j)$ , concluding the proof. ◀

Now we are ready to describe how to construct  $T'$  in  $G'$ . For any  $l \in \{0, \dots, m-1\}$ , let  $C \in V(F_l)$  be a vertex of minimum degree. If  $C$  is originally  $\mathbf{a}_i$ -heavy for some  $\mathbf{a}_i \in A$  and we contract  $C$  to  $\mathbf{a}_i$ , we pick an arbitrary edge  $e$  between  $C$  and  $\mathbf{a}_i$  and we add it to  $T'$ . Otherwise, by Lemma 22 we have  $C \in V(P_l)$ . This means that  $C$  has a parent  $C'$ . In this case, we pick an arbitrary edge  $e$  between  $C$  and  $C'$  and we add it to  $T'$ .

**Phase 5.** We construct a forest  $T$  in  $G$  the same way as in the 1-apex case. We set  $E(T) = E(T') \cup \bigcup_{C \in V(F)} E(T_C)$ .

This completes the description of the algorithm.

## 7.1 Analysis

By the construction,  $T$  has  $a$  connected components. We will show that  $T$  is a  $O(a)$ -thin spanning forest. Let  $U$  be a cut in  $G$ . Similar to the 1-apex case, we partition  $E(T) \cap \delta(U)$  into three subsets:

1.  $T_1 = \{\{\mathbf{a}_i, v\} \in E(T) \cap \delta(U) : \mathbf{a}_i \in A, v \in V(\Gamma)\}$ .
2.  $T_2 = \{\{u, v\} \in E(T) \cap \delta(U) : u \text{ and } v \text{ are in the same component of } \Gamma'\}$ .
3.  $T_3 = \{\{u, v\} \in E(T) \cap \delta(U) : u \text{ and } v \text{ are in different components of } \Gamma'\}$ .

► **Lemma 23.** *There exists a constant  $\alpha_1$  such that  $|T_1| \leq \alpha_1 \cdot a \cdot z(\delta(U))$ .*

**Proof.** A similar argument as in the case of 1-apex graph applies here with two differences. First, a cut  $U$  is tiny if  $z(\delta(U)) < 1/(100 \cdot a)$ . Second, for any  $\mathbf{a}_i \in A$  and  $C \in V(F)$  where  $C$  is originally  $\mathbf{a}_i$ -heavy, we have that  $z(C, \{\mathbf{a}_i\}) \geq 1/a$ . Therefore, we get  $|T_1| \leq 100 \cdot a \cdot z(\delta(U))$ . ◀

► **Lemma 24.** *There exists a constant  $\alpha_2$  such that  $|T_2| \leq \alpha_2 \cdot a \cdot z(\delta(U))$ .*

**Proof.** Again, a similar argument as in the case of 1-apex graphs applies here. The only difference here is the definition of tiny cut. Therefore, we get  $|T_2| \leq 100 \cdot a \cdot z(\delta(U))$ . ◀

► **Lemma 25.** *There exists a constant  $\alpha_3$  such that  $|T_3| \leq \alpha_3 \cdot a \cdot z(\delta(U))$ .*

**Proof.** Similar to the 1-apex case, we partition  $T_3$  into three subsets:

1.  $T_{31} = \{\{u, v\} \in T_3 : u \in U, v \notin U, \exists C_u, C_v \in V(F) \text{ s.t. } u \in C_u, v \in C_v, U \cap C_v \neq \emptyset\}$ .
2.  $T_{32} = \{\{u, v\} \in T_3 : u \in U, v \notin U, \exists C_u, C_v \in V(F) \text{ s.t. } u \in C_u, v \in C_v, U \cap C_u \neq \emptyset\}$ .
3.  $T_{33} = \{\{u, v\} \in T_3 : u \in U, v \notin U, \exists C_u, C_v \in V(F) \text{ s.t. } u \in C_u, v \in C_v, U \cap C_v = \emptyset, C_u \subseteq U\}$ .

The arguments for  $T_{31}$  and  $T_{32}$  are the same as in 1-apex graphs. The only difference here is that a cut  $U$  is tiny if  $z(\delta(U)) < 1/(100 \cdot a)$ . Therefore, we have  $|T_{31}| \leq 600 \cdot a \cdot z(\delta(U))$  and  $|T_{32}| \leq 600 \cdot a \cdot z(\delta(U))$ .

Now for  $T_{33}$  we want to find a constant  $\alpha_{33}$  such that  $|T_{33}| \leq \alpha_{33} \cdot a \cdot z(\delta(U))$ . We define two new cuts  $U_1$  and  $U_2$  as follows. For every  $C \in V(F)$  with  $C \cap U \neq \emptyset$ , if  $C \cap U \neq C$ , we add all other vertices of  $C$  to  $U$  (delete all other vertices of  $C$  from  $U$ ) to obtain  $U_1$  ( $U_2$ ) and we say that  $C$  is  $U$ -important. Let  $U'_1 = \{C \in V(F) : C \subseteq U_1\}$  and  $U'_2 = \{C \in V(F) : C \subseteq U_2\}$ .

For any  $e = \{u, v\} \in T_{33}$  where  $u \in U$ ,  $v \notin U$ ,  $u \in C_u$  and  $v \in C_v$  for some  $C_u, C_v \in V(F)$ , by the construction of  $T_3$ , we have that both  $C_u$  and  $C_v$  have been contracted to the same apex  $\mathbf{a}_i$  for some  $\mathbf{a}_i \in A$ . Let  $j \in \{0, \dots, m\}$  be the step during which  $C_u$  is contracted to  $\mathbf{a}_i$ . Let  $B = \{C \in V(F) : C \text{ is contracted to } \mathbf{a}_i\}$ . Let  $D$  be the connected component of  $F[B]$  containing  $C_u$ . Let  $D_{U'_1}^{\text{in}} = D[U'_1]$ ,  $D_{U'_1}^{\text{out}} = D[V(D) \setminus U'_1]$ ,  $D_{U'_2}^{\text{in}} = D[U'_2]$ ,  $D_{U'_2}^{\text{out}} = D[V(D) \setminus U'_2]$ .

We consider the following two cases:

**Case 1:**  $\mathbf{a}_i \notin U$ . We know that  $C_u \in U$ . By the construction, we have that  $z_j(C_u, \{\mathbf{a}_i\}) \geq 1/a$ . If  $z_0(C_u, \{\mathbf{a}_i\}) \geq 1/(100 \cdot a)$ , we can charge  $e$  to  $z_0(C_u, \{\mathbf{a}_i\})$  and we know that there are at most six edges in  $T_{33}$  with a vertex in  $C_u$ .

Otherwise, we have that  $z_0(C_u, (V(D) \setminus C_u)) \geq 99/(100 \cdot a)$ . If  $z_0(C_u, (V(D_{U'_1}^{\text{out}}) \setminus C_u)) \geq 1/(100 \cdot a)$ , then by the construction of  $D_{U'_1}^{\text{out}}$  we get  $z_0(C_u, (G \setminus U)) \geq 1/(100 \cdot a)$ . Therefore we can charge  $e$  to  $z_0(C_u, (G \setminus U))$ .

Otherwise, we have that  $z_0(C_u, (V(D_{U'_1}^{\text{in}}) \setminus C_u)) \geq 98/(100 \cdot a)$ . This implies that  $\deg_{D_{U'_1}^{\text{in}}}(C_u) \geq 98$ . Now note that  $D_{U'_1}^{\text{in}}$  is a planar graph and its average degree is at most 6. Now a similar argument as in the 1-apex case applies here. Let  $V_1 = \{C \in D_{U'_1}^{\text{in}} : \deg_{D_{U'_1}^{\text{in}}}(C) \geq 98\}$ . Let  $V_2 = \{C \in D_{U'_1}^{\text{in}} : \deg_{D_{U'_1}^{\text{in}}}(C) \leq 5\}$ . By planarity of  $D_{U'_1}^{\text{in}}$ ,

we have that  $|V_1| \leq |V_2|$ . For any  $C \in V_2$ , if  $C$  is  $U$ -important, then  $C \cap U$  is a cut for  $C$  which is not tiny. Therefore we have  $z(C \cap U, G \setminus U) \geq 1/(100 \cdot a)$ . If  $C$  is not  $U$ -important, we have  $z(C, G \setminus U) \geq 95/(100 \cdot a)$ . Now note that for any  $C' \in V_1$  there are at most six edges in  $T_{33}$  with a vertex in  $C'$ . Therefore we have  $|T_{33}| \leq 1000 \cdot a \cdot z(\delta(U))$ .

**Case 2:**  $\mathbf{a}_i \in U$ . Let  $X_1 = \{C \in D_{U'_2}^{\text{out}} : \deg_{D_{U'_2}^{\text{out}}}(C) \geq 98\}$ . Let  $X_2 = \{C \in D_{U'_2}^{\text{out}} : \deg_{D_{U'_2}^{\text{out}}}(C) \leq 5\}$ . We know that  $C_v \notin U$ . We follow a similar approach as in the first case by considering  $U'_2$ ,  $D_{U'_2}^{\text{in}}$  and  $D_{U'_2}^{\text{out}}$ . The same argument applies here by replacing  $U'_1$ ,  $D_{U'_1}^{\text{in}}$ ,  $D_{U'_1}^{\text{out}}$ ,  $X_1$  and  $X_2$  with  $U'_2$ ,  $D_{U'_2}^{\text{in}}$ ,  $D_{U'_2}^{\text{out}}$ ,  $V_1$  and  $V_2$  respectively. Therefore, we get  $|T_{33}| \leq 1000 \cdot a \cdot z(\delta(U))$ .

Now from what we have discussed, we have  $|T_{31}| \leq 600 \cdot a \cdot z(\delta(U))$ ,  $|T_{32}| \leq 600 \cdot a \cdot z(\delta(U))$  and  $|T_{33}| \leq 1000 \cdot a \cdot z(\delta(U))$ . Therefore, we get  $|T_3| \leq 2200 \cdot a \cdot z(\delta(U))$  completing the proof.  $\blacktriangleleft$

► **Lemma 26.**  $T$  is a  $O(a)$ -thin spanning forest in  $G$  with at most  $a$  connected components.

**Proof.** By the construction,  $T$  is a spanning forest and has at most  $a$  connected components. Let  $\alpha = \alpha_1 + \alpha_2 + \alpha_3$ , where  $\alpha_1$ ,  $\alpha_2$  and  $\alpha_3$  are obtained by Lemmas 23, 24, 25. Therefore for any  $U \subset V(G)$ , we have  $|T| \leq \alpha \cdot a \cdot z(\delta(U)) = 2400 \cdot a \cdot z(\delta(U))$  completing the proof.  $\blacktriangleleft$

We are now ready to prove the main result of this Section.

► **Theorem 27.** Let  $a \geq 1$  and let  $G$  be a  $a$ -apex graph with set of apices  $A = \{\mathbf{a}_1, \dots, \mathbf{a}_a\}$ . Let  $z : E(G) \rightarrow \mathbb{R}_{\geq 0}$  be  $\beta$ -thick for some  $\beta > 0$ . Then there exists a polynomial time algorithm which given  $G$ ,  $A$  and  $z$  outputs a  $O(a/\beta)$ -thin spanning forest in  $G$  (w.r.t.  $z$ ) with at most  $a$  connected components.

**Proof.** By Lemma 26, for  $\beta \geq 2$ , we know that we can find a  $(2400a)$ -thin spanning forest in  $G$  (w.r.t.  $z$ ) with at most  $a$  connected components. For any other  $0 < \beta < 2$ , the claim follows by scaling  $z$  by a factor of  $2/\beta$ .  $\blacktriangleleft$

## 8 Thin forests in higher genus graphs with many apices

In this section we generalize our algorithm for computing a thin tree on a graph with many apices, to compute a thin forest in a graph of higher genus and with many apices. The following theorem is implicit in [8].

► **Theorem 28** (Erickson and Sidiropoulos [8]). *Let  $G$  be a graph with  $\text{eg}(G) = g$ , and let  $z$  be a  $\beta$ -thick weight function on the edges of  $G$  for some  $\beta \geq 0$ . Then there exists a polynomial time algorithm which given  $G$ ,  $z$ , and an embedding of  $G$  into a surface of Euler genus  $g$ , outputs a  $O(1/\beta)$ -thin spanning forest in  $G$  (w.r.t.  $z$ ), with at most  $g$  connected components.*

In this section, we study the problem in higher genus graphs. First, the following two Lemmas can be obtained by Euler's formula.

► **Lemma 29.** *Let  $G$  be a graph of genus  $g \geq 1$  with  $|V(G)| \geq 10g$ . Then there exists  $v_0 \in V(G)$  with  $\deg_G(v_0) \leq 7$ .*

► **Lemma 30.** *Let  $G$  be an  $n$  vertex graph of genus  $g \geq 1$ . Then the average degree of vertices of  $G$  is at most  $6 + 12(g - 1)/n$ .*

For the remainder of this section, let  $G$  be an  $a$ -apex graph with the set of apices  $A = \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_a\}$  on a surface of genus  $g$ . Let  $\Gamma = G \setminus A$ , where  $\Gamma$  is a graph of genus  $g$ . Let  $z$  be a 2-thick weight function on the edges of  $G$ . We will find a  $O(a \cdot g)$ -thin (w.r.t.  $z$ ) spanning forest in  $G$  with at most  $O(a + g)$  connected components. The high level approach is similar to the case where  $\Gamma$  was planar. The algorithm proceeds in 5 phases.

**Phase 1.** We say that a cut  $U$  is *tiny* (w.r.t.  $z$ ) if  $z(\delta(U)) < 1/(1000 \cdot a \cdot g)$ . We construct  $\Gamma'$  the same way as in Section 7. The only difference here is the definition of tiny cut.

**Phase 2.** Similar to the planar case, for each connected component  $C$  of  $\Gamma'$  we find a  $O(a \cdot g)$ -thin forest  $T_C$ , with at most  $g$  connected components, using Theorem 28

**Phase 3.** We define  $F$  and  $G'$  the exact same way as in Section 7.

► **Lemma 31.** *For every  $\{C, C'\} \in E(F)$ , we have  $z(C, C') \leq 1/(1000 \cdot a \cdot g)$ .*

**Proof.** The same argument as in Lemma 10 applies here. The only difference here is the definition of tiny cut. ◀

**Phase 4.** We construct a spanning forest  $T'$  on  $G'$ , with at most  $a + 10g$  connected components. We follow a similar approach as in the planar case. Let  $m = |V(F)| - 10g$ . If  $m \leq 0$ , we set  $E(T') = \emptyset$  and we skip to the next phase. Otherwise, we define two sequences of graphs  $F_0, F_1, \dots, F_m$ ,  $G'_0, G'_1, \dots, G'_m$ , a sequence of weight functions  $z_0, z_1, \dots, z_m$ , a sequence of forests  $P_0, P_1, \dots, P_m$  satisfying the inductive invariant (I1) the exact same way as in Section 7. For any  $j \in \{0, 1, \dots, m - 1\}$ ,  $C \in V(F_j)$  and  $\mathbf{a}_i \in A$ , we also define the notion of  $\mathbf{a}_i$ -heavy and *originally*  $\mathbf{a}_i$ -heavy the same way as in Section 7. The only differences here is that  $m = |V(F)| - 10g$  instead of  $|V(F)|$ .

► **Lemma 32.** *Let  $j \in \{0, 1, \dots, m\}$ . Let  $C \in V(F_j)$  be a vertex of minimum degree. Then there exists  $\mathbf{a}_i \in A$  such that  $C$  is  $\mathbf{a}_i$ -heavy in  $F_j$ .*

**Proof.** By the construction,  $|V(F_j)| \geq 10g$ . Therefore by Lemma 29, we have  $\deg_{F_j}(C) \leq 7$ . Therefore by Lemma 31, we get  $z_j(C, A) \geq 2 - 7/(1000 \cdot a \cdot g) \geq 1$ . This implies that there exists  $\mathbf{a}_i \in A$  such that  $z_j(C, \{\mathbf{a}_i\}) \geq 1/a$ . ◀

► **Lemma 33.** *For any  $j \in \{0, \dots, m\}$  and for any  $v \in \Gamma \cap P_j$ , we have  $\deg_{P_j}(v) \leq 8$ .*

**Proof.** A similar argument as in the planar case applies here. The only difference here is that the minimum degree is at most 7. Therefore, every vertex can be the parent of at most 7 other vertices and can have at most one parent. ◀

► **Lemma 34.** *The inductive invariant (I1) is maintained.*

**Proof.** The exact same argument as in Section 7 applies here. ◀

Now we construct a forest  $T'$  on  $G'$  the same way as in Section 7.

► **Lemma 35.**  *$T'$  has at most  $a + 10g$  connected components.*

**Proof.** If  $|V(F)| \leq 10g$ , then we are done. Otherwise, we have  $|F_m| = 10g$ . Now since  $|A| = a$ , by the construction, we have that the number of connected components of  $T'$  is at most  $a + 10g$ . ◀

**Phase 5.** We construct a forest  $T$  in  $G$  the exact same way as in Section 7, by setting  $E(T) = E(T') \cup \bigcup_{C \in V(F)} E(T_C)$ .

This completes the description of the algorithm.

## 8.1 Analysis

► **Lemma 36.**  *$T$  is a spanning forest in  $G$ , with at most  $O(a + g)$  connected components.*

**Proof.** For any  $C \in V(F)$ , let  $g_C$  be the genus of  $\Gamma[C]$ . By Theorem 28 we know that the number of connected components in  $T_C$  is at most  $g_C$ . Therefore, by Lemma 35 we have that the number of connected components in  $T$  is at most  $a + 10g + \sum_{C \in V(F)} g_C \leq a + 11g$ . ◀

For the thinness of  $T$ , we follow a similar approach as in the planar case. There are two main differences here: First for any  $j \in \{0, 1, \dots, m\}$ , by Lemma 30 we have that the average degree of  $F_j$  is at most  $20g$ . Second a cut  $U$  is tiny if  $z(\delta(U)) < 1/(1000 \cdot a \cdot g)$ .

Let  $U$  be a cut in  $G$ . Similar to the planar case, we partition  $E(T) \cap \delta(U)$  into three subsets:

1.  $T_1 = \{\{\mathbf{a}_i, v\} \in E(T) \cap \delta(U) : \mathbf{a}_i \in A, v \in V(\Gamma)\}$ .
2.  $T_2 = \{\{u, v\} \in E(T) \cap \delta(U) : u \text{ and } v \text{ are in the same component of } \Gamma'\}$ .
3.  $T_3 = \{\{u, v\} \in E(T) \cap \delta(U) : u \text{ and } v \text{ are in different components of } \Gamma'\}$ .

Also similar to the planar case, we partition  $T_3$  into three subsets:

1.  $T_{31} = \{\{u, v\} \in T_3 : u \in U, v \notin U, \exists C_u, C_v \in V(F) \text{ s.t. } u \in C_u, v \in C_v, U \cap C_v \neq \emptyset\}$ .
2.  $T_{32} = \{\{u, v\} \in T_3 : u \in U, v \notin U, \exists C_u, C_v \in V(F) \text{ s.t. } u \in C_u, v \in C_v, U \cap C_u \neq \emptyset\}$ .
3.  $T_{33} = \{\{u, v\} \in T_3 : u \in U, v \notin U, \exists C_u, C_v \in V(F) \text{ s.t. } u \in C_u, v \in C_v, U \cap C_v = \emptyset, C_u \subseteq U\}$ .

► **Lemma 37.** *For any index  $i \in \{1, 2, 31, 32\}$ , there exists a constant  $\alpha_i$  such that  $|T_i| \leq \alpha_i \cdot a \cdot g \cdot z(\delta(U))$ .*

**Proof.** A similar argument as in Lemmas 23, 24 and 25 applies here. There are two differences here. First the definition of a tiny cut is different. Second, for each  $C \in V(F)$ ,  $C$  can be the parent of at most seven vertices and can have at most one parent. Therefore we get  $|T_1| \leq 1000 \cdot a \cdot g \cdot z(\delta(U))$ ,  $|T_2| \leq 1000 \cdot a \cdot g \cdot z(\delta(U))$ ,  $|T_{31}| \leq 8000 \cdot a \cdot g \cdot z(\delta(U))$  and  $|T_{32}| \leq 8000 \cdot a \cdot g \cdot z(\delta(U))$ . ◀

► **Lemma 38.** *There exists a constant  $\alpha_{33}$  such that  $|T_{33}| \leq \alpha_{33} \cdot a \cdot g \cdot z(\delta(U))$ .*

**Proof.** Let  $e = \{u, v\} \in T_{33}$ . We follow a similar approach as in the planar case. We define  $U_1, U_2, U'_1, U'_2, B, D_{U'_1}^{\text{in}}, D_{U'_1}^{\text{out}}, D_{U'_2}^{\text{in}}$  and  $D_{U'_2}^{\text{out}}$  the exact same way as in Lemma 25. Let  $V_1 = \{C \in D_{U'_1}^{\text{in}} : \deg_{D_{U'_1}^{\text{in}}}(C) \geq 98g\}$ ,  $V_2 = \{C \in D_{U'_1}^{\text{in}} : \deg_{D_{U'_1}^{\text{in}}}(C) \leq 20g\}$ ,  $X_1 = \{C \in D_{U'_2}^{\text{out}} : \deg_{D_{U'_2}^{\text{out}}}(C) \geq 98g\}$ , and  $X_2 = \{C \in D_{U'_2}^{\text{out}} : \deg_{D_{U'_2}^{\text{out}}}(C) \leq 5g\}$ . By Lemma 30 we have that  $|V_1| \leq |V_2|$  and  $|X_1| \leq |X_2|$ . With these definitions, the rest of the proof is the same as in Lemma 25, and thus we get  $|T_{33}| \leq 10000 \cdot a \cdot g \cdot z(\delta(U))$ . ◀

► **Lemma 39.**  *$T$  is a  $O(a \cdot g)$ -thin spanning forest in  $G$ , with at most  $O(a + g)$  connected components.*

**Proof.** By combining Lemmas 36, 37 and 38 we get  $|T| \leq 24000 \cdot a \cdot g \cdot z(\delta(U))$ , which proves the assertion. ◀

We are now ready to prove the main result of this Section.

► **Theorem 40.** *Let  $a, g \geq 1$ . Let  $G$  be a graph and  $A \subseteq V(G)$ , with  $|A| = a$ , such that  $H = G \setminus A$  is a graph of genus  $g$ . Let  $z : E(G) \rightarrow \mathbb{R}_{\geq 0}$  be  $\beta$ -thick for some  $\beta > 0$ . Then there exists a polynomial time algorithm which given  $G, A$ , an embedding of  $H$  on a surface of genus  $g$ , and  $z$  outputs a  $O((a \cdot g)/\beta)$ -thin spanning forest in  $G$  (w.r.t.  $z$ ) with at most  $O(a + g)$  connected components.*

**Proof.** For  $\beta \geq 2$ , by Lemma 39, we can find a  $(24000 \cdot a \cdot g)$ -thin spanning forest with at most  $a + 11g$  connected components. For  $0 < \beta < 2$ , the claim follows by scaling  $z$  by a factor of  $2/\beta$ . ◀

## 9 Thin subgraphs in nearly-embeddable graphs

In this section we give our algorithm for computing a thin subgraph in a  $(a, g, 1, p)$ -nearly embeddable graph, proving Lemma 8. We first handle graphs without any apices, and then proceed to the general case.

### 9.1 $(0, g, 1, p)$ -nearly embeddable graphs

We now describe our algorithm for computing a thin subgraph in a  $(0, g, 1, p)$ -nearly embeddable graph.

For the remainder of this subsection, let  $\vec{G}$  be a  $(0, g, 1, p)$ -nearly embeddable digraph and let  $G$  be its symmetrization. Let  $\vec{H}$  be the single vortex of  $\vec{G}$  of width  $p$ , attached to some face  $\vec{F}$  of  $\vec{G}$ . Let  $H$  and  $F$  be the symmetrizations of  $\vec{H}$  and  $\vec{F}$  respectively. Let  $\{B_v\}_{v \in V(F)}$  be a path-decomposition of  $H$  of width  $p$ . Let  $\vec{W}$  be a closed walk in  $\vec{G}$  visiting all vertices in  $V(\vec{H})$  and let  $W$  be its symmetrization. Let  $z : E(G) \rightarrow \mathbb{R}_{\geq 0}$  be  $\alpha$ -thick, for some  $\alpha \geq 2$ , and  $\vec{W}$ -dense. Let  $G'$  be the graph obtained by contracting  $F$  to a single vertex  $v^*$  in  $G \setminus H$ .

Following [8] we introduce the following notation. For any  $u, v \in V(G)$ , a *ribbon*  $R$  between  $u$  and  $v$  is the set of all parallel edges  $e = \{u, v\}$  such that for every  $e, e' \in R$ , there

exists a homeomorphism between  $e$  and  $e'$  on the surface. Let  $R'$  be a set of parallel edges in  $G$ . We say that an edge  $e \in R'$  is central if the total weight of edges on each side of  $e$  in  $R'$  (containing  $e$ ), is at least  $z(R')/2$ .

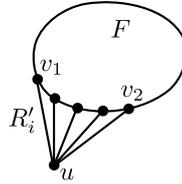
We will find a  $O(1)$ -thin spanning forest  $S$  in  $G'$  (w.r.t.  $z$ ), with at most  $g$  connected components, such that  $S$  is  $O(1)$ -thin in  $G$  (w.r.t.  $z$ ). We follow a similar approach to [8] to construct  $S$ . We apply some modifications that assure  $S$  is  $O(1)$ -thin in  $G$  (w.r.t.  $z$ ).

### 9.1.1 The modified ribbon-contraction argument

If  $|V(G')| \leq g$ , then we set  $E(S) = \emptyset$  and we are done. Otherwise, let  $l = |V(G')| - g$ . We define two sequences of graphs  $G_0, \dots, G_l$  and  $G'_0, \dots, G'_l$ , with  $G_0 = G$  and  $G'_0 = G'$ . For each  $j \in \{0, \dots, l\}$ ,  $G_j$  is obtained by uncontracting  $v^* \in V(G'_j)$ . Let  $i \geq 0$  and suppose we have defined  $G'_i$ . Let  $R_i$  be the heaviest ribbon in  $G'_i$  (w.r.t.  $z$ ). Let  $R'_i \subseteq E(G_i)$  be the corresponding set of edges in  $G_i$ . We contract all the edges in  $R_i$  and we let  $G'_{i+1}$  be the graph obtained after contracting  $R_i$ . We also perform the contraction in a way such that for all  $i \in \{0, \dots, l-1\}$  we have  $v^* \in G'_i$ .

Let  $i \in \{0, \dots, l-1\}$ . If  $R_i = \{u, v\}$  where  $u, v \neq v^*$ , similar to [8], we let  $e_i$  be a central edge in  $R_i$  and we add  $e_i$  to  $S$ . Otherwise, suppose that  $R_i = \{u, v^*\}$  for some  $u \in V(G'_i)$ . If there exists an edge  $e \in R_i$  with  $e \in W$  or  $z(e) \geq 0.1$ , we let  $e_i = e$  and we add it to  $S$ . Otherwise, we can assume that there is no edge  $e \in R$  with  $e \in W$  or  $z(e) \geq 0.1$ .

Let  $Q_i$  be the set of vertices  $v \in V(F)$  with an endpoint in  $R'_i$ . By the construction,  $Q_i$  is a subpath of  $F$ . Let  $v_1, v_2 \in V(F)$  be the endpoints of  $Q_i$ . Let  $W'_i$  be the restriction of  $W$  on  $\bigcup_{v \in Q_i} B_v$ . Let  $W''_i$  be the subgraph of  $W'_i$  obtained by deleting all edges  $e$  with both endpoints in  $B_{v_1}$  or  $B_{v_2}$ .



For any subgraph  $C$  of  $W$ , we define the  $i$ -load of  $C$  as follows. The  $i$ -load of  $C$  is the total weight of all edges in  $R'_i$  with an endpoint in  $C$ . Let  $C_i$  be the connected component of  $W''_i$  with the maximum  $i$ -load. Let  $Y_i = \{e \in R'_i : e \text{ has an endpoint in } C_i\}$ . We let  $e_i$  be a central edge in  $Y_i$  and we add  $e_i$  to  $S$ .

We set  $T = S \cup W$ . We will show that  $T$  is a  $O(p^2)$ -thin spanning subgraph of  $G$  (w.r.t.  $z$ ), with at most  $g$  connected components.

► **Lemma 41.**  $T$  is a spanning subgraph of  $G$  with at most  $g$  connected components.

**Proof.** By the construction,  $S$  has at most  $g$  connected components in  $G'$ . Now note that  $W$  is a closed walk visiting  $H$  in  $G$ . Therefore, all vertices of  $F$  are in the same connected component in  $T$ . This means that  $T$  has at most  $g$  connected components. ◀

► **Lemma 42.** For any  $i \in \{0, \dots, l-1\}$ ,  $W''_i$  has at most  $2p$  connected components.

**Proof.** This follows immediately from the fact that  $\{B_v\}_{v \in V(F)}$  is a path-decomposition of width  $p$  and there is no edge in  $R_i \cap W$ . ◀

► **Lemma 43.** For any  $i \in \{0, \dots, l-1\}$ ,  $W''_i$  has at most  $p(p+1)$  connected components.

**Proof.** By Lemma 42 we know that  $W_i'$  has at most  $2p$  connected components.  $W_i''$  is obtained by deleting at most  $p(p-1)$  edges of  $W_i'$ . Therefore,  $W_i''$  has at most  $p(p+1) = 2p + p(p-1)$  connected components. ◀

► **Lemma 44.** *For any  $i \in \{0, \dots, l-1\}$ , the  $i$ -load of  $W_i'$  is at least 0.4.*

**Proof.** The  $i$ -load of  $W_i'$  is  $z(R_i')$ . Following [8] we know that  $z(R_i) \geq 2/5$  and thus  $z(R_i') \geq 2/5$ . ◀

► **Lemma 45.** *For any  $i \in \{0, \dots, l-1\}$ , the  $i$ -load of  $W_i''$  is at least 0.2.*

**Proof.** By Lemma 44 the  $i$ -load of  $W'$  is at least 0.4. By the construction, we have  $z(\{u, v_1\}) \leq 0.1$  and  $z(\{u, v_2\}) \leq 0.1$ . By deleting edges with both endpoints in  $B_{v_1}$  or  $B_{v_2}$ , we decrease the  $i$ -load by at most 0.2. Therefore, the  $i$ -load of  $W_i''$  is at least 0.2. ◀

► **Lemma 46.** *For any  $i \in \{0, \dots, l-1\}$ , the  $i$ -load of  $C_i$  is at least  $1/5p(p+1)$ .*

**Proof.** By Lemma 45 we know that the  $i$ -load of  $W_i''$  is at least 0.2. By Lemma 43 there are at most  $p(p+1)$  connected components in  $W_i''$ . Therefore the  $i$ -load of  $C_i$  is at least  $1/5p(p+1)$ . ◀

► **Lemma 47.** *There exists a constant  $\beta$  such that for any  $U \subseteq V(G)$ , we have  $|S \cap \delta(U)| \leq \beta \cdot p^2 \cdot z(\delta(U))$ .*

**Proof.** First we partition  $S \cap \delta(U)$  into two subsets:

1.  $S_1 = \{\{u, v\} \in S \cap \delta(U) : u, v \notin V(F)\}$ .
2.  $S_2 = \{\{u, v\} \in S \cap \delta(U) : v \in V(F)\}$ .

By the construction, following [8] we have  $|S_1| \leq 20 \cdot z(\delta(U))$ . Let  $e = \{u, v\} \in S_2$ . Let  $i \in \{0, \dots, l-1\}$  be the step that we add  $e$  to  $S$ . If  $e \in W$ , we can charge it to  $z(e) \geq 1/2$  and we are done. Suppose  $e \notin W$ . If there exists an edge  $e' \in E(C_i) \cap \delta(U)$ , we know that by the construction,  $e'$  does not have both endpoints in  $B_{v_1}$  or  $B_{v_2}$ . Therefore, for all  $j \neq i$  we have  $e' \notin E(C_j)$ . Thus we can charge  $e$  to  $z(e') \geq 1/2$  and we are done. Otherwise, suppose there is no edge in  $E(C_i) \cap \delta(U)$ . In this case, by the construction, for all  $e'' \in R_i$  with an endpoint in  $C_i$ , we have  $e'' \in \delta(U)$ . Now we know that  $e$  is the central edge in  $Y_i$ . By Lemma 46 we know that the  $i$ -load of  $C_i$  is at least  $1/5p(p+1)$ . Therefore, we can charge  $e$  to the  $i$ -load of  $C_i$  and we get  $|S_2| \leq 10 \cdot p^2 z(\delta(U))$ . Therefore, we have  $|S \cap \delta(U)| \leq 20 \cdot p^2 \cdot z(\delta(U))$ . ◀

► **Lemma 48.** *Let  $\vec{G}$  be a  $(0, g, 1, p)$ -nearly embeddable digraph, let  $\vec{H}$  be its vortex, and let  $\vec{W}$  be a walk in  $\vec{G}$  visiting all vertices in  $V(\vec{H})$ . Let  $G, H,$  and  $W$  be the symmetrizations of  $\vec{G}, \vec{H},$  and  $\vec{W}$  respectively. Let  $z : E(G) \rightarrow \mathbb{R}_{\geq 0}$  be  $\alpha$ -thick for some  $\alpha \geq 2$ , and  $\vec{W}$ -dense. Then there exists a polynomial time algorithm which given  $\vec{G}, \vec{H}, \vec{W}, z,$  and an embedding of  $\vec{G} \setminus \vec{H}$  into a surface of genus  $g$ , outputs a subgraph  $S \subseteq G \setminus H$ , satisfying the following conditions:*

1.  $W \cup S$  is a spanning subgraph of  $G$  and has  $O(g)$  connected components.
2.  $W \cup S$  is  $O(p^2)$ -thin w.r.t.  $z$ .

**Proof.** The assertion follows immediately by Lemmas 41 and 47. ◀

## 9.2 $(a, g, 1, p)$ -nearly embeddable graphs

For the remainder of this subsection, let  $a, g, k, p \geq 0$  and  $\vec{G}$  be an  $n$ -vertex  $(a, g, 1, p)$ -nearly embeddable digraph and let  $G$  be its symmetrization. Let  $\vec{H}$  be the single vortex of width  $p$ , attached to a face  $\vec{F}$  of  $\vec{G}$ . Let  $H$  and  $F$  be the symmetrization of  $\vec{H}$  and  $\vec{F}$  respectively. Let  $A \subseteq V(G)$  with  $|A| = a$  be the set of apices of  $G$ , where  $\Gamma = G \setminus (A \cup H)$  is a graph of genus  $g$ . Let  $\vec{W}$  be a walk in  $\vec{G}$  visiting all vertices in  $V(\vec{H})$  and let  $W$  be its symmetrization. Let  $z : E(G) \rightarrow \mathbb{R}_{\geq 0}$  be  $\alpha$ -thick, for some  $\alpha \geq 2$ , and  $\vec{W}$ -dense. Let  $G'$  be the graph obtained by contracting  $F$  to a single vertex  $v^*$  in  $G \setminus H$ .

We follow a similar algorithm as in Section 8 to find a  $O(a \cdot g + p^2)$ -thin spanning forest  $S$  in  $G'$ , with at most  $O(a + g)$  connected components. We modify the algorithm such that  $S$  is a  $O(a \cdot g + p^2)$ -thin subgraph of  $G$ .

We first start with  $\Gamma$  and construct  $\Gamma'$  the same way as in Section 8. For each connected component  $C$  of  $\Gamma'$ , we want to find a  $O(p^2)$ -thin spanning forest  $T_C$ , with at most  $g$  connected components. Let  $C_{v^*}$  be the connected component of  $\Gamma'$  with  $v^* \in C_{v^*}$ .  $C_{v^*}$  is a graph of genus at most  $g$ . For this component, we apply the modified ribbon-contraction argument on Subsection 9.1 to find  $T_{C_{v^*}}$ . Therefore,  $T_{C_{v^*}}$  is a  $O(p^2)$ -thin spanning forest in  $C_{v^*}$  with at most  $g$  connected components. The rest of the algorithm is the same as in Section 8 and we find a  $O(a \cdot g + p^2)$ -thin spanning forest  $S$  in  $G'$ , with at most  $O(a + g)$  connected components. Let  $T = S \cup W$ .

► **Lemma 49.**  *$T$  is a spanning subgraph of  $G$  with at most  $O(a + g)$  connected components.*

**Proof.** The same proof as in Lemma 41 applies here. The only difference here is that  $S$  has at most  $O(a + g)$  connected components in  $G'$ . Therefore,  $T$  has at most  $O(a + g)$  connected components in  $G$ . ◀

► **Lemma 50.**  *$S$  is a  $O(a \cdot g + p^2)$ -thin subgraph of  $G$  (w.r.t.  $z$ ).*

**Proof.** Let  $U \subseteq V(G)$  be a cut. Similar to Subsection 9.1, we partition  $S \cap \delta(U)$  into two subsets.

1.  $S_1 = \{\{u, v\} \in S \cap \delta(U) : u, v \notin V(F)\}$ .
2.  $S_2 = \{\{u, v\} \in S \cap \delta(U) : v \in V(F)\}$ .

First, by the construction and Lemma 39, we have  $|S_1| \leq 24000 \cdot a \cdot g \cdot z(\delta(U))$ . Now we partition  $S_2$  into three subsets.

1.  $S_{21} = \{\{a_j, v\} \in S_2 : a_j \in A, v \in V(F)\}$ .
2.  $S_{22} = \{\{u, v\} \in S_2 : v \in V(F), u \text{ and } v \text{ are in different components of } \Gamma'\}$ .
3.  $S_{23} = \{\{u, v\} \in S_2 : v \in V(F), u, v \in C_{v^*}\}$ .

By the construction, we know that  $|S_{21}| \leq 1$  and  $|S_{22}| \leq 7$ . Also, by Lemma 47 we have  $|S_{23}| \leq 20 \cdot p^2 \cdot z'(\delta(U))$ . Therefore, we have  $|S \cap \delta(U)| \leq 8(24000 \cdot a \cdot g + 20p^2)z(\delta(U))$ . ◀

► **Lemma 51.**  *$T$  is a  $O(a \cdot g + p^2)$ -thin subgraph of  $G$  (w.r.t.  $z$ ).*

**Proof.** By Lemma 50 we know that  $S$  is a  $O(a \cdot g + p^2)$ -thin subgraph of  $G$  (w.r.t.  $z$ ). Now note that  $z$  is  $\vec{W}$ -dense. Therefore,  $T = S \cup W$  is a  $O(a \cdot g + p^2)$ -thin subgraph of  $G$  (w.r.t.  $z$ ). ◀

We are now ready to prove the main result of this Section.

**Proof of Lemma 8.** It follows by Lemmas 49 and 51. ◀

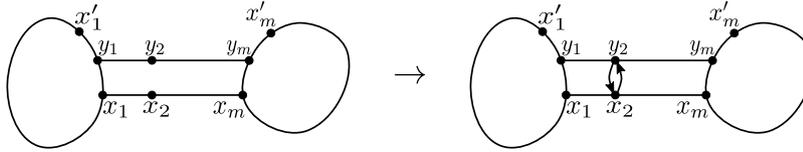
## 10 A preprocessing step for the dynamic program

► **Definition 52** (Facial normalization). Let  $g \geq 0$ ,  $p \geq 1$  and let  $\vec{G}$  be a  $(0, g, 1, p)$ -nearly embeddable graph. Let  $\vec{F}$  be the face on which the vortex is attached. We say that  $\vec{G}$  is *facially normalized* if the symmetrization of  $\vec{F}$  is a simple cycle and every  $v \in V(\vec{F})$  has at most one incident edge that is not in  $E(\vec{F})$ .

► **Lemma 53.** Let  $g \geq 0$ ,  $p \geq 1$  and let  $\vec{G}$  be a  $(0, g, 1, p)$ -nearly embeddable graph. There exists a polynomial-time computable  $(0, g, 1, p)$ -nearly embeddable facially normalized graph  $\vec{G}'$  such that the following holds. Let  $\vec{H}$  be the vortex in  $\vec{G}$  and let  $\vec{H}'$  be the vortex in  $\vec{G}'$ . Then  $\text{OPT}_{\vec{G}}(V(\vec{H})) = \text{OPT}_{\vec{G}'}(V(\vec{H}'))$ . Moreover there exists a polynomial-time algorithm which given any closed walk  $\vec{W}'$  in  $\vec{G}'$  that visits all vertices in  $V(\vec{H}')$ , outputs some closed walk  $\vec{W}$  in  $\vec{G}$  that visits all vertices in  $V(\vec{H})$  with  $\text{cost}_{\vec{G}}(\vec{W}) = \text{cost}_{\vec{G}'}(\vec{W}')$ .

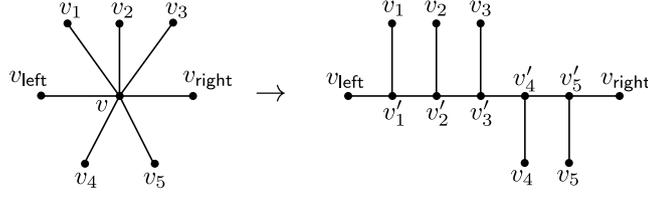
**Proof.** Let  $\vec{F}$  be the face in  $\vec{G}$  that  $\vec{H}$  is attached to. Let  $F$  be the symmetrization of  $\vec{F}$ . We first construct a  $(0, g, 1, p)$ -nearly embeddable graph  $\vec{G}''$ , with a vortex  $\vec{H}''$  attached to a face  $\vec{F}''$  such that the symmetrization of  $\vec{F}''$  is a simple cycle. Initially we set  $\vec{G}'' = \vec{G}$ . If  $\vec{F}$  is a simple cycle then there is nothing to be done. Otherwise, suppose that  $\vec{F}$  is not a simple cycle. Therefore,  $\partial F$  contains a family of simple cycles  $\mathcal{C} = \{C_1, \dots, C_k\}$  for some  $k$ , and a family of simple paths  $\mathcal{P} = \{P_1, \dots, P_l\}$  for some  $l$ , such that every  $P_i \in \mathcal{P}$  is a path  $x_1, x_2, \dots, x_m$  where  $x_1 \in C_\alpha$  and  $x_m \in C_\beta$  for some  $C_\alpha, C_\beta \in \mathcal{C}$ . We allow  $\mathcal{P}$  to contain paths of length 0.

For every  $P = x_1, x_2, \dots, x_m \in \mathcal{P}$ , where  $x_1 \in C$  and  $x_m \in C'$  for some  $C, C' \in \mathcal{C}$ , we update  $\vec{G}''$  as follows. Let  $x'_1 \in V(C)$  and  $x'_m \in V(C')$  be neighbors of  $x_1$  and  $x_m$ . We first duplicate  $P$  to get a new path  $P' = y_1, y_2, \dots, y_m$ . For every edge  $e \in E(P)$ , we set the cost of the corresponding edge  $e' \in P'$  equal to the cost of  $e$ . Also, for every  $j \in \{1, \dots, m\}$ , we add two edges  $(x_i, y_i)$  and  $(y_i, x_i)$  to  $\vec{G}''$  with  $\text{cost}_{\vec{G}''}(x_i, y_i) = \text{cost}_{\vec{G}''}(y_i, x_i) = 0$ . Also, we delete edges  $(x_1, x'_1)$ ,  $(x'_1, x_1)$ ,  $(x_m, x'_m)$ , and  $(x'_m, x_m)$  and we add edges  $(y_1, x'_1)$ ,  $(x'_1, y_1)$ ,  $(y_m, x'_m)$  and  $(x'_m, y_m)$  with the same cost respectively.



By the construction,  $\vec{G}''$  is a  $(0, g, 1, p)$ -nearly embeddable graph, such that  $\vec{F}''$  is a simple cycle. Also suppose that  $\vec{W}''$  is a closed walk in  $\vec{G}''$  that visits all vertices in  $V(\vec{H}'')$ . Then we can find a closed walk  $\vec{W}$  in  $\vec{G}$  that visits all vertices in  $V(\vec{H})$  with  $\text{cost}_{\vec{G}}(\vec{W}) = \text{cost}_{\vec{G}''}(\vec{W}'')$ .

Now we construct a facially normalized graph  $\vec{G}'$ . Initially we set  $\vec{G}' = \vec{G}''$ . For every  $v \in V(\vec{F}'')$  that has more than one incident edge in  $E(\vec{G}'') \setminus E(\vec{F}'')$  we update  $\vec{G}'$  as follows. Let  $v_{\text{left}}, v_{\text{right}} \in V(\vec{F}'')$  be the left and right neighbors of  $v$  on  $\vec{F}''$ . Let  $\mathcal{V} = \{v_1, \dots, v_m\}$  be the set of all neighbors of  $v$  in  $V(\vec{G}'') \setminus V(\vec{F}'')$ . Let  $\mathcal{V}' = \{v'_1, \dots, v'_m\}$ . First we delete  $v$  from  $\vec{G}'$  and we add  $\mathcal{V}'$  to  $V(\vec{G}')$ . For every  $(v, v_i) \in E(\vec{G}'')$  we add  $(v'_i, v_i)$  to  $E(\vec{G}')$  with  $\text{cost}_{\vec{G}'}(v'_i, v_i) = \text{cost}_{\vec{G}''}(v, v_i)$ . Also, for every  $j \in \{1, \dots, m-1\}$ , we add  $(v'_j, v'_{j+1})$  and  $(v'_{j+1}, v'_j)$  to  $E(\vec{G}')$  with  $\text{cost}_{\vec{G}'}(v'_j, v'_{j+1}) = 0$ . Finally we add  $(v'_1, v_{\text{left}})$ ,  $(v_{\text{left}}, v'_1)$ ,  $(v'_m, v_{\text{right}})$  and  $(v_{\text{right}}, v'_m)$  to  $\vec{G}'$  with the same costs as in  $\vec{G}''$ .



It is immediate that  $\vec{G}'$  is the desired graph.  $\blacktriangleleft$

► **Lemma 54.** *Let  $g \geq 0$ ,  $p \geq 1$  and let  $\vec{G}$  be a  $(0, g, 1, p)$ -nearly embeddable graph. There exists a polynomial-time computable  $(0, g, 1, p)$ -nearly embeddable facially normalized graph  $\vec{G}''$  such that the following conditions hold:*

1. *Let  $\vec{H}$  be the vortex in  $\vec{G}$  and let  $\vec{H}''$  be the vortex in  $\vec{G}''$ . Then  $\text{OPT}_{\vec{G}}(V(\vec{H})) = \text{OPT}_{\vec{G}''}(V(\vec{H}''))$ .*
2. *There exists a polynomial-time algorithm which given any closed walk  $\vec{W}''$  in  $\vec{G}''$  that visits all vertices in  $V(\vec{H}'')$ , outputs some closed walk  $\vec{W}$  in  $\vec{G}$  that visits all vertices in  $V(\vec{H})$  with  $\text{cost}_{\vec{G}}(\vec{W}) = \text{cost}_{\vec{G}''}(\vec{W}'')$ .*
3. *Let  $\vec{\Gamma}$  be the genus- $g$  piece of  $\vec{G}''$ . Let  $\vec{F}''$  be the face of  $\vec{\Gamma}$  on which the vortex  $\vec{H}''$  is attached. Then any  $v \in V(\vec{\Gamma}) \setminus V(\vec{F}'')$  has degree at most 4.*
4. *There exists some closed walk  $\vec{W}^*$  in  $\vec{G}''$  that visits all vertices in  $V(\vec{H}'')$ , with  $\text{cost}_{\vec{G}''}(\vec{W}^*) = \text{OPT}_{\vec{G}''}(V(\vec{H}''))$ , and such that every edge in  $\vec{\Gamma}$  is traversed at most once by  $\vec{W}^*$ .*

We say that a graph  $\vec{G}''$  satisfying the above conditions is cross normalized.

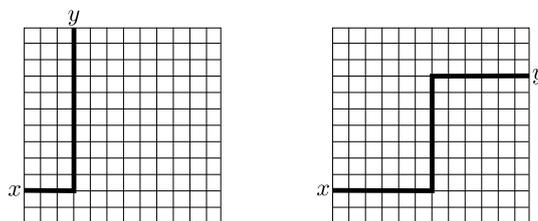
**Proof.** We begin with computing the facially normalized graph  $\vec{G}'$  given by Lemma 53. Clearly  $\vec{G}'$  satisfies conditions (1) and (2).

We next modify  $\vec{G}'$  so that it also satisfies (3). This can be done as follows. Let  $\vec{\Gamma}'$  be the genus- $g$  piece of  $\vec{G}'$  and let  $\vec{F}'$  be the face on which the vortex is attached. We replace each  $v \in V(\vec{\Gamma}') \setminus V(\vec{F}')$  of degree  $d > 4$  by a tree  $T_v$  with  $d$  leaves and with maximum degree 4; we replace each edge incident to  $v$  an edge incident to a unique leaf, and we set the length of every edge in  $E(T_v)$  to 0.

It remains to modify  $\vec{G}'$  so that it also satisfies (4). Let  $\vec{H}'$  be the vortex in  $\vec{G}'$ . Let  $\vec{W}'$  be a walk in  $\vec{G}'$  that visits all vertices in  $\vec{H}'$  with  $\text{cost}_{\vec{G}'}(\vec{W}') = \text{OPT}_{\vec{G}'}(V(\vec{H}'))$ . We may assume w.l.o.g. that  $\vec{W}'$  contains at most  $n^2$  edges. Thus, every vertex in  $v \in V(\vec{\Gamma}') \setminus V(\vec{F}')$  is visited at most  $n^2$  times by  $\vec{W}'$ . We replace each  $v \in V(\vec{\Gamma}') \setminus V(\vec{F}')$  by a grid  $A_v$  of size  $3n^2 \times 3n^2$ , with each edge having length 0. Each edge incident to  $v$  in  $\vec{G}'$ , corresponds to a unique sides of  $A_v$  so that the ordering of the sides agrees with the ordering of the edges around  $v$  (in  $\psi$ ). We replace each  $(u, v) \in E(\vec{\Gamma}')$ , by a matching of size  $3n^2$  between the corresponding sides of  $A_u$  and  $A_v$ , where each edge in the matching has length equal to the length of  $(u, v)$ . Let  $\vec{G}''$  be the resulting graph.

We obtain the desired walk  $\vec{W}^*$  in  $\vec{G}''$  as follows. Let  $x_1, \dots, x_{4\ell}$  be the vertices in the boundary of  $A_v$ , with  $\ell = 3n^2 - 1$ , appearing in this order along a clockwise traversal of  $A_v$ , and such that  $x_1$  is the lower left corner. Then for any  $i \in \{1, \dots, 4\}$ , the vertices  $x_{(i-1)\ell+1}, \dots, x_{(i-1)\ell+n^2}$  correspond to the copies of  $v$  on the  $i$ -th side of  $A_v$ . We traverse  $\vec{G}'$  starting at some arbitrary vertex in  $V(\vec{H}')$ , and we inductively construct the walk  $\vec{W}^*$ . We consider each edge  $(u, v)$  in the order that it is traversed by  $\vec{W}'$ . Suppose that  $(u, v)$  is the  $t$ -th edge traversed by  $\vec{W}'$ , for some  $t \in \{1, \dots, n^2\}$ . For each  $i \in \{1, \dots, 4\}$ , we let the  $t$ -th copy of  $v$  on the  $i$ -th side of  $A_v$  to be  $x_{(i-1)\ell+t}$ . We distinguish between the following cases: (i) If  $u, v \in V(\vec{H}')$ , then  $(u, v) \in E(\vec{G}'')$  and we simply traverse  $(u, v)$  in  $\vec{G}''$ . (ii) If  $u \in V(\vec{H}')$  and  $v \notin V(\vec{H}')$  then we traverse the edge in  $\vec{G}''$  that connects  $u$  to the  $t$ -th copy of  $v$  in

the appropriate side of  $A_v$ . (iii) If  $u \notin V(\vec{H}')$  and  $v \in V(\vec{H}')$  then we traverse the edge in  $\vec{G}''$  that connects the  $t$ -th copy of  $u$  in the appropriate side of  $A_u$  to  $v$ . (iv) If  $u, v \notin V(\vec{H}')$  then we traverse the edge in  $\vec{G}''$  that connects the  $t$ -th copy of  $u$  to the  $t$ -th copy of  $v$  in the appropriate sides of  $A_u$  and  $A_v$  respectively. Finally, for any pair of consecutive edges  $(u, v), (v, w)$  traversed by  $\vec{W}'$ , with  $v \notin V(\vec{H}')$ , we need to add a path  $P$  in  $A_v$  connecting two copies of  $v$  in the corresponding sides of  $A_v$ . Since  $A_v$  is a grid of size  $3n^2 \times 3n^2$  this can be done so that all these paths are edge-disjoint. More precisely, this can be done as follows. Suppose that  $(u, v)$  is the  $t$ -th edge traversed by  $\vec{W}'$ , for some  $t \in \{1, \dots, n^2\}$ . If  $P$  connects vertices  $x$  and  $y$  in consecutive sides of  $A_v$ , then we proceed as follows. We may assume w.l.o.g that  $x = x_t$  and  $y = x_{\ell+t+1}$ , since other cases can be handled in a similar way. We set  $P$  to be the unique path starting at  $x_t$ , following  $t + 1$  horizontal edges in  $A_v$ , and finally following  $\ell - t$  vertical edges to  $x_{\ell+t+1}$ . Otherwise, if  $P$  connects vertices  $x$  and  $y$  in opposite sides of  $A_v$ , then we proceed as follows. We may assume w.l.o.g that  $x = x_t$  and  $y = x_{2\ell+t+1}$ , since other cases can be handled in a similar way. We set  $P$  to be the unique path starting at  $x_t$ , following  $t + n^2$  horizontal edges in  $A_v$ , and then following  $\ell - 2t$  vertical edges, and finally following  $\ell - t - n^2 - 1$  horizontal edges to  $x_{2\ell+t+1}$ .



By the construction, it is immediate that all the paths  $P$  constructed above are pairwise edge-disjoint, which implies that every edge in  $E(\vec{G}'')$  is visited by  $\vec{W}^*$  at most once, concluding the proof.  $\blacktriangleleft$

## 11 Uncrossing an optimal walk traversing a vortex

Let  $g \geq 0, p \geq 1$  and let  $\vec{G}$  be a  $(0, g, 1, p)$ -nearly embeddable graph. By Lemma 54 we may assume w.l.o.g. that  $\vec{G}$  is facially normalized and cross normalized. Let  $\vec{G}' \subseteq \vec{G}$  be the piece of genus  $g$  and fix a drawing  $\psi$  of  $\vec{G}'$  into a surface of genus  $g$ . Let  $\vec{H}$  be the single vortex in  $\vec{G}$  and suppose that  $\vec{H}$  is attached to some face  $\vec{F}$  of  $\vec{G}'$ . Fix an optimal solution  $\vec{W}_{\text{OPT}}$ , that is a closed walk in  $\vec{G}$  that visits all vertices in  $\vec{H}$  minimizing  $\text{cost}_{\vec{G}}(\vec{W})$ ; if there are multiple such walks pick consistently one with a minimum number of edges. Since  $\vec{G}$  is cross normalized we may assume w.l.o.g. that  $\vec{W}_{\text{OPT}}$  traverses every edge in  $E(\vec{G}') \setminus E(\vec{F})$  at most once.

### 11.1 The structure of an optimal solution

► **Definition 55** (Shadow). Let  $\mathcal{W}$  be a collection of walks in  $\vec{G}$ . We define *shadow* of  $\mathcal{W}$  (w.r.t.  $\vec{G}'$ ) to be the collection of open and closed walks obtained by restricting every walk in  $\mathcal{W}$  on  $\vec{G}'$  (note that a walk in  $\mathcal{W}$  can give rise to multiple walks in  $\mathcal{W}'$ , and every open walk in  $\mathcal{W}'$  must have both endpoints in  $\vec{F}$ ).

We say that two edge-disjoint paths  $P, P'$  in  $\vec{G}'$  *cross* (w.r.to  $\psi$ ) if there exists  $v \in V(P) \cap V(P')$  such that  $v$  has degree 4 (recall that  $\vec{G}$  is cross normalized), with neighbors  $u_1, \dots, u_4$ , such that the edges  $\{v, u_1\}, \dots, \{v, u_4\}$  appear in this order around  $v$  in the embedding  $\psi$ ,  $P$  contains the subpath  $u_1, v, u_3$ , and  $P'$  contains the subpath  $u_2, v, u_4$ . We

say that two walks in  $\vec{G}'$  cross (at  $v$ , w.r.to  $\psi$ ) if they contain crossing subpaths. Finally, a walk is *self-crossing* if it contains two disjoint crossing subpaths.

► **Lemma 56** (Uncrossing an optimal walk of a vortex). *There exists a collection  $\mathcal{W} = \{\vec{W}_1, \dots, \vec{W}_\ell\}$  of closed walks in  $\vec{G}$  satisfying the following conditions:*

1. *Every edge in  $E(\vec{G}') \setminus E(\vec{F})$  is traversed in total at most once by all the walks in  $\mathcal{W}$ .*
2.  *$V(\vec{W}_1) \cup \dots \cup V(\vec{W}_\ell)$  is a strongly-connected subgraph of  $\vec{G}$ .*
3.  *$V(\vec{H}) \subseteq V(\vec{W}_1) \cup \dots \cup V(\vec{W}_\ell)$ .*
4.  *$\sum_{i=1}^{\ell} \text{cost}_{\vec{G}}(\vec{W}_i) \leq \text{OPT}_{\vec{G}}(V(\vec{H}))$ .*
5. *Let  $\mathcal{W}'$  be the shadow of  $\mathcal{W}$ . Then the walks in  $\mathcal{W}'$  are non-self-crossing and pairwise non-crossing.*

**Proof.** Initially, we set  $\mathcal{W} = \{\vec{W}_{\text{OPT}}\}$ . Recall that since  $\vec{G}$  is cross normalized, every edge in  $E(\vec{G}') \setminus E(\vec{F})$  is traversed at most once by  $\vec{W}_{\text{OPT}}$ . Clearly, this choice of  $\mathcal{W}$  satisfies conditions (1)–(4). We proceed to iteratively modify  $\mathcal{W}$  until condition (4) is also satisfied, while inductively maintaining (1)–(4).

Suppose that the current choice for  $\mathcal{W}$  does not satisfy (5). This means that either there exist two distinct crossing walks in  $\mathcal{W}$ , or there exists some self-crossing walk in  $\mathcal{W}$ . In either case, it follows that there exist subpaths  $P, P'$  of the walks in  $\mathcal{W}$  that are crossing (w.r.to  $\phi$ ). This means that there exists  $v \in V(P) \cap V(P')$  and  $e_1, e_2 \in E(P)$ ,  $e_3, e_4 \in E(P')$  such that  $\psi(e_1), \psi(e_4), \psi(e_2), \psi(e_3)$  appear in this order around  $\psi(v)$ . We modify  $P$  and  $P'$  by swapping  $e_1$  and  $e_3$ . It is immediate that the above operation preserves conditions (1)–(4). Moreover, after performing the operation, the total number of crossings and self-crossings (counted with multiplicities) between the walks in  $\mathcal{W}$  decreases by at least one. Since the original number of crossings is finite, it follows that the process terminates after a finite number of iterations. By the inductive condition, it is immediate that when the process terminates the collection  $\mathcal{W}$  satisfies condition (5), concluding the proof. ◀

For the remainder of this section let  $\mathcal{W}$  and  $\mathcal{W}'$  be as in Lemma 56. Let  $\mathcal{I}$  be a graph with  $V(\mathcal{I}) = \mathcal{W}'$  and with  $E(\mathcal{I}) = \left\{ \{W, W'\} \in \binom{\mathcal{W}'}{2} : V(W) \cap V(W') \neq \emptyset \right\}$ .

Let  $\vec{W}, \vec{Z}$  be distinct closed walks in some digraph, and let  $v \in V(\vec{W}) \cap V(\vec{Z})$ . Suppose that  $\vec{W} = x_1, \dots, x_k, v, x_{k+1}, \dots, x_{k'}, x_1$  and  $\vec{Z} = y_1, \dots, y_r, v, y_{r+1}, \dots, y_{r'}, y_1$ . Let  $\vec{S}$  be the closed walk  $x_1, \dots, x_k, v, y_{r+1}, \dots, y_{r'}, y_1, \dots, y_r, v, x_{k+1}, \dots, x_{k'}, x_1$ . We say that  $\vec{S}$  is obtained by *shortcutting*  $\vec{W}$  and  $\vec{Z}$  (at  $v$ ).

Let  $\mathcal{J}$  be a subgraph of  $\mathcal{I}$ . Let  $\mathcal{W}^{\mathcal{J}}$  be a collection of walks in  $\vec{G}$  constructed inductively as follows. Initially we set  $\mathcal{W}^{\mathcal{J}} = \mathcal{W}$ . We consider all  $\{W, W'\} \in E(\mathcal{J})$  in an arbitrary order. Note that since  $\{W, W'\} \in E(\mathcal{J})$ , it follows that  $W$  and  $W'$  cross at some  $v \in V(\vec{G}')$ . Let  $R$  and  $R'$  be the walks in  $\mathcal{W}^{\mathcal{J}}$  such that  $W$  and  $W'$  are sub-walks of  $R$  and  $R'$  respectively. If  $R \neq R'$  then we replace  $R$  and  $R'$  in  $\mathcal{W}^{\mathcal{J}}$  by the walk obtained by shortcutting  $R$  and  $R'$  at  $v$ . This completes the construction of  $\mathcal{W}^{\mathcal{J}}$ . We say  $\mathcal{W}^{\mathcal{J}}$  is obtained by *shortcutting*  $\mathcal{W}$  at  $\mathcal{J}$ .

► **Lemma 57.** *There exists some forest  $\mathcal{F}$  in  $\mathcal{I}$  such that the collection of walks obtained by shortcutting  $\mathcal{W}$  at  $\mathcal{F}$  contains a single walk.*

**Proof.** Let  $\mathcal{F}$  be a forest obtained by taking a spanning subtree in each connected component of  $\mathcal{I}$ . Let  $W, W' \in \mathcal{W}$ . Let  $\mathcal{W}^{\mathcal{F}}$  be obtained by shortcutting  $\mathcal{W}$  at  $\mathcal{F}$ . It suffices to show that  $W$  and  $W'$  become parts of the same walk in  $\mathcal{W}^{\mathcal{F}}$ . By condition (2) of Lemma 56 we have that there exists a sequence of walks  $W_1, \dots, W_t \in \mathcal{W}$ , with  $W_1 = W$ ,  $W_t = W'$ , and such that for any  $i \in \{1, \dots, t-1\}$ , there exists some walk  $A_i \in W_i \cap \vec{G}'$ , and some walk  $B_{i+1} \in W_{i+1} \cap \vec{G}'$  such that  $\{A_i, B_{i+1}\} \in E(\mathcal{I})$ . Therefore  $A_i$  and  $B_{i+1}$  are in the same

connected component of  $\mathcal{I}$ . Thus there exists some tree  $\mathcal{T}_i$  in  $\mathcal{F}$  such that  $A_i, B_{i+1} \in V(\mathcal{T}_i)$ . It follows that after shortcutting  $\mathcal{W}$  at  $\mathcal{F}$ , the walks  $A_i$  and  $B_{i+1}$  become parts of the same walk. By induction on  $i \in \{1, \dots, t-1\}$ , it follows that  $A_1$  and  $B_t$  become parts of the same walk in  $\mathcal{W}^{\mathcal{F}}$ , and thus so do  $W$  and  $W'$ , concluding the proof.  $\blacktriangleleft$

For the remainder let  $\mathcal{F}$  be the forest given by Lemma 57.

► **Lemma 58.** *All leaves of  $\mathcal{F}$  intersect  $F$ .*

**Proof.** Suppose that there exists some leaf  $\vec{W}$  of  $\mathcal{F}$  with  $V(\vec{W}) \cap V(F) = \emptyset$ . Then simply removing  $\vec{W}$  from  $\mathcal{W}$  leaves a new collection of walks that visits all vertices in  $V(\vec{H})$  and such that the union of all walks is a strongly-connected subgraph of  $\vec{G}$ . Thus after shortcutting all these walks we may obtain a new single walk  $\vec{R}$  that visits all vertices in  $V(\vec{H})$  with  $\text{cost}_{\vec{G}}(\vec{R}) \leq \text{cost}_{\vec{G}}(\vec{W}_{\text{OPT}})$  and with fewer edges than  $\vec{W}_{\text{OPT}}$ , contradicting the choice of  $\vec{W}_{\text{OPT}}$ .  $\blacktriangleleft$

## 12 The dynamic program for traversing a vortex in a planar graph

For the remainder of this section let  $\vec{G}$  be a  $n$ -vertex  $(0, 0, 1, p)$ -nearly embeddable graph (that is, planar with a single vortex). Let  $\vec{H}$  be the vortex in  $\vec{G}$  and suppose it is attached on some face  $\vec{F}$ . Fix an optimal solution  $\vec{W}_{\text{OPT}}$ , that is a closed walk in  $\vec{G}$  that visits all vertices in  $\vec{H}$  minimizing  $\text{cost}_{\vec{G}}(\vec{W})$ ; if there are multiple such walks pick consistently one with a minimum number of edges. Let  $F$  be the symmetrization of  $\vec{F}$ . We present an algorithm for computing a walk traversing all vertices in  $V(\vec{H})$  based on dynamic programming. By Lemma 54 we may assume w.l.o.g. that  $\vec{G}$  is facially normalized and cross normalized.

Fix a path-decomposition  $\{B_v\}_{v \in V(F)}$  of  $\vec{H}$  of width  $p$ .

Let  $\mathcal{S}$  be a collection of walks in  $\vec{G}$ . For any  $v \in V(\vec{G})$  we denote by  $\text{in-degree}_{\mathcal{S}}(v)$  the number of times that the walks in  $\mathcal{S}$  enter  $v$ ; similarly, we denote by  $\text{out-degree}_{\mathcal{S}}(v)$  the number of times that the walks in  $\mathcal{S}$  exit  $v$ . We define  $\vec{G}[\mathcal{S}]$  to be the graph with  $V(\vec{G}[\mathcal{S}]) = \bigcup_{W \in \mathcal{S}} V(W)$  and  $E(\vec{G}[\mathcal{S}]) = \bigcup_{W \in \mathcal{S}} E(W)$ .

### 12.1 The dynamic program

Let  $\mathcal{P}$  be the set of all subpaths of  $F$ , where we allow for simplicity in notation that a path be closed. Let  $u, v$  be the endpoints of  $P$ . Let  $\vec{H}_P = \vec{H} \left[ \bigcup_{x \in V(P)} B_x \right]$ . Let  $\mathcal{C}_P$  be the set of all possible partitions of  $B_u \cup B_v$ . Let  $\mathcal{D}_P^{\text{in}} = \{0, \dots, n\}^{B_u \cup B_v}$ ,  $\mathcal{D}_P^{\text{out}} = \{0, \dots, n\}^{B_u \cup B_v}$ , that is, every element of  $\mathcal{D}_P^{\text{in}} \cup \mathcal{D}_P^{\text{out}}$  is a function  $f : B_u \cup B_v \rightarrow \{0, \dots, n\}$ . Let  $\mathcal{A} = V(F)^2$ .

#### 12.1.1 The dynamic programming table

The dynamic programming table is indexed by all pairs  $(P, \phi)$  where  $P \in \mathcal{P}$  and  $\phi = (C, f^{\text{in}}, f^{\text{out}}, a, l, r, p) \in \mathcal{I}_P$ , where

$$\mathcal{I}_P = \mathcal{C}_P \times \mathcal{D}_P^{\text{in}} \times \mathcal{D}_P^{\text{out}} \times (\mathcal{A} \cup (\mathcal{A} \times \mathcal{A}) \cup \text{nil}) \times (V(\vec{G}) \cup \text{nil}) \times (V(\vec{G}) \cup \text{nil}) \times (V(\vec{G}) \cup \text{nil}).$$

A *partial solution* is a collection of walks in  $\vec{G}$ .

We say that a partial solution  $\mathcal{S}$  is *compatible* with  $(P, \phi)$  if the following conditions are satisfied:

(T1) For every  $x \in V(\vec{H}_P)$  there exists some walk in  $\mathcal{S}$  that visits  $x$ . That is  $V(\vec{H}_P) \subseteq \bigcup_{Q \in \mathcal{S}} V(Q)$ .

- (T2) If  $a \neq \text{nil}$  and  $a \in \mathcal{A}$ , let  $a = (u', v')$ . Let  $Q_1$  be the shortest path from  $u'$  to  $l$  in  $\vec{G}$ . Let  $Q_2$  be the shortest path from  $l$  to  $r$  in  $\vec{G}$ . Let  $Q_3$  be the shortest path from  $r$  to  $v'$  in  $\vec{G}$ . Let  $Q_1^*$  be the walk from  $u'$  to  $v'$  obtained by the concatenation of  $Q_1, Q_2$  and  $Q_3$ . Then  $Q_1^*$  is a sub-walk of some walk in  $\mathcal{S}$ . We refer to  $Q_1^*$  as the *grip* of  $\mathcal{S}$ , and in this case we say that it is an *unbroken grip*. If  $a \in V(P)^2$  then we say that the unbroken grip is *closed* and otherwise we say that it is *open*. Otherwise, if  $a \in (\mathcal{A} \times \mathcal{A})$ , let  $a = ((u'_1, v'_1), (u'_2, v'_2))$ . Let  $Q'_1$  be the shortest path from  $u'_1$  to  $l$  in  $\vec{G}$ . Let  $Q'_2$  be the shortest path from  $l$  to  $v'_1$  in  $\vec{G}$ . Let  $Q_2^*$  be the path from  $u'_1$  to  $v'_1$  obtained by the concatenation of  $Q'_1$  and  $Q'_2$ . Let  $Q''_1$  be the shortest path from  $u'_2$  to  $r$  in  $\vec{G}$ . Let  $Q''_2$  be the shortest path from  $r$  to  $v'_2$  in  $\vec{G}$ . Let  $Q_3^*$  be the path from  $u'_2$  to  $v'_2$  obtained by the concatenation of  $Q''_1$  and  $Q''_2$ . Then  $Q_2^*$  and  $Q_3^*$  are sub-walks of some walks in  $\mathcal{S}$ . We refer to  $(Q_2^*, Q_3^*)$  as the *broken grip* of  $\mathcal{S}$ .
- (T3) If  $a = \text{nil}$  or  $a \in \mathcal{A}$ , then every open walk in  $\mathcal{S}$  has both endpoints in  $B_u \cup B_v$ , except possibly for one walk  $W \in \mathcal{S}$  that contains the grip as a sub-walk. If  $a = (Q_1, Q_2) \in (\mathcal{A} \times \mathcal{A})$ , then every open walk in  $\mathcal{S}$  has both endpoints in  $B_u \cup B_v$ , except possibly for at most two walks  $W_1, W_2 \in \mathcal{S}$  that contain  $Q_1$  and  $Q_2$  as sub-walks (note that  $Q_1$  and  $Q_2$  might be sub-walks of the same walk in  $\mathcal{S}$ ).
- (T4) For all  $x \in B_u \cup B_v$  we have  $f^{\text{in}}(x) = \text{in-degree}_{\mathcal{S}}(x)$  and  $f^{\text{out}}(x) = \text{out-degree}_{\mathcal{S}}(x)$ .
- (T5) For any  $x, y \in B_u \cup B_v$  we have that if  $x$  and  $y$  are in the same set of the partition  $C$  then they are in the same weakly-connected component of  $\bigcup_{W \in \mathcal{S}} W$ . Moreover for any  $z \in V(\vec{H}_P)$  there exists  $z' \in B_u \cap B_v$  such that  $z$  and  $z'$  are in the same weakly-connected component of  $\bigcup_{W \in \mathcal{S}} W$ .

### 12.1.2 Merging partial solutions

We compute the values of the dynamic programming table inductively as follows. Let  $P, P_1, P_2 \in \mathcal{P}$  such that  $E(P_1) \neq \emptyset$ ,  $E(P_2) \neq \emptyset$ ,  $E(P_1) \cap E(P_2) = \emptyset$ , and  $P = P_1 \cup P_2$ . Let  $u \in V(P_1)$ ,  $w \in V(P_1) \cap V(P_2)$ ,  $v \in V(P_2)$  such that  $u, w$  are the endpoints of  $P_1$  and  $w, v$  are the endpoints of  $P_2$ . Let

$$\phi = (C, f^{\text{in}}, f^{\text{out}}, a, l, r, p) \in \mathcal{I}_P,$$

$$\phi_1 = (C_1, f_1^{\text{in}}, f_1^{\text{out}}, a_1, l_1, r_1, p_1) \in \mathcal{I}_{P_1},$$

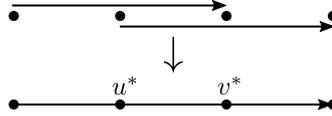
$$\phi_2 = (C_2, f_2^{\text{in}}, f_2^{\text{out}}, a_2, l_2, r_2, p_2) \in \mathcal{I}_{P_2}.$$

For any  $i \in \{1, 2\}$  let  $\mathcal{S}_i$  be a partial solution that is compatible with  $(P_i, \phi_i)$ . We proceed to compute a collection of walks  $\mathcal{S}$  that is compatible with  $(P, \phi)$ . This is done in phases, as follows:

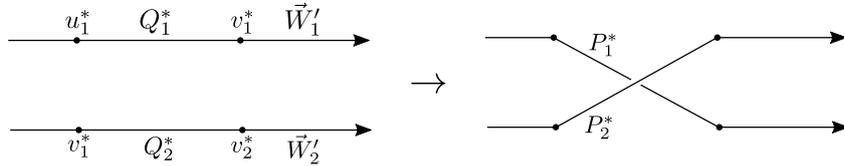
**Merging phase 1: Joining the walks.** We check that for all  $x \in B_w$  we have  $f_1^{\text{in}}(x) = f_2^{\text{out}}(x)$  and  $f_2^{\text{in}}(x) = f_1^{\text{out}}(x)$ . If not then the merging procedure return nil. For any  $x \in B_w$  and for any  $i \in \{1, 2\}$  let  $E_i(x)^{\text{in}}$  (resp.  $E_i(x)^{\text{out}}$ ) be the multiset of all edges in all walks in  $\mathcal{S}_i$  that are incoming to (resp. outgoing from)  $x$  counted with multiplicities. Since  $f_1^{\text{out}}(x) = f_2^{\text{in}}(x)$  and  $f_2^{\text{out}}(x) = f_1^{\text{in}}(x)$  it follows that  $|E_1^{\text{in}}(x) \cup E_2^{\text{in}}(x)| = |E_1^{\text{out}}(x) \cup E_2^{\text{out}}(x)|$ . Pick an arbitrary bijection  $\sigma_x : E_1^{\text{in}}(x) \cup E_2^{\text{in}}(x) \rightarrow E_1^{\text{out}}(x) \cup E_2^{\text{out}}(x)$ . We initially set  $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ . For each  $x \in B_w$  we proceed as follows. For each  $e \in E_1^{\text{in}}(x) \cup E_2^{\text{in}}(x)$  we modify the walk traversing  $e$  so that immediately after traversing  $e$  it continues with the walk traversing  $\sigma_x(e) \in E_1^{\text{out}} \cup E_2^{\text{out}}$ .

**Merging phase 2: Updating the grip.** We check that at least one of the following conditions is satisfied:

1. Suppose that  $a = l = r = p = a_1 = l_1 = r_1 = p_1 = a_2 = l_2 = r_2 = p_2 = \text{nil}$ . Then there is nothing to do.
2. Suppose that  $a_1 = l_1 = r_1 = p_1 = \text{nil}$ ,  $l = l_2$ ,  $r = r_2$ ,  $p = p_2$  and  $a = a_2 = (u_2^*, v_2^*) \in \mathcal{A}$  with  $\{u_2^*, v_2^*\} \cap V(P_1) \subseteq \{u, w\}$ , or  $a_2 = l_2 = r_2 = p_2 = \text{nil}$ ,  $l = l_1$ ,  $r = r_1$ ,  $p = p_1$  and  $a = a_1 = (u_1^*, v_1^*) \in \mathcal{A}$  with  $\{u_1^*, v_1^*\} \cap V(P_2) \subseteq \{w, v\}$ . Then there is nothing to do.
3. Suppose that  $a_1 \neq \text{nil}$ ,  $a_2 \neq \text{nil}$ , and  $a \neq \text{nil}$ . Suppose  $a_1 = a_2 = a = (u^*, v^*) \in \mathcal{A}$ , with  $a \in V(P_1) \times V(P_2)$  or  $a \in V(P_2) \times V(P_1)$ . Suppose  $l_1 = l_2 = l$ ,  $r_1 = r_2 = r$  and  $p_1 = p_2 = p$ . Then we proceed as follows to ensure that (T2) holds. We may assume w.l.o.g. that  $a \in V(P_1) \times V(P_2)$  since the remaining case can be handled in a similar way. Let  $Q^*$  be the grip between  $u^*$  and  $v^*$  in  $\vec{G}$ . It follows by (T2) that for any  $i \in \{1, 2\}$  there exists a walk  $\vec{W}_i \in \mathcal{S}_i$  that contains  $Q^*$  as a sub-walk. It follows by the definition of the merging phase 1 that for any  $i \in \{1, 2\}$  there exists  $\vec{W}'_i \in \mathcal{S}$  that contains  $Q^*$  as a sub-walk. We will modify  $\mathcal{S}$  in order to ensure that (T2) holds. We remove  $Q^*$  from  $\vec{W}'_2$  and we merge  $\vec{W}'_1$  with  $\vec{W}'_2 \setminus Q^*$  (via concatenation).

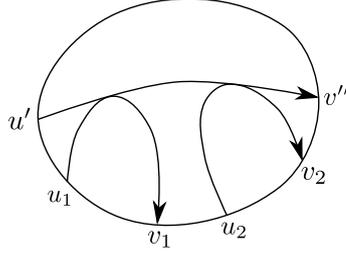


4. Suppose that  $a_1, a_2, a \in \mathcal{A}$ . Suppose that  $a_1 = (u_1^*, v_1^*)$ ,  $a_2 = (u_2^*, v_2^*)$ ,  $a = (u^*, v^*)$ , with  $u^* \in \{u_1^*, u_2^*\}$ , and  $v^* \in \{v_1^*, v_2^*\}$ . Suppose that  $l_1 = l$ ,  $l_2 = r_2 = r$  and  $p_1 = p_2 = p$ , or  $l_2 = l$ ,  $l_1 = r_1 = r$  and  $p_1 = p_2 = p$ , or  $l = r = p_1 = p_2$  and  $l_2 = r_2$ , or  $l = r = p_1 = p_2$  and  $l_1 = r_1$ . Then we proceed as follows to ensure that (T2) holds. We may assume w.l.o.g. that  $a = (u_2^*, v_1^*)$ ,  $l_1 = l$ ,  $l_2 = r_2 = r$  and  $p_1 = p_2 = p$  since the other cases can be handled in a similar way. For any  $i \in \{1, 2\}$  let  $Q_i^*$  be the grip of  $\mathcal{S}_i$ . It follows by (T2) that for any  $i \in \{1, 2\}$  there exists a walk  $\vec{W}_i \in \mathcal{S}_i$  that contains  $Q_i^*$  as a sub-walk. It follows that for any  $i \in \{1, 2\}$  there exists  $\vec{W}'_i \in \mathcal{S}$  that contains  $Q_i^*$  as a sub-walk. We will modify  $\mathcal{S}$  in order to ensure that (T2) holds. If  $a = a_1$  or  $a = a_2$  then there is nothing left to do. Otherwise, let  $R'_1$  be the shortest path in  $\vec{G}$  from  $u_1^*$  to  $r_1$ . Let  $R''_1$  be the shortest path in  $\vec{G}$  from  $r_1$  to  $l_2$ . Let  $R'''_1$  be the shortest path in  $\vec{G}$  from  $l_2$  to  $v_2^*$ . Let  $R_1^*$  be the path in  $\vec{G}$  from  $u_1^*$  to  $v_2^*$  obtained by concatenation of  $R'_1$ ,  $R''_1$  and  $R'''_1$ . Let  $R'_2$  be the shortest path in  $\vec{G}$  from  $u_2^*$  to  $p$ . Let  $R''_2$  be the shortest path in  $\vec{G}$  from  $p$  to  $l_1$ . Let  $R'''_2$  be the shortest path in  $\vec{G}$  from  $l_1$  to  $v_1^*$ . Let  $R_2^*$  be the path in  $\vec{G}$  from  $u_2^*$  to  $v_1^*$  obtained by concatenation of  $R'_2$ ,  $R''_2$  and  $R'''_2$ . We remove  $Q_1^*$  and  $Q_2^*$  from  $\vec{W}'_1$  and  $\vec{W}'_2$  and replace them by  $R_1^*$  and  $R_2^*$ .



5. Suppose that  $a_1 = \text{nil}$  and  $a = a_2 \in (\mathcal{A} \times \mathcal{A})$ , or  $a_2 = \text{nil}$  and  $a = a_1 \in (\mathcal{A} \times \mathcal{A})$ . Then there is nothing to do.
6. Suppose that  $a_1 \in \mathcal{A}$ ,  $a_2 \in \mathcal{A}$  and  $a \in (\mathcal{A} \times \mathcal{A})$ . Suppose  $a_1 = (u_1, v_1)$ ,  $a_2 = (u_2, v_2)$  and  $a = ((u', v'), (u'', v''))$ , with  $v' \in \{v_1, v_2\}$  and  $u'' \in \{u_1, u_2\}$ . We may assume w.l.o.g. that  $a = ((u', v_1), (u_2, v''))$ . Suppose that  $l = l_1 = r_1$ ,  $r = l_2 = r_2$  and  $p = p_1 = p_2$ . For any  $i \in \{1, 2\}$ , let  $Q_i$  be the grip of  $\mathcal{S}_i$ . It follows by (T2) that for

any  $i \in \{1, 2\}$  there exists a walk  $\vec{W}_i \in \mathcal{S}_i$  that contains  $Q_i$  as a sub-walk. It follows that for any  $i \in \{1, 2\}$  there exists  $\vec{W}'_i \in \mathcal{S}$  that contains  $Q_i$  as a sub-walk. Let  $Q'_1$  be the shortest path in  $\vec{G}$  from  $u_1$  to  $l_1$ . Let  $Q'_2$  be the shortest path in  $\vec{G}$  from  $l_1$  to  $l_2$ . Let  $Q'_3$  be the shortest path in  $\vec{G}$  from  $l_2$  to  $v_2$ . Let  $Q_1^*$  be the path in  $\vec{G}$  from  $u_1$  to  $v_2$  obtained by concatenation of  $Q'_1$ ,  $Q'_2$  and  $Q'_3$ . Let  $Q''_1$  be the shortest path in  $\vec{G}$  from  $u'$  to  $l_1$ . Let  $Q''_2$  be the shortest path in  $\vec{G}$  from  $l_1$  to  $v_1$ . Let  $Q_2^*$  be the path in  $\vec{G}$  from  $u'$  to  $v_1$  obtained by concatenation of  $Q''_1$  and  $Q''_2$ . Let  $Q'''_1$  be the shortest path in  $\vec{G}$  from  $u_2$  to  $l_2$ . Let  $Q'''_2$  be the shortest path in  $\vec{G}$  from  $l_2$  to  $v''$ . Let  $Q_3^*$  be the path in  $\vec{G}$  from  $u_2$  to  $v''$  obtained by concatenation of  $Q'''_1$  and  $Q'''_2$ . Then we remove  $Q_1$  and  $Q_2$  from  $\vec{W}'_1$  and  $\vec{W}'_2$ , and replace them by  $Q_1^*$ ,  $Q_2^*$  and  $Q_3^*$ .



7. Suppose that  $a_1 \in (\mathcal{A} \times \mathcal{A})$ ,  $a_2 \in \mathcal{A}$  and  $a \in (\mathcal{A} \times \mathcal{A})$ , or  $a_1 \in \mathcal{A}$ ,  $a_2 \in (\mathcal{A} \times \mathcal{A})$  and  $a \in (\mathcal{A} \times \mathcal{A})$ . We may assume w.l.o.g that  $a_1 \in (\mathcal{A} \times \mathcal{A})$ ,  $a_2 \in \mathcal{A}$  and  $a \in (\mathcal{A} \times \mathcal{A})$ . Suppose that  $a_1 = ((u_1, v_1), (u'_1, v'_1))$ ,  $a_2 = (u_2, v_2)$  and  $a = ((u, v), (u', v'))$ , with  $(u, v) = (u_1, v_1)$ ,  $u' \in \{u_1, u_2\}$ , and  $v' \in \{v_1, v_2\}$ , or  $(u, v) = (u_2, v_2)$ ,  $u' \in \{u_1, u_2\}$ , and  $v' \in \{v_1, v_2\}$ . We may assume w.l.o.g that  $(u, v) = (u_1, v_1)$  and  $(u', v') = (u_2, v'_1)$ . Suppose that  $l = l_1$ ,  $r = l_2 = r_2$  and  $p = p_1 = p_2$ . Let  $(Q_1, Q'_1)$  be the grip of  $\mathcal{S}_1$  and let  $Q_2$  be the grip of  $\mathcal{S}_2$ . Let  $R_1$  be the shortest path in  $\vec{G}$  from  $u'_1$  to  $r_1$ . Let  $R_2$  be the shortest path in  $\vec{G}$  from  $r_1$  to  $r_2$ . Let  $R_3$  be the shortest path in  $\vec{G}$  from  $r_2$  to  $v_2$ . Let  $R'$  be the path in  $\vec{G}$  from  $u'_1$  to  $v_2$  obtained by concatenation of  $R_1$ ,  $R_2$  and  $R_3$ . Let  $R'_1$  be the shortest path in  $\vec{G}$  from  $u_2$  to  $r_2$ . Let  $R'_2$  be the shortest path in  $\vec{G}$  from  $r_2$  to  $v'_1$ . Let  $R''$  be the path in  $\vec{G}$  from  $u_2$  to  $v'_1$  obtained by concatenation of  $R'_1$  and  $R'_2$ . Then we remove  $Q'_1$  and  $Q_2$ , and replace them by  $R'$  and  $R''$ .

If none of the above holds then the merging procedure returns nil.

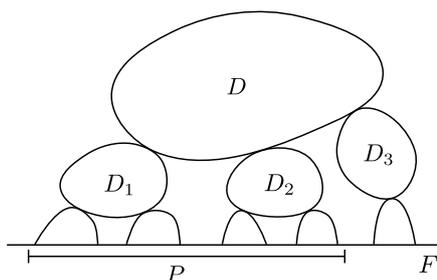
**Merging phase 3: Checking connectivity.** We check that condition (T5) holds for  $\mathcal{S}$  and we return nil if it does not.

► **Lemma 59.** *If the merging procedure outputs some partial solution  $\mathcal{S}$  then  $\mathcal{S}$  is compatible with  $\phi$ .*

**Proof.** It follows immediately by the definition of compatibility. ◀

### 12.1.3 Initializing the dynamic programming table

For all  $P \in \mathcal{P}$  containing at most one edge and for all  $\phi \in \mathcal{I}_P$  with  $\phi = (C, f^{\text{in}}, f^{\text{out}}, a, l, r, p)$  we proceed as follows. We enumerate all partial solutions  $\mathcal{S}$  that are compatible with  $(P, \phi)$  and have minimum cost. Any walk in any such partial solution can intersect  $\vec{G} \setminus \vec{H}$  only on the at most two oppositely-directed edges in  $E(P)$ . Moreover there are at most  $O(n^7)$  possibilities for  $a$ ,  $l$ ,  $r$  and  $p$ . Thus the enumeration can clearly be done in time  $n^{O(1)}$  by ensuring that for all walks  $W \in \mathcal{S}$ , their sub-walks that do not intersect  $E(P)$  are shortest paths between vertices in  $B_u \cup B_v$ . The total running time of this initialization step is therefore  $n^{O(p)}$ .



■ **Figure 1** Example of a basic path.

### 12.1.4 Updating the dynamic programming table

For all  $P \in \mathcal{P}$  containing  $m > 1$  edges, and for all  $P_1, P_2 \in \mathcal{P}$  with  $E(P_1) \neq \emptyset$ ,  $E(P_2) \neq \emptyset$ ,  $E(P_1) \cap E(P_2) = \emptyset$  and  $P_1 \cup P_2 = P$ , and for all  $\phi_1 \in \mathcal{I}_{P_1}$  and  $\phi_2 \in \mathcal{I}_{P_2}$  we proceed as follows. Suppose that for all paths  $P'$  containing  $m' < m$  edges and all  $\phi' \in \mathcal{I}_{P'}$  we have computed the partial solutions in the dynamic programming table at  $(P', \phi')$ . If there exist partial solutions  $\mathcal{S}_1$  and  $\mathcal{S}_2$  at  $(P_1, \phi_1)$  and  $(P_2, \phi_2)$  respectively, we call the merging process to merge  $\mathcal{S}_1$  and  $\mathcal{S}_2$ . Suppose that the merging process returns a partial solution  $\mathcal{S}$  at  $(P, \phi)$  for some  $\phi \in \mathcal{I}_P$ . If there is no partial solution stored currently at  $(P, \phi)$  then we store  $\mathcal{S}$  at that location. Otherwise if there there exists a partial solution  $\mathcal{S}'$  stored at  $(P, \phi)$  and the cost of  $\mathcal{S}$  is smaller than the cost of  $\mathcal{S}'$  then we replace  $\mathcal{S}'$  with  $\mathcal{S}$ .

## 12.2 Analysis

Let  $\mathcal{W}$  be the collection of walks given by Lemma 56. Let  $\mathcal{W}'$  be the shadow of  $\mathcal{W}$ . Let  $\mathcal{F}$  be the forest obtained by Lemma 57. For every connected component  $\mathcal{T}$  of  $\mathcal{F}$  pick some  $v_{\mathcal{T}} \in V(\mathcal{T})$  and consider  $\mathcal{T}$  to be rooted at  $v_{\mathcal{T}}$ .

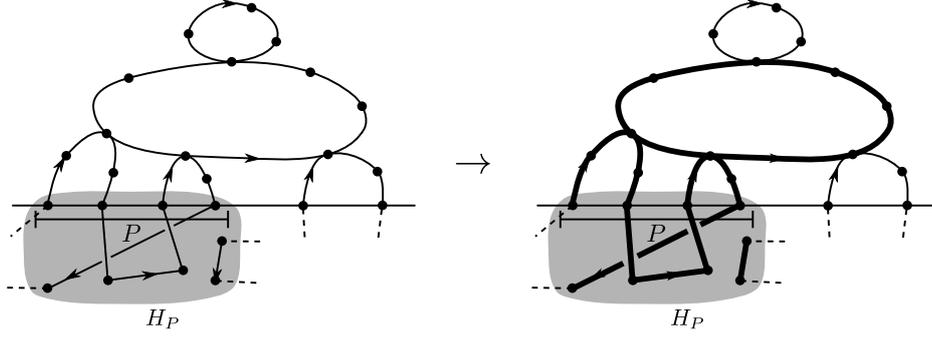
Let  $\vec{G}'$  be the planar piece of  $\vec{G}$ , that is  $\vec{G}' = \vec{G} \setminus (V(\vec{H}) \setminus (F))$ . Fix some planar drawing  $\psi$  of  $G'$ . Let  $\mathcal{D}$  be the disk with  $\partial\mathcal{D} = \psi(F)$  with  $\psi(\vec{G}') \subset \mathcal{D}$ .

Let  $P \in \mathcal{P}$  with endpoints  $u, v$ , and let  $\mathcal{T}$  be a subtree of some tree in  $\mathcal{F}$ . We say that  $P$  covers  $\mathcal{T}$  if for all  $D \in V(\mathcal{T})$  we have  $V(D) \cap V(F) \subseteq V(P)$ . We say that  $P$  avoids  $\mathcal{T}$  if for all  $D \in V(\mathcal{T})$  we have  $V(D) \cap V(P) \subseteq \{u, v\}$ .

► **Definition 60** (Basic path). Let  $P \in \mathcal{P}$ . Let  $u, v \in V(P)$  be the endpoints of  $P$ . We say that  $P$  is *basic* (w.r.t.  $\mathcal{W}$ ) if either  $P \setminus \{u, v\}$  does not intersect any of the walks in  $\mathcal{W}$  (in this case we call  $P$  *empty basic*) or the following holds. There exists some tree  $\mathcal{T}$  in  $\mathcal{F}$  and some  $D \in V(\mathcal{T})$ , with children  $D_1, \dots, D_k$ , intersecting  $D$  in this order along a traversal of  $D$ , such that the following conditions are satisfied (see Figure 1):

1. For any  $i \in \{1, \dots, k\}$ , let  $\mathcal{T}_{D_i}$  be the subtree of  $\mathcal{T}$  rooted at  $D_i$  and let  $\mathcal{T}_D$  be the subtree of  $\mathcal{T}$  rooted at  $D$ . Then at least one of the following two conditions is satisfied:
  - 1.1.  $P$  covers  $\mathcal{T}_D$  and avoids  $\mathcal{T} \setminus \mathcal{T}_D$ .
  - 1.2. There exists  $j \in \{1, \dots, k\}$  such that for all  $l \leq j$ ,  $P$  covers  $\mathcal{T}_{D_l}$  and  $P$  avoids  $\mathcal{T}_D \setminus \bigcup_{m=1}^j \mathcal{T}_{D_m}$ .
2. Let  $\mathcal{T}'$  be a tree in  $\mathcal{F}$  with  $\mathcal{T}' \neq \mathcal{T}$ . Then either  $P$  covers  $\mathcal{T}'$  or  $P$  avoids  $\mathcal{T}'$ .

► **Definition 61** (Facial restriction). Let  $P \in \mathcal{P}$  be basic. Let  $\mathcal{W}'$  be the collection of walks obtained by restricting every  $W \in \mathcal{W}$  on  $H_P$ . If  $P$  is empty, we say that  $\mathcal{W}'$  is the *P-facial restriction* of  $\mathcal{W}$ . Suppose that  $P$  is not empty. Let  $\mathcal{T}, D, \mathcal{T}_D, k$  and  $j$  be as in Definition 60.



■ **Figure 2** Part of the collection of walks in  $\mathcal{W}$  (left) and the corresponding  $P$ -facial restriction of  $\mathcal{W}$  depicted in bold (right).

Let  $\mathcal{F}' = \{\mathcal{T}' \in \mathcal{F} : P \text{ covers } \mathcal{T}'\}$  and let  $\mathcal{R} = \bigcup_{\mathcal{T}' \in \mathcal{F}'} V(\mathcal{T}')$ . If  $P$  covers  $\mathcal{T}_D$  and avoids  $\mathcal{T} \setminus \mathcal{T}_D$ , we say that  $\mathcal{R} \cup \mathcal{W}' \cup V(\mathcal{T}_D)$  is the  $P$ -facial restriction of  $\mathcal{W}$ . Otherwise, we say that  $\mathcal{R} \cup \mathcal{W}' \cup \{D\} \cup \bigcup_{i=1}^j V(\mathcal{T}_{D_i})$  is the  $P$ -facial restriction of  $\mathcal{W}$ . Figure 2 depicts an example of a  $P$ -facial restriction.

► **Definition 62** (Important walk). Let  $P \in \mathcal{P}$  be a basic path. Let  $\mathcal{W}'$  be the  $P$ -facial restriction of  $\mathcal{W}$ . Let  $\mathcal{W}''$  be the shadow of  $\mathcal{W}'$ . Let  $Q$  be a walk in  $G[\mathcal{W}'']$ . We say that  $Q$  is  $P$ -important (w.r.to.  $\mathcal{W}$ ) if the following conditions hold:

1. Both endpoints of  $Q$  are in  $V(P)$ .
2.  $Q$  is the concatenation of walks  $Q_1, \dots, Q_\ell$  such that for each  $i \in \{1, \dots, \ell\}$  there exists some  $W_i \in \mathcal{W}$  such that  $Q_i$  is a sub-walk of  $W_i$ , and for each  $j \in \{1, \dots, \ell - 1\}$  we have  $\{W_j, W_{j+1}\} \in E(\mathcal{F})$ .

► **Proposition 63.** For any  $u, v \in V(P)$ , there is at most one  $P$ -important walk from  $u$  to  $v$ .

**Proof.** It follows immediately by the fact that  $\mathcal{F}$  is a forest. ◀

► **Lemma 64.** Let  $P \in \mathcal{P}$  be basic w.r.t.  $\mathcal{W}$  with endpoints  $u, v \in V(F)$ , where  $u = v$  if  $P$  is closed. Let  $\mathcal{W}_P$  be the  $P$ -facial restriction of  $\mathcal{W}$ . Let  $\Gamma = \bigcup_{\vec{W} \in \mathcal{W}_P} \vec{W}$ . Let  $C$  be the partition of  $B_u \cup B_v$  that corresponds to the weakly-connected components of  $\Gamma$ . For any  $x \in B_u \cup B_v$  let  $f^{\text{in}}(x) = \text{in-degree}_{\mathcal{W}_P}(x)$  and  $f^{\text{out}}(x) = \text{out-degree}_{\mathcal{W}_P}(x)$ . Then there exists some  $a \in \mathcal{A} \cup (\mathcal{A} \times \mathcal{A}) \cup \text{nil}$  and  $l, r, p \in (V(\vec{G}) \cup \text{nil})$  such that the dynamic programming table contains some partial solution  $\mathcal{S}$  at location  $(P, (C, f^{\text{in}}, f^{\text{out}}, a, l, r, p))$ , such that  $\text{cost}_{\vec{G}}(\mathcal{S}) \leq \text{cost}_{\vec{G}}(\mathcal{W}_P)$ .

**Proof.** Let us first assume that  $\mathcal{F}$  contains only one tree. We will deal with the more general case later on. First suppose that  $P$  is an empty basic path. Let  $P = x_1, x_2, \dots, x_m$ , where  $x_1 = u$  and  $x_m = v$ . We will prove the assertion by induction on  $m$ . For the base case, suppose that  $m = 2$ , and thus  $P$  contains only one edge. Let  $a = l = r = p = \text{nil}$ . In this case, a partial solution  $\mathcal{S}$  at location  $(P, (C, f^{\text{in}}, f^{\text{out}}, a, l, r, p))$  is computed in the initialization step of the dynamic programming table and clearly we have  $\text{cost}_{\vec{G}}(\mathcal{S}) \leq \text{cost}_{\vec{G}}(\mathcal{W}_P)$  and we are done. Now suppose that  $m > 2$  and we have proved the assertion for all  $m' < m$ . We first decompose  $P$  into two edge-disjoint paths  $P_1$  and  $P_2$ , such that  $V(P_1) \cap V(P_2) = w$  for some  $1 < j < m$  and  $w = x_j$ . For  $i \in \{1, 2\}$  let  $\mathcal{W}_{P_i}$  be the  $P_i$ -facial restriction of  $\vec{W}_{\text{OPT}}$  and let  $\Gamma_i = \bigcup_{\vec{W} \in \mathcal{W}_{P_i}} \vec{W}$ . Let  $C_1$  be the partition of  $B_u \cup B_w$  that corresponds to the weakly-connected components of  $\Gamma_1$  and let  $C_2$  be the partition of  $B_w \cup B_v$  that corresponds to the weakly-connected components

of  $\Gamma_2$ . For any  $x \in B_u \cup B_w$  let  $f_1^{\text{in}}(x) = \text{in-degree}_{\mathcal{W}_{P_1}}(x)$  and  $f_1^{\text{out}}(x) = \text{out-degree}_{\mathcal{W}_{P_1}}(x)$ . Also for any  $x \in B_w \cup B_v$  let  $f_2^{\text{in}}(x) = \text{in-degree}_{\mathcal{W}_{P_2}}(x)$  and  $f_2^{\text{out}}(x) = \text{out-degree}_{\mathcal{W}_{P_2}}(x)$ . By the induction hypothesis, there exists partial solutions  $\mathcal{S}_1$  and  $\mathcal{S}_2$  that are compatible with  $(P_1, (C_1, f_1^{\text{in}}, f_1^{\text{out}}, \text{nil}, \text{nil}, \text{nil}, \text{nil}))$  and  $(P_2, (C_2, f_2^{\text{in}}, f_2^{\text{out}}, \text{nil}, \text{nil}, \text{nil}, \text{nil}))$  respectively, and we have  $\text{cost}_{\bar{G}}(\mathcal{S}_1) \leq \text{cost}_{\bar{G}}(\mathcal{W}_{P_1})$ ,  $\text{cost}_{\bar{G}}(\mathcal{S}_2) \leq \text{cost}_{\bar{G}}(\mathcal{W}_{P_2})$ . The dynamic program will merge  $\mathcal{S}_1$  and  $\mathcal{S}_2$  to get the desired  $\mathcal{S}$ . Note that by the construction, for every  $x \in B_w$  we have  $f_1^{\text{in}}(x) = f_2^{\text{out}}(x)$  and  $f_2^{\text{in}}(x) = f_1^{\text{out}}(x)$ . Therefore, by the first merging phase, we can merge walks in  $\mathcal{S}_1$  and  $\mathcal{S}_2$ . Also, we let  $a = l = r = p = \text{nil}$  and we proceed the second phase of merging. Finally, by the construction and definition of  $P$ -facial restriction, (T5) holds and the merging process returns a partial solution  $\mathcal{S}$  compatible with  $(P, (C, f^{\text{in}}, f^{\text{out}}, a, l, r, p))$  with  $\text{cost}_{\bar{G}}(\mathcal{S}) \leq \text{cost}_{\bar{G}}(\mathcal{W}_P)$ , as desired.

Now suppose that  $P$  is not empty basic. Let  $\mathcal{T}$ ,  $D$ ,  $\mathcal{T}_D$ ,  $k$  and  $j$  be as in Definition 60. We will prove the assertion by induction on  $\mathcal{T}$ . For the base case, suppose that  $D$  is a leaf of  $\mathcal{T}$ . Suppose that  $P = x_1, x_2, \dots, x_m$  for some  $m > 0$ , where  $x_1 = u$  and  $x_m = v$ , and  $D$  is a (possibly closed) walk from  $x_i \in V(P)$  to  $x_j \in V(P)$ . We may assume w.l.o.g. that  $i \leq j$ . There are some possible cases here based on  $i$  and  $j$ . First suppose that  $i > 1$ ,  $j < m$  and  $j - i \geq 3$ . In this case, let  $P_1 = x_1, \dots, x_i$ ,  $P_2 = x_i, x_{i+1}$ ,  $P_3 = x_{i+1}, \dots, x_{j-1}$ ,  $P_4 = x_{j-1}, x_j$  and  $P_5 = x_j, \dots, x_m$ . Let  $P_6 = P_1 \cup P_2$ ,  $P_7 = P_6 \cup P_3$  and  $P_8 = P_7 \cup P_4$ . For  $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$  let  $\mathcal{W}_{P_i}$  be the  $P_i$ -facial restriction of  $\bar{W}_{\text{OPT}}$  and let  $\Gamma_i = \bigcup_{\bar{W} \in \mathcal{W}_{P_i}} \bar{W}$ . We also define  $C_i$ ,  $f_i^{\text{in}}$  and  $f_i^{\text{out}}$  as in the previous case. Note that  $P_1$ ,  $P_3$  and  $P_5$  are empty basic paths. We let  $a_1 = a_3 = a_5 = \text{nil}$ ,  $l_1 = l_3 = l_5 = \text{nil}$ ,  $r_1 = r_3 = r_5 = \text{nil}$ ,  $p_1 = p_3 = p_5 = \text{nil}$  and thus we can find partial solutions  $\mathcal{S}_1$ ,  $\mathcal{S}_3$  and  $\mathcal{S}_5$  compatible with  $(P_1, (C_1, f_1^{\text{in}}, f_1^{\text{out}}, a_1, l_1, r_1, p_1))$ ,  $(P_3, (C_3, f_3^{\text{in}}, f_3^{\text{out}}, a_3, l_3, r_3, p_3))$  and  $(P_5, (C_5, f_5^{\text{in}}, f_5^{\text{out}}, a_5, l_5, r_5, p_5))$  respectively. We also have  $\text{cost}_{\bar{G}}(\mathcal{S}_1) \leq \text{cost}_{\bar{G}}(\mathcal{W}_{P_1})$ ,  $\text{cost}_{\bar{G}}(\mathcal{S}_3) \leq \text{cost}_{\bar{G}}(\mathcal{W}_{P_3})$  and  $\text{cost}_{\bar{G}}(\mathcal{S}_5) \leq \text{cost}_{\bar{G}}(\mathcal{W}_{P_5})$ . Let  $a_2 = a_4 = a_6 = a_7 = a_8 = (x_i, x_j)$ . If  $D$  does not have a parent in  $\mathcal{T}$ , then we let  $l_2 = l_4 = l_6 = l_7 = l_8 = r_2 = r_4 = r_6 = r_7 = r_8 = p_2 = p_4 = p_6 = p_7 = p_8 \in V(D)$  to be an arbitrary vertex of  $D$ . Otherwise, suppose that  $D$  has a parent  $D'$  in  $\mathcal{T}$ . If  $D'$  does not have a parent in  $\mathcal{T}$ , then we let  $l_2 = l_4 = l_6 = l_7 = l_8 = r_2 = r_4 = r_6 = r_7 = r_8 = p_2 = p_4 = p_6 = p_7 = p_8 \in V(D) \cap V(D')$ . Otherwise, suppose that  $D'$  has a parent  $D''$  in  $\mathcal{T}$ . In this case, we let  $l_2 = l_4 = l_6 = l_7 = l_8 = r_2 = r_4 = r_6 = r_7 = r_8 \in V(D) \cap V(D')$  and  $p_2 = p_4 = p_6 = p_7 = p_8 \in V(D') \cap V(D'')$ . Therefore, by computing the initialization step, we can find a partial solution  $\mathcal{S}_2$  compatible with  $(P_2, (C_2, f_2^{\text{in}}, f_2^{\text{out}}, a_2, l_2, r_2, p_2))$  and a partial solution  $\mathcal{S}_4$  compatible with  $(P_4, (C_4, f_4^{\text{in}}, f_4^{\text{out}}, a_4, l_4, r_4, p_4))$ , and we have  $\text{cost}_{\bar{G}}(\mathcal{S}_2) \leq \text{cost}_{\bar{G}}(\mathcal{W}_{P_2})$  and  $\text{cost}_{\bar{G}}(\mathcal{S}_4) \leq \text{cost}_{\bar{G}}(\mathcal{W}_{P_4})$ . Now by merging  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , we get a partial solution  $\mathcal{S}_6$  compatible with  $(P_6, (C_6, f_6^{\text{in}}, f_6^{\text{out}}, a_6, l_6, r_6, p_6))$ . By merging  $\mathcal{S}_6$  and  $\mathcal{S}_3$ , we get a partial solution  $\mathcal{S}_7$  compatible with  $(P_7, (C_7, f_7^{\text{in}}, f_7^{\text{out}}, a_7, l_7, r_7, p_7))$ . By merging  $\mathcal{S}_7$  and  $\mathcal{S}_4$ , we get a partial solution  $\mathcal{S}_8$  compatible with  $(P_8, (C_8, f_8^{\text{in}}, f_8^{\text{out}}, a_8, l_8, r_8, p_8))$ , and finally by merging  $\mathcal{S}_8$  and  $\mathcal{S}_5$ , we get the desired partial solution  $\mathcal{S}$  compatible with  $(P, (C, f^{\text{in}}, f^{\text{out}}, a, l, r, p))$  with  $\text{cost}_{\bar{G}}(\mathcal{S}) \leq \text{cost}_{\bar{G}}(\mathcal{W}_P)$ . If  $i = 1$  or  $j = m$ , we will follow a similar approach. The only different is that instead of dividing  $P$  into five paths, we divide it into four paths. Finally, the last case is when  $j - i < 3$ . In this case, if  $i \neq j$ , we define the same subpaths  $P_1$ ,  $P_2$ ,  $P_4$  and  $P_5$ , and we follow a similar approach. Otherwise, suppose that  $i = j$ . In this case, we let  $P_1 = x_1, \dots, x_i$  and  $P_2 = x_i, \dots, x_m$  and by following the same approach by merging two partial solutions, we get the desired  $\mathcal{S}$ .

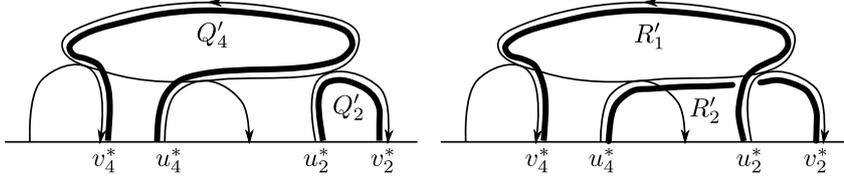
Now suppose that  $D \in V(\mathcal{T})$  is non-leaf. In this case, we prove the assertion by induction on  $j$ , where  $j$  comes from Definition 60. Note that we perform a second induction inside the first induction. For the base case, suppose that  $j = 1$ . In this case,  $D_1$  is a child of  $D$  and  $P$  covers  $\mathcal{T}_{D_1}$  and avoids  $\mathcal{T}_D \setminus \mathcal{T}_{D_1}$ . Therefore, by using the first induction hypothesis

on  $D_1$ , there exists  $a \in \mathcal{A} \cup (\mathcal{A} \times \mathcal{A}) \cup \text{nil}$  and  $l, r, p \in V(\vec{G}) \cup \text{nil}$  such that the dynamic programming table contains some partial solution  $\mathcal{S}$  at location  $(P, (C, f^{\text{in}}, f^{\text{out}}, a, l, r, p))$  with  $\text{cost}_{\vec{G}}(\mathcal{S}) \leq \text{cost}_{\vec{G}}(\mathcal{W}_P)$ . Now for the same  $a, l, r, p$  and  $\mathcal{S}$ , we have that  $\mathcal{S}$  is compatible with  $(P, (C, f^{\text{in}}, f^{\text{out}}, a, l, r, p))$ , as desired. Now suppose that we have proved the assertion for all  $1 \leq j' < j$ . By Definition 60, there exists a basic path  $P_1 \subseteq P$ , where  $u$  is the first vertex of  $P_1$ , such that for all  $l \leq j-1$ ,  $P_1$  covers  $\mathcal{T}_{D_l}$  and avoids  $\mathcal{T}_D \setminus (\bigcup_{m=1}^{j-1} \mathcal{T}_{D_m})$ . Also there exists a basic path  $P_2 \subseteq P$ , where  $v$  is the last vertex of  $P_2$ , such that  $P_2$  covers  $\mathcal{T}_{D_j}$  and avoids  $\mathcal{T} \setminus \mathcal{T}_{D_j}$ . Let  $u' \in V(F)$  and  $v' \in V(F)$  be the other endpoints of  $P_1$  and  $P_2$  respectively. Let  $P_3 \in \mathcal{P}$  be the path between  $u'$  and  $v'$  that does not contain  $v$  and let  $P_4 = P_1 \cup P_3$ . By the construction,  $P_3$  and  $P_4$  are basic. For  $i \in \{1, 2, 3, 4\}$ , let  $\mathcal{W}_{P_i}$  be the  $P_i$ -facial restriction of  $\vec{W}_{OPT}$  and let  $\Gamma_i = \bigcup_{\vec{W} \in \mathcal{W}_{P_i}} \vec{W}$ . Let  $C_1, C_2, C_3$  and  $C_4$  be the partitions of  $B_u \cup B_{u'}$ ,  $B_{v'} \cup B_v$ ,  $B_{u'} \cup B_{v'}$  and  $B_u \cup B_{v'}$  that corresponds to the weakly connected components of  $\Gamma_1, \Gamma_2, \Gamma_3$  and  $\Gamma_4$  respectively. For any  $x \in B_u \cup B_{u'}$  let  $f_1^{\text{in}}(x) = \text{in-degree}_{\mathcal{W}_{P_1}}(x)$  and  $f_1^{\text{out}}(x) = \text{out-degree}_{\mathcal{W}_{P_1}}(x)$ , for any  $x \in B_{v'} \cup B_v$  let  $f_2^{\text{in}}(x) = \text{in-degree}_{\mathcal{W}_{P_2}}(x)$  and  $f_2^{\text{out}}(x) = \text{out-degree}_{\mathcal{W}_{P_2}}(x)$ , for any  $x \in B_{u'} \cup B_{v'}$ , let  $f_3^{\text{in}}(x) = \text{in-degree}_{\mathcal{W}_{P_3}}(x)$  and  $f_3^{\text{out}}(x) = \text{out-degree}_{\mathcal{W}_{P_3}}(x)$ , and for any  $x \in B_u \cup B_{v'}$  let  $f_4^{\text{in}}(x) = \text{in-degree}_{\mathcal{W}_{P_4}}(x)$  and  $f_4^{\text{out}}(x) = \text{out-degree}_{\mathcal{W}_{P_4}}(x)$ . By the second induction hypothesis, there exists some  $a_1 \in \mathcal{A} \cup (\mathcal{A} \times \mathcal{A}) \cup \text{nil}$  and  $l_1, r_1, p_1 \in V(\vec{G}) \cup \text{nil}$ , such that the dynamic programming table contains some partial solution  $\mathcal{S}_1$  at location  $(P_1, (C_1, f_1^{\text{in}}, f_1^{\text{out}}, a_1, l_1, r_1, p_1))$ , with  $\text{cost}_{\vec{G}}(\mathcal{S}_1) \leq \text{cost}_{\vec{G}}(\mathcal{W}_{P_1})$ . Also by the first induction hypothesis, there exists some  $a_2 \in \mathcal{A} \cup (\mathcal{A} \times \mathcal{A}) \cup \text{nil}$  and  $l_2, r_2, p_2 \in V(\vec{G}) \cup \text{nil}$ , such that the dynamic programming table contains some partial solution  $\mathcal{S}_2$  at location  $(P_2, (C_2, f_2^{\text{in}}, f_2^{\text{out}}, a_2, l_2, r_2, p_2))$ , with  $\text{cost}_{\vec{G}}(\mathcal{S}_2) \leq \text{cost}_{\vec{G}}(\mathcal{W}_{P_2})$ . Let  $a_3 = l_3 = r_3 = p_3 = \text{nil}$ . Let  $a_4 = a_1, l_4 = l_1, r_4 = r_1$  and  $p_4 = p_1$ . Since  $P_3$  is basic, there exists a partial solution  $\mathcal{S}_3$  compatible with  $(P_3, (C_3, f_3^{\text{in}}, f_3^{\text{out}}, a_3, l_3, r_3, p_3))$ . Now we merge  $\mathcal{S}_1$  and  $\mathcal{S}_3$  to get a partial solution  $\mathcal{S}_4$  compatible with  $(P_4, (C_4, f_4^{\text{in}}, f_4^{\text{out}}, a_4, l_4, r_4, p_4))$ . Note that for every  $x \in B_{u'}$ , we have  $f_3^{\text{in}}(x) = f_1^{\text{in}}(x)$  and  $f_3^{\text{out}}(x) = f_1^{\text{out}}(x)$ . Therefore, we can apply the first merging phase. Also we have  $a_4 = a_1, l_4 = l_1, r_4 = r_1$  and  $p_4 = p_1$ , and thus we can apply the second merging phase. Finally, by the construction (T5) holds and we get a partial solution  $\mathcal{S}_4$  compatible with  $(P_4, (C_4, f_4^{\text{in}}, f_4^{\text{out}}, a_4, l_4, r_4, p_4))$ . Now, we merge two partial solutions  $\mathcal{S}_4$  and  $\mathcal{S}_2$  to get the desired  $\mathcal{S}$ . Clearly, for every  $x \in B_{v'}$  we have  $f_4^{\text{in}}(x) = f_2^{\text{out}}(x)$  and  $f_4^{\text{out}}(x) = f_2^{\text{in}}(x)$ . Therefore, we can apply the first phase of merging. If  $a_2 = l_2 = r_2 = p_2 = a_4 = l_4 = r_4 = p_4 = \text{nil}$ , then we let  $a = l = r = p = \text{nil}$ . Otherwise, if  $a_4 = l_4 = r_4 = p_4 = \text{nil}$  and  $a_2 = (u_2^*, v_2^*) \in \mathcal{A}$  with  $\{u_2^*, v_2^*\} \cap V(P_4) \subseteq \{u, v'\}$ , then we let  $a = a_2, l = l_2, r = r_2$  and  $p = p_2$ . If  $a_2 = l_2 = r_2 = p_2 = \text{nil}$  and  $a_4 = (u_4^*, v_4^*)$  with  $\{u_4^*, v_4^*\} \cap V(P_2) \subseteq \{v', v\}$ , then we let  $a = a_4, l = l_4, r = r_4$  and  $p = p_4$ . Otherwise, if  $a_2 \neq \text{nil}, a_4 \neq \text{nil}, a_4 = a_2, l_4 = l_2, r_4 = r_2$  and  $p_4 = p_2$ , then we let  $a = a_4, l = l_4, r = r_4$  and  $p = p_4$ . Otherwise, if  $a_2 = (u_2^*, v_2^*) \in \mathcal{A}, a_4 = (u_4^*, v_4^*) \in \mathcal{A}, l_4 = r_4$  and  $p_1 = p_2$ , then we let  $a = (u^*, v^*)$ , where  $u^* \in \{u_2^*, u_4^*\}$  and  $v^* \in \{v_2^*, v_4^*\}, l = l_2, r = r_4$  and  $p = p_2$ . Otherwise, if  $a_2 = \text{nil}$  and  $a_4 \in (\mathcal{A} \times \mathcal{A})$ , then we let  $a = a_4, l = l_4, r = r_4$  and  $p = p_4$ . Otherwise, if  $a_4 = \text{nil}$  and  $a_2 \in (\mathcal{A} \times \mathcal{A})$ , then we let  $a = a_2, l = l_2, r = r_2$  and  $p = p_2$ . Otherwise, if  $a_2 = (u_2^*, v_2^*) \in \mathcal{A}, a_4 = (u_4^*, v_4^*) \in \mathcal{A}, l_2 = r_2, l_4 = r_4$  and  $p_2 = p_4$ , then we let  $a = ((u', v'), (u'', v''))$  where  $v' \in \{v_2, v_4\}$  and  $u'' \in \{u_2, u_4\}, l = l_2, r = r_2$  and  $p = p_2$ . Otherwise, if  $a_4 = ((u_4, v_4), (u'_4, v'_4)) \in (\mathcal{A} \times \mathcal{A}), a_2 = (u_2, v_2) \in \mathcal{A}, l_2 = r_2$  and  $p_2 = p_4$ , then we let  $a = ((u_4, v_4), (u_2, v'_4)), l = l_4, r = r_2$  and  $p = p_2$ . Therefore, after applying the second merging phase, we get a partial solution  $\mathcal{S}$  compatible with  $(P, (C, f^{\text{in}}, f^{\text{out}}, a, l, r, p))$ .

Now we have to show that  $\text{cost}_{\vec{G}}(\mathcal{S}) \leq \text{cost}_{\vec{G}}(\mathcal{W}_P)$ . Let us first suppose that  $D$  is a closed walk. We will deal with the case where  $D$  is an open walk later on. If  $a_2 = l_2 = r_2 = p_2 = \text{nil}$  or  $a_4 = l_4 = r_4 = p_4 = \text{nil}$ , then this is immediate. If  $a_2 = a_4, l_2 = l_4, r_2 = r_4$  and  $p_2 = p_4$ ,

then we have  $a = a_2$ ,  $l = l_2$ ,  $r = r_2$  and  $p = p_2$ , and thus this case is also immediate. Suppose that  $a_2 = (u_2^*, v_2^*) \neq \text{nil}$ ,  $a_4 = (u_4^*, v_4^*) \neq \text{nil}$ ,  $l_4 = r_4$  and  $p_1 = p_2$ , and  $a = (u^*, v^*)$ , where  $u^* \in \{u_2^*, u_4^*\}$  and  $v^* \in \{v_2^*, v_4^*\}$ . We may assume w.l.o.g that  $a = (u_2^*, v_4^*)$ . We have that  $l = l_2$ ,  $r = r_4$  and  $p = p_2$ . For  $i \in \{2, 4\}$  let  $Q_i^*$  be the grip of  $\mathcal{S}_i$ , and let  $\vec{W}_i \in \mathcal{S}_i$  that contains  $Q_i^*$  as a sub-walk. Let  $Y_1$  be the shortest-path in  $\vec{G}$  from  $u_2^*$  to  $l_2$ . Let  $Y_2$  be the shortest-path in  $\vec{G}$  from  $l_2$  to  $l_4$ . Let  $Y_3$  be the shortest-path in  $\vec{G}$  from  $l_4$  to  $v_4^*$ . Let  $R_1^*$  be the path in  $\vec{G}$  from  $u_2^*$  to  $v_4^*$  obtained by concatenation of  $Y_1$ ,  $Y_2$  and  $Y_3$ . Let  $Y_1'$  be the shortest-path in  $\vec{G}$  from  $u_4^*$  to  $r_4$ . Let  $Y_2'$  be the shortest-path in  $\vec{G}$  from  $r_4$  to  $l_2$ . Let  $Y_3'$  be the shortest-path in  $\vec{G}$  from  $l_2$  to  $v_2^*$ . Let  $R_2^*$  be the path in  $\vec{G}$  from  $u_4^*$  to  $v_2^*$  obtained by concatenation of  $Y_1'$ ,  $Y_2'$  and  $Y_3'$ . Now by the construction, we have that  $\text{cost}_{\vec{G}}(\mathcal{S}) = \text{cost}_{\vec{G}}(\mathcal{S}_2) + \text{cost}_{\vec{G}}(\mathcal{S}_4) + \text{cost}_{\vec{G}}(R_1^*) + \text{cost}_{\vec{G}}(R_2^*) - \text{cost}_{\vec{G}}(Q_2^*) - \text{cost}_{\vec{G}}(Q_4^*)$ . Note that by the construction, there exists a  $P$ -important walk from  $u_2^*$  to  $v_4^*$  (w.r.t.  $\mathcal{W}$ ) and a  $P$ -important walk from  $u_4^*$  to  $v_2^*$  (w.r.t.  $\mathcal{W}$ ). By Proposition 63 these walks are unique. Let  $R_1'$  be the  $P$ -important walk from  $u_2^*$  to  $v_4^*$  (w.r.t.  $\mathcal{W}$ ) and let  $R_2'$  be the  $P$ -important walk from  $u_4^*$  to  $v_2^*$  (w.r.t.  $\mathcal{W}$ ). Also, there exists a  $P_2$ -important walk from  $u_2^*$  to  $v_2^*$  (w.r.t.  $\mathcal{W}$ ) and a  $P_4$ -important walk from  $u_4^*$  to  $v_4^*$  (w.r.t.  $\mathcal{W}$ ), and thus by Proposition 63 these walks are unique. Let  $Q_2'$  be the  $P_2$ -important walk from  $u_2^*$  to  $v_2^*$  (w.r.t.  $\mathcal{W}$ ) and let  $Q_4'$  be the  $P_4$ -important walk from  $u_4^*$  to  $v_4^*$  (w.r.t.  $\mathcal{W}$ ). By the definition of important walks, we have that  $\text{cost}_{\vec{G}}(R_1^*) \leq \text{cost}_{\vec{G}}(R_1')$ ,  $\text{cost}_{\vec{G}}(R_2^*) \leq \text{cost}_{\vec{G}}(R_2')$ ,  $\text{cost}_{\vec{G}}(Q_2^*) \leq \text{cost}_{\vec{G}}(Q_2')$  and  $\text{cost}_{\vec{G}}(Q_4^*) \leq \text{cost}_{\vec{G}}(Q_4')$ . Also by the construction, we have that  $\text{cost}_{\vec{G}}(R_1') + \text{cost}_{\vec{G}}(R_2') = \text{cost}_{\vec{G}}(Q_2') + \text{cost}_{\vec{G}}(Q_4')$ . Therefore, we have

$$\begin{aligned}
\text{cost}_{\vec{G}}(\mathcal{S}) &= \text{cost}_{\vec{G}}(\mathcal{S}_2) + \text{cost}_{\vec{G}}(\mathcal{S}_4) + \text{cost}_{\vec{G}}(R_1^*) + \text{cost}_{\vec{G}}(R_2^*) - \text{cost}_{\vec{G}}(Q_2^*) - \text{cost}_{\vec{G}}(Q_4^*) \\
&\leq \text{cost}_{\vec{G}}(\mathcal{S}_2) + \text{cost}_{\vec{G}}(\mathcal{S}_4) + \text{cost}_{\vec{G}}(R_1') + \text{cost}_{\vec{G}}(R_2') - \text{cost}_{\vec{G}}(Q_2^*) - \text{cost}_{\vec{G}}(Q_4^*) \\
&= (\text{cost}_{\vec{G}}(\mathcal{S}_2) - \text{cost}_{\vec{G}}(Q_4^*)) + (\text{cost}_{\vec{G}}(\mathcal{S}_4) - \text{cost}_{\vec{G}}(Q_2^*)) + \text{cost}_{\vec{G}}(R_1') + \text{cost}_{\vec{G}}(R_2') \\
&\leq \text{cost}_{\vec{G}}(\mathcal{W}_{P_1}) + \text{cost}_{\vec{G}}(\mathcal{W}_{P_2}) \\
&= \text{cost}_{\vec{G}}(\mathcal{W}_P).
\end{aligned}$$



Now suppose that  $D$  is an open walk. If  $a_2 = \text{nil}$  and  $a_4 \in (\mathcal{A} \times \mathcal{A})$ , or  $a_4 = \text{nil}$  and  $a_2 \in (\mathcal{A} \times \mathcal{A})$ , then this is immediate. If  $a_2 = (u_2^*, v_2^*) \in \mathcal{A}$ ,  $a_4 = (u_4^*, v_4^*) \in \mathcal{A}$ ,  $l_2 = r_2$ ,  $l_4 = r_4$  and  $p_2 = p_4$ , then we have  $a = ((u', v'), (u'', v''))$  where  $v' \in \{v_2^*, v_4^*\}$  and  $u'' \in \{u_2^*, u_4^*\}$ . We may assume w.l.o.g that  $v' = v_2^*$  and  $u'' = u_4^*$ . Then we have  $l = l_2$ ,  $r = r_2$  and  $p = p_2$ . Let  $R_1$  be the shortest-path in  $\vec{G}$  from  $u'$  to  $l_2$ . Let  $R_2$  be the shortest-path in  $\vec{G}$  from  $l_1$  to  $v'$ . Let  $R'$  be the path from  $u'$  to  $v'$  obtained by the concatenation of  $R_1$  and  $R_2$ . Let  $Y_1$  be the shortest path in  $\vec{G}$  from  $u''$  to  $l_4$ . Let  $Y_2$  be the shortest path in  $\vec{G}$  from  $l_4$  to  $v''$ . Let  $Y'$  be the path from  $u''$  to  $v''$  obtained by the concatenation of  $Y_1$  and  $Y_2$ . Let  $Z_1$  be the shortest path in  $\vec{G}$  from  $u_2^*$  to  $l_2$ . Let  $Z_2$  be the shortest path in  $\vec{G}$  from  $l_2$  to  $l_4$ . Let  $Z_3$  be the shortest path in  $\vec{G}$  from  $l_4$  to  $v_4^*$ . Let  $Z'$  be the path from  $u_2^*$  to  $v_4^*$  obtained by the concatenation of  $Z_1$ ,  $Z_2$  and  $Z_3$ . Let  $Q_2^*$  be the grip of  $\mathcal{S}_2$  and let  $Q_4^*$  be the grip of  $\mathcal{S}_4$ . By the construction, we have that  $\text{cost}_{\vec{G}}(\mathcal{S}) = \text{cost}_{\vec{G}}(\mathcal{S}_2) + \text{cost}_{\vec{G}}(\mathcal{S}_4) + \text{cost}_{\vec{G}}(R') + \text{cost}_{\vec{G}}(Y') + \text{cost}_{\vec{G}}(Z') - \text{cost}_{\vec{G}}(Q_2^*) - \text{cost}_{\vec{G}}(Q_4^*)$ . By the construction, there exists a  $P$ -important walk from  $u_2^*$  to  $v_4^*$ , a  $P$ -important walk from  $u'$  to  $v_2^*$ , and a  $P$ -important walk from  $u_4^*$  to  $v''$ . Therefore by Proposition 63, these

important walks are unique. Let  $R''$ ,  $Y''$  and  $Z''$  be the important walks from  $u'$  to  $v'$ ,  $u''$  to  $v''$ , and  $u_2^*$  to  $v_4^*$  respectively. Therefore, we have that

$$\begin{aligned}
 \text{cost}_{\vec{G}}(\mathcal{S}) &= \text{cost}_{\vec{G}}(\mathcal{S}_2) + \text{cost}_{\vec{G}}(\mathcal{S}_4) + \text{cost}_{\vec{G}}(R') + \text{cost}_{\vec{G}}(Y') + \text{cost}_{\vec{G}}(Z') \\
 &\quad - \text{cost}_{\vec{G}}(Q_2^*) - \text{cost}_{\vec{G}}(Q_4^*) \\
 &\leq \text{cost}_{\vec{G}}(\mathcal{S}_2) + \text{cost}_{\vec{G}}(\mathcal{S}_4) + \text{cost}_{\vec{G}}(R'') + \text{cost}_{\vec{G}}(Y'') + \text{cost}_{\vec{G}}(Z'') \\
 &\quad - \text{cost}_{\vec{G}}(Q_2^*) - \text{cost}_{\vec{G}}(Q_4^*) \\
 &\leq \text{cost}_{\vec{G}}(\mathcal{W}_{P_1}) + \text{cost}_{\vec{G}}(\mathcal{W}_{P_2}) = \text{cost}_{\vec{G}}(\mathcal{W}_P).
 \end{aligned}$$

If  $a_4 = ((u_4, v_4), (u'_4, v'_4)) \in (\mathcal{A} \times \mathcal{A})$ ,  $a_2 = (u_2, v_2) \in \mathcal{A}$ ,  $l_2 = r_2$  and  $p_2 = p_4$ , then we have  $a = ((u_4, v_4), (u_2, v'_4))$ ,  $l = l_4$ ,  $r = r_2$  and  $p = p_2$ . Let  $(Q_4^*, Q_4^{**})$  be the grip of  $\mathcal{S}_4$ , and let  $Q_2^*$  be the grip of  $\mathcal{S}_2$ . We may assume w.l.o.g that  $Q_4^*$  is a path from  $u_4$  to  $v_4$ , and  $Q_4^{**}$  is a path from  $u'_4$  to  $v'_4$ . Let  $Y_1$  be the shortest path in  $\vec{G}$  from  $u_2$  to  $l_2$ . Let  $Y_2$  be the shortest path in  $\vec{G}$  from  $l_2$  to  $v'_4$ . Let  $Y'$  be the path from  $u_2$  to  $v'_4$  obtained by the concatenation of  $Y_1$  and  $Y_2$ . Let  $Z_1$  be the shortest path in  $\vec{G}$  from  $u'_4$  to  $r_4$ . Let  $Z_2$  be the shortest path in  $\vec{G}$  from  $r_4$  to  $l_2$ . Let  $Z_3$  be the shortest path in  $\vec{G}$  from  $l_2$  to  $v_2$ . Let  $Z'$  be the path from  $u'_4$  to  $v_2$  obtained by the concatenation of  $Z_1$ ,  $Z_2$  and  $Z_3$ . By the construction, we have that  $\text{cost}_{\vec{G}}(\mathcal{S}) = \text{cost}_{\vec{G}}(\mathcal{S}_2) + \text{cost}_{\vec{G}}(\mathcal{S}_4) + \text{cost}_{\vec{G}}(Y') + \text{cost}_{\vec{G}}(Z') - \text{cost}_{\vec{G}}(Q_2^*) - \text{cost}_{\vec{G}}(Q_4^{**})$ . By the construction, there exists a  $P$ -important walk from  $u'_4$  to  $v_2$ , and a  $P$ -important walk from  $u_2$  to  $v'_4$ . Therefore by Proposition 63, these important walks are unique. Let  $Y''$  and  $Z''$  be the important walks from  $u_2$  to  $v'_4$ , and  $u'_4$  to  $v_2$  respectively. Therefore we have

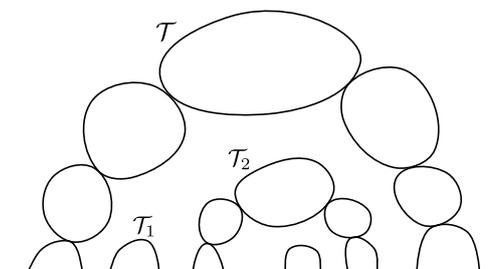
$$\begin{aligned}
 \text{cost}_{\vec{G}}(\mathcal{S}) &= \text{cost}_{\vec{G}}(\mathcal{S}_2) + \text{cost}_{\vec{G}}(\mathcal{S}_4) + \text{cost}_{\vec{G}}(Y') + \text{cost}_{\vec{G}}(Z') - \text{cost}_{\vec{G}}(Q_2^*) - \text{cost}_{\vec{G}}(Q_4^{**}) \\
 &\leq \text{cost}_{\vec{G}}(\mathcal{S}_2) + \text{cost}_{\vec{G}}(\mathcal{S}_4) + \text{cost}_{\vec{G}}(Y'') + \text{cost}_{\vec{G}}(Z'') - \text{cost}_{\vec{G}}(Q_2^*) - \text{cost}_{\vec{G}}(Q_4^{**}) \\
 &\leq \text{cost}_{\vec{G}}(\mathcal{W}_{P_1}) + \text{cost}_{\vec{G}}(\mathcal{W}_{P_2}) = \text{cost}_{\vec{G}}(\mathcal{W}_P).
 \end{aligned}$$

Now suppose that  $\mathcal{F}$  contains more than one tree. Let  $A = \{\mathcal{T}_1, \dots, \mathcal{T}_m\}$  be the set of all trees in  $\mathcal{F}$ . For every  $\mathcal{T} \in A$  we define the *level* of  $\mathcal{T}$ ,  $L(\mathcal{T})$ , as follows. Let  $D$  be the root of  $\mathcal{T}$ . We set  $L(\mathcal{T}) = 0$ , if there exists a basic path  $P'$  that covers  $\mathcal{T}$  and avoids all  $\mathcal{T}' \in \mathcal{F} \setminus \{\mathcal{T}\}$ . We call a minimal such path, a *corresponding* basic path for  $\mathcal{T}$  and we denote it by  $P_{\mathcal{T}}$ ; it is immediate that there is a unique such minimal path. Let  $\mathcal{F}_0 = \{\mathcal{T} \in \mathcal{F} : L(\mathcal{T}) = 0\}$ . Now for  $i \geq 0$ , suppose that we have defined trees of level  $i$  and  $\mathcal{F}_i$ . Suppose that  $L(\mathcal{T}) \notin \{0, \dots, i\}$ . We set  $L(\mathcal{T}) = i + 1$  if there exists a basic path  $P'$  that covers  $\mathcal{T}$  such that for all  $\mathcal{T}' \in \bigcup_{j=0}^i \mathcal{F}_j$ ,  $P'$  either avoids or covers  $\mathcal{T}'$ , and for all  $\mathcal{T}'' \in (\mathcal{F} \setminus \{\mathcal{T}\}) \setminus \bigcup_{j=0}^i \mathcal{F}_j$ ,  $P'$  avoids  $\mathcal{T}''$ . We also call a minimal such path corresponding basic for  $\mathcal{T}$  and we denote it by  $P_{\mathcal{T}}$ . Let  $\mathcal{F}_{i+1} = \{\mathcal{T} \in \mathcal{F} : L(\mathcal{T}) = i + 1\}$ .

We say that some  $\mathcal{T} \in \mathcal{F}$  is *outer-most* if there is no  $\mathcal{T}' \in \mathcal{F}$  with  $L(\mathcal{T}') > L(\mathcal{T})$  and  $P_{\mathcal{T}} \subset P_{\mathcal{T}'}$ .

Let us first suppose that there exists only one outer-most tree  $\mathcal{T} \in \mathcal{F}$ , such that  $P_{\mathcal{T}} \subseteq P$ . We will deal with the more general case later. Also, suppose that  $\mathcal{T} \in \mathcal{F}_m$  for some  $m \geq 0$ . We will prove the assertion by induction on  $m$ . We also prove that for this case, we have  $a = l = r = p = \text{nil}$ . For the base case, if  $m = 0$ , then by the construction,  $\mathcal{T}$  is the only tree with a corresponding basic path  $P_{\mathcal{T}} \subseteq P$ . For this case, we have already established the assertion and we are done. Now suppose that we have proved the assertion for all  $m' < m$ . Let  $\mathcal{F}' = \mathcal{F} \setminus \{\mathcal{T}\}$ . Let  $\mathcal{T}_1, \dots, \mathcal{T}_t \in \mathcal{F}'$  be all outer-most trees in  $\mathcal{F}'$ , such that for  $j \in \{1, \dots, t\}$  we have  $P_{\mathcal{T}_j} \subseteq P$ , and they intersect  $F$  in this order. By the construction, for every  $j \in \{1, \dots, t\}$  we have  $L(\mathcal{T}_j) < m$ . For every  $j \in \{1, \dots, t\}$  let  $P_{\mathcal{T}_j}$  be a corresponding basic path for  $\mathcal{T}_j$ . By the induction hypothesis, for every  $j \in \{1, \dots, t\}$  there exists a partial

solution  $\mathcal{S}_j$  for  $P_{\mathcal{T}_j}$ . Now, we can apply the same argument when we had only one tree  $\mathcal{T} \in \mathcal{F}$  for  $\mathcal{T}$ . Note that for each  $j \in \{1, \dots, t\}$ , we have  $a_j = l_j = r_j = p_j = \text{nil}$ . The only difference is that the intermediate basic paths here are not necessarily empty basic paths, and each  $\mathcal{T}_j$  appears as an intermediate basic path, with a partial solution  $\mathcal{S}_j$ . Therefore, by following a similar approach to the previous cases and merging appropriately we get a partial solution  $\mathcal{S}$ , as desired.



Now, suppose that there exist more than one outer-most trees  $\mathcal{T} \in \mathcal{F}$ , where  $P_{\mathcal{T}} \subseteq P$ . Let  $B = \{\mathcal{T}_1, \dots, \mathcal{T}_t\}$  be the set of all such trees, where they intersect consecutive subpaths of  $P$  in this order. In this case, we prove the assertion by induction on  $t$ . For the base case where  $t = 1$ , we have already proved the assertion. Now suppose that we have proved the assertion for all  $t' < t$ . Let  $B' = \{\mathcal{T}_1, \dots, \mathcal{T}_{t-1}\}$ . Let  $P' \subseteq P$  be the subpath of  $P$  such that  $B'$  is the set of all outer-most trees, with  $P_{\mathcal{T}_j} \subseteq P'$  for all  $1 \leq j \leq t-1$ . By the induction hypothesis, there exists a partial solution  $\mathcal{S}'$  for  $P'$ . Let  $P'' = P \setminus P'$ . By the construction,  $P''$  is a corresponding basic path for  $\mathcal{T}_t$ , and thus by the induction hypothesis, there exists a partial solution  $\mathcal{S}''$  for  $P''$ . Therefore, by merging  $\mathcal{S}'$  and  $\mathcal{S}''$  we get a partial solution  $\mathcal{S}$  for  $P$ , as desired.  $\blacktriangleleft$

► **Theorem 65.** *Let  $\vec{G}$  be an  $n$ -vertex  $(0, 0, 1, p)$ -nearly embeddable graph (that is, planar with a single vortex) and let  $\vec{H}$  be the vortex of  $\vec{G}$ . Then there exists an algorithm which computes a walk  $\vec{W}$  visiting all vertices in  $V(\vec{H})$  of total length  $\text{OPT}_{\vec{G}}(V(\vec{H}))$  in time  $n^{O(p)}$ .*

**Proof.** We have  $F \in \mathcal{P}$  and it is immediate to check that  $F$  is basic. Since  $F$  is a cycle, both the endpoints of  $F$  are some vertex  $v^\circ$ . It follows by Lemma 64 that there exists some  $\phi \in \mathcal{I}_F$  such that the dynamic programming table contains a partial solution  $\mathcal{S}$  at location  $(F, \phi)$ . Let  $\phi = (C, f^{\text{in}}, f^{\text{out}}, a, l, r, p)$ . It follows by condition (T4) that for all  $x \in B_{v^\circ}$  we have  $\text{in-degree}_{\mathcal{S}}(x) = \text{out-degree}_{\mathcal{S}}(x)$ . Thus by repeatedly merging pairs of walks that respectively terminate to and start from the same vertex, we obtain a collection  $\mathcal{S}'$  of closed walks with  $\text{cost}_{\vec{G}}(\mathcal{S}') = \text{cost}_{\vec{G}}(\mathcal{S})$  that visit all vertices in  $V(\vec{H})$ . By Lemma 57 we can assume that  $C$  is the trivial partition containing only one cluster, that is  $C = \{\{B_{v^\circ}\}\}$ . Since  $C$  is trivial, it follows by (T5) that all vertices in  $V(\vec{H})$  are in the same weakly-connected component of  $\bigcup_{W \in \mathcal{S}} W$ . Thus all vertices in  $V(\vec{H})$  are in the same strongly-connected component of  $\bigcup_{W \in \mathcal{S}'} W$ . Since all walks in  $\mathcal{S}'$  are closed, by repeatedly shortcutting pairs of intersecting walks, we obtain a walk  $\vec{W}$  with that visits all vertices in  $V(\vec{H})$  with  $\text{cost}_{\vec{G}}(\vec{W}) = \text{cost}_{\vec{G}}(\mathcal{S}') = \text{cost}_{\vec{G}}(\mathcal{S}) \leq \text{cost}_{\vec{G}}(\vec{W}_{\text{OPT}}) = \text{OPT}_{\vec{G}}$ .

The running time is polynomial in the size of the dynamic programming table, which is at most  $|\mathcal{P}| \cdot \max_{P \in \mathcal{P}} |\mathcal{I}_P| = O(n^2) \cdot p^{O(p)} \cdot n^{O(p)} \cdot O(n^2) = n^{O(p)}$ .  $\blacktriangleleft$

### 13 The dynamic program for traversing a vortex in a bounded genus graph

For the remainder of this section let  $\vec{G}$  be a  $n$ -vertex  $(0, g, 1, p)$ -nearly embeddable graph. Let  $\vec{H}$  be the vortex in  $\vec{G}$ , attached to some face  $\vec{F}$ . Let  $\vec{G}' = \vec{G} \setminus (V(\vec{H}) \setminus (\vec{F}))$  and fix some embedding  $\psi$  of  $\vec{G}'$  on a surface  $S$  of genus  $g$ . Let  $F$  be the symmetrization of  $\vec{F}$ . Let  $\vec{W}_{\text{OPT}}$  be a closed walk in  $\vec{G}$  that visits all vertices in  $\vec{H}$  with minimum  $\text{cost}_{\vec{G}}(\vec{W})$ . Fix a path-decomposition  $\{B_v\}_{v \in V(F)}$  of  $\vec{H}$  of width  $p$ . We present a similar algorithm as in Section 12 for computing a walk traversing all vertices in  $V(\vec{H})$  based on dynamic programming. By Lemma 54 we may assume w.l.o.g. that  $\vec{G}$  is facially normalized and cross normalized.

#### 13.1 The dynamic program

Let  $\mathcal{Q}$  be the set of all (possible closed) subpaths of  $F$ . For any integer  $m$ , let

$$\mathcal{P}_m = \{A \subseteq \mathcal{Q} : |A| \leq m, \text{ for every } Q, Q' \in A \text{ we have } V(Q) \cap V(Q') = \emptyset\}$$

and let

$$\mathcal{P}_\infty = \{A \subseteq \mathcal{Q} : \text{For every } Q, Q' \in A \text{ we have } V(Q) \cap V(Q') = \emptyset\}.$$

For every  $P = \{Q_1, \dots, Q_m\} \in \mathcal{P}_\infty$ , let  $E(P) = \bigcup_{i=1}^m E(Q_i)$  and let  $V(P) = \bigcup_{i=1}^m V(Q_i)$ . For each  $i \in \{1, \dots, m\}$ , let  $u_i$  and  $v_i$  be the endpoints of  $Q_i$ . Similar to the planar case, let  $\vec{H}_P = \vec{H} \left[ \bigcup_{x \in V(P)} B_x \right]$ . Let  $\mathcal{B} = \bigcup_{i=1}^m (B_{u_i} \cup B_{v_i})$ . Let  $\mathcal{C}_P$  be the set of all possible partitions of  $\mathcal{B}$ . Let  $\mathcal{D}_P^{\text{in}} = \{0, \dots, n\}^{\mathcal{B}}$ ,  $\mathcal{D}_P^{\text{out}} = \{0, \dots, n\}^{\mathcal{B}}$ , that is, every element of  $\mathcal{D}_P^{\text{in}} \cup \mathcal{D}_P^{\text{out}}$  is a function  $f : \mathcal{B} \rightarrow \{0, \dots, n\}$ .

##### 13.1.1 The dynamic programming table.

Let  $\mathcal{P} = \mathcal{P}_{324000g^4}$ . With these definitions, the dynamic programming table is indexed the exact same way as in the planar case. Also, a *partial solution* is a collection of walks in  $\vec{G}$ .

We say that a partial solution  $\mathcal{S}$  is *compatible* with  $(P, \phi)$  if the same conditions (T1)-(T5) as in the planar case are satisfied. The only difference is that instead of  $B_u \cup B_v$ , we have  $\mathcal{B}$ .

###### 13.1.1.1 Merging partial solutions

We follow a similar approach as in the planar case. Let  $P = \{Q_1, \dots, Q_m\}$ ,  $P_1 = \{Q'_1, \dots, Q'_{m'}\}$ ,  $P_2 = \{Q''_1, \dots, Q''_{m''}\} \in \mathcal{P}$  such that  $E(P_1) \neq \emptyset$ ,  $E(P_2) \neq \emptyset$ ,  $E(P_1) \cap E(P_2) = \emptyset$ , and  $E(P) = E(P_1) \cup E(P_2)$ . Let  $\phi = (C, f^{\text{in}}, f^{\text{out}}, a, l, r, p) \in \mathcal{I}_P$ ,  $\phi_1 = (C_1, f_1^{\text{in}}, f_1^{\text{out}}, a_1, l_1, r_1, p_1) \in \mathcal{I}_{P_1}$ ,  $\phi_2 = (C_2, f_2^{\text{in}}, f_2^{\text{out}}, a_2, l_2, r_2, p_2) \in \mathcal{I}_{P_2}$ .

Let  $\mathcal{S}_1$  and  $\mathcal{S}_2$  be partial solutions compatible with  $(P_1, \phi_1)$  and  $(P_2, \phi_2)$  respectively. Similar to the planar case, we compute a partial solution  $\mathcal{S}$  compatible with  $(P, \phi)$  as follows.

**Merging phase 1: Joining the walks.** For every  $w \in V(P_1) \cap V(P_2)$ , we check that for all  $x \in B_w$  we have  $f_1^{\text{in}}(x) = f_2^{\text{out}}(x)$  and  $f_2^{\text{in}}(x) = f_1^{\text{out}}(x)$ . If not then the merging procedure returns nil. Otherwise, for every  $w \in V(P_1) \cap V(P_2)$  and every  $x \in B_w$ , we follow the exact same approach as in the planar case.

**Merging phase 2: Updating the grip.** This phase is identical to the planar case.

**Merging phase 3: Checking connectivity.** Similar to the planar case, we check that condition (T5) holds for  $\mathcal{S}$  and we return nil if it does not.

### 13.1.2 Initializing the dynamic programming table.

For all  $P \in \mathcal{P}_1$  with  $|E(P)| \leq 1$ , we follow the same approach as in the planar case.

### 13.1.3 Updating the dynamic programming table.

For all  $P \in \mathcal{P}$  with  $|E(P)| > 1$ , and for all  $P_1, P_2 \in \mathcal{P}$  with  $E(P_1) \neq \emptyset$ ,  $E(P_2) \neq \emptyset$ ,  $E(P_1) \cap E(P_2) = \emptyset$  and  $E(P_1) \cup E(P_2) = E(P)$ , and for all  $\phi_1 \in \mathcal{I}_{P_1}$  and  $\phi_2 \in \mathcal{I}_{P_2}$  we proceed as follows. Suppose that for all  $P' \in \mathcal{P}$  with  $|E(P')| < |E(P)|$  and all  $\phi' \in \mathcal{I}_{P'}$ , we have computed the partial solutions in the dynamic programming table at  $(P', \phi')$ . Now similar to the planar case, if there exists partial solutions  $\mathcal{S}_1$  and  $\mathcal{S}_2$  at  $(P_1, \phi_1)$  and  $(P_2, \phi_2)$  respectively, we call the merging process to (possibly) get a partial solution  $\mathcal{S}$  at  $(P, \phi)$  for some  $\phi \in \mathcal{I}_P$ . Now similar to the planar case, if there is no partial solution at  $(P, \phi)$  then we store  $\mathcal{S}$  at  $(P, \phi)$ . Otherwise if there exists a partial solution  $\mathcal{S}'$  stored at  $(P, \phi)$  and  $\text{cost}_{\vec{G}}(\mathcal{S}) < \text{cost}_{\vec{G}}(\mathcal{S}')$  then we replace  $\mathcal{S}'$  with  $\mathcal{S}$ .

## 13.2 Analysis

Let  $\mathcal{W}$  be the collection of walks given by Lemma 56. Let  $\mathcal{F}$  be the forest given by Lemma 57. Let  $\mathcal{T}$  be a subtree of  $\mathcal{F}$ . We say that  $\mathcal{T}$  is *trivial* if  $\psi(F) \cup \bigcup_{D \in V(\mathcal{T})} \psi(D)$  is contractible. Otherwise, we say that  $\mathcal{T}$  is *non-trivial*.

Let  $Q \in \mathcal{Q}$ , and let  $\mathcal{T}$  be a subtree of some tree in  $\mathcal{F}$ . We define the terms  $Q$  *covers*  $\mathcal{T}$  and  $Q$  *avoids*  $\mathcal{T}$  the exact same way as in the planar case. Let  $P = \{Q_1, \dots, Q_m\} \in \mathcal{P}_\infty$ . We say that  $P$  *covers*  $\mathcal{T}$  if for all  $D \in V(\mathcal{T})$  we have  $V(D) \cap V(F) \subseteq V(Q_1 \cup \dots \cup Q_m)$ . We say that  $P$  *avoids*  $\mathcal{T}$  if for all  $Q_i \in P$  we have that  $Q_i$  avoids  $\mathcal{T}$ .

► **Definition 66** (Basic family of paths). Let  $P = \{Q_1, \dots, Q_m\} \in \mathcal{P}_\infty$ . For each  $i \in \{1, \dots, m\}$ , let  $u_i$  and  $v_i$  be the endpoints of  $Q_i$ . We say that  $P$  is *basic* (w.r.t.  $\mathcal{W}$ ) if either  $V(P) \setminus (\bigcup_{i=1}^m \{u_i, v_i\})$  does not intersect any of the walks in  $\mathcal{W}$  (in which case we call it *empty basic*) or there exists  $\mathcal{T} \in \mathcal{F}$  and  $D \in V(\mathcal{T})$ , with children  $D_1, \dots, D_k$ , intersecting  $D$  in this order along a traversal of  $D$ , such that the exact same conditions (1) & (2) as in Definition 60 hold, and  $|P|$  is minimal subject to the following:

3. If  $P$  covers  $\mathcal{T}_D$  and avoids  $\mathcal{T} \setminus \mathcal{T}_D$ , let  $\mathcal{T}[P] = \mathcal{T}_D$ , and otherwise let  $\mathcal{T}[P] = \bigcup_{i=1}^j \mathcal{T}_{D_i}$ . For every two disjoint subtrees  $\mathcal{T}_1$  and  $\mathcal{T}_2$  of  $\mathcal{T}[P]$ , the following holds. If there exists  $Q_i \in P$  such that  $Q_i$  covers  $\mathcal{T}_1 \cup \mathcal{T}_2$  and avoids  $\mathcal{T} \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)$ , then there exist edge-disjoint subpaths  $Q'_i, Q''_i$  of  $Q_i$  such that  $Q_i = Q'_i \cup Q''_i$ ,  $Q'_i$  covers  $\mathcal{T}_1$  and avoids  $\mathcal{T} \setminus \mathcal{T}_1$ , and  $Q''_i$  covers  $\mathcal{T}_2$  and avoids  $\mathcal{T} \setminus \mathcal{T}_2$  (see Figure 3 for an example).

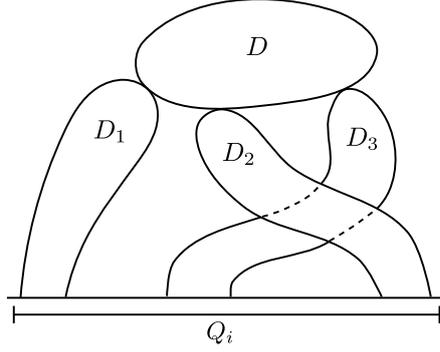
Moreover if for each non-trivial tree  $\mathcal{T}'$  in  $\mathcal{F}$  with  $\mathcal{T}' \neq \mathcal{T}$ , we have that  $P$  avoids  $\mathcal{T}'$ , then we say that  $P$  is *elementary*. Furthermore, if for each non-trivial tree  $\mathcal{T}'$  in  $\mathcal{F}$ , we have that  $P$  avoids  $\mathcal{T}'$ , then we say that  $P$  is *trivial*.

► **Definition 67** (Twins). Let  $P, P' \in \mathcal{P}_\infty$ . We say that  $P$  and  $P'$  are *twins* if for each subtree  $\mathcal{T}$  of  $\mathcal{F}$ ,  $P$  covers  $\mathcal{T}$  if and only if  $P'$  covers  $\mathcal{T}$ , and  $P$  avoids  $\mathcal{T}$  if and only if  $P'$  avoids  $\mathcal{T}$ .

► **Definition 68** (Succinctness). Let  $P = \{Q_1, \dots, Q_m\} \in \mathcal{P}_\infty$  be basic. For each  $i \in \{1, \dots, m\}$ , let  $u_i$  and  $v_i$  be the endpoints of  $Q_i$ , let  $Q'_i = Q_i \setminus \{u_i\}$  and let  $Q''_i = Q_i \setminus \{v_i\}$ . We say that  $P$  is *succinct* if for all  $i \in \{1, \dots, m\}$ ,  $P$  and  $\{Q_1, \dots, Q_{i-1}, Q'_i, Q_{i+1}, \dots, Q_m\}$  are not twins, and moreover  $P$  and  $\{Q_1, \dots, Q_{i-1}, Q''_i, Q_{i+1}, \dots, Q_m\}$  are not twins.

► **Lemma 69.** *Let  $P \in \mathcal{P}_\infty$  be non-empty basic. Then there exists some succinct and basic  $P' \in \mathcal{P}_\infty$  such that  $P$  and  $P'$  are twins with  $E(P') \subseteq E(P)$ .*

**Proof.** It is immediate by Definition 68. ◀



■ **Figure 3** Example of a non-basic family of paths.  $P = \{Q_i\}$  is not basic. Let  $\mathcal{T}_1 = D_1 \cup D_2$  and  $\mathcal{T}_2 = D_3$ . Then  $Q_i$  covers  $\mathcal{T}_1 \cup \mathcal{T}_2$  and avoids  $\mathcal{T} \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)$ , but there is no edge-disjoint subpaths  $Q'_i, Q''_i$  of  $Q_i$  satisfying the third condition.

► **Lemma 70.** *Let  $P \in \mathcal{P}_\infty$  be non-empty basic elementary and succinct. Let  $\mathcal{T}, D, j, D_1, \dots, D_j$  be as in Definition 66. Then there exist  $P_1, P_2 \in \mathcal{P}_\infty$  satisfying the following conditions:*

1.  $P_1$  and  $P_2$  are non-empty basic elementary and succinct.
2.  $P_1$  covers  $\bigcup_{i=1}^{j-1} \mathcal{T}_{D_i}$  and avoids  $\mathcal{T} \setminus (\bigcup_{i=1}^{j-1} \mathcal{T}_{D_i})$ .
3.  $P_2$  covers  $\mathcal{T}_{D_j}$  and avoids  $\mathcal{T} \setminus \mathcal{T}_{D_j}$ .
4.  $E(P_1) \subseteq E(P)$ ,  $E(P_2) \subseteq E(P)$ , and  $E(P_1) \cap E(P_2) = \emptyset$ .

**Proof.** We begin by defining auxiliary  $Z_1, Z_2 \in \mathcal{P}_\infty$ . The desired  $P_1$  and  $P_2$  will be succinct twins of  $Z_1$  and  $Z_2$ . First we define  $Z_1$ . Initially, we set  $Z_1 = P$  and we inductively modify  $Z_1$  until it covers  $C_1 = \bigcup_{i=1}^{j-1} \mathcal{T}_{D_i}$  and avoids  $A_1 = \mathcal{T} \setminus (\bigcup_{i=1}^{j-1} \mathcal{T}_{D_i})$  as follows: If  $Z_1$  contains a path  $Q$  that does not intersect  $C_1$  then we remove  $Q$  from  $Z_1$ . If  $Z_1$  contains a path  $Q$  that intersects both  $C_1$  and  $A_1$  then we proceed as follows: let  $R$  be the collection of paths obtained from  $Q$  by deleting all vertices in  $A_1 \cap Q$ ; let  $R'$  be the collection obtained from  $R$  by removing all paths that do not intersect  $C_1$ ; let  $R''$  be the collection obtained from  $R'$  by replacing each  $Q' \in R'$  by the minimal subpath  $Q'' \subseteq Q'$  with  $V(Q'') \cap C_1 = V(Q') \cap C_1$ . We repeat the above process until the resulting  $Z_1$  covers  $C_1 = \bigcup_{i=1}^{j-1} \mathcal{T}_{D_i}$  and avoids  $A_1 = \mathcal{T} \setminus (\bigcup_{i=1}^{j-1} \mathcal{T}_{D_i})$ . In a similar fashion we define  $Z_2$  that covers  $C_2 = \mathcal{T}_{D_j}$  and avoids  $A_2 = \mathcal{T} \setminus \mathcal{T}_{D_j}$ . It is immediate by construction that  $E(Z_1) \subseteq E(P)$ ,  $E(Z_2) \subseteq E(P)$  and  $E(Z_1) \cap E(Z_2) = \emptyset$ .

Next we argue that  $Z_1$  is basic. It is immediate that conditions (1) & (2) of Definition 66 are satisfied. It remains to establish condition (3) of Definition 66. Let  $Q \in P_1$ . By construction there exists  $Q' \in P$  such that  $Q \subseteq Q'$ . Let  $\mathcal{T}[Z_1]$  and  $\mathcal{T}[P]$  be as in Definition 66. Let  $\mathcal{T}_1$  and  $\mathcal{T}_2$  be disjoint subtrees of  $\mathcal{T}[Z_1]$  such that  $Q$  covers  $\mathcal{T}_1 \cup \mathcal{T}_2$  and avoids  $\mathcal{T} \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)$ . Let  $\mathcal{T}_{Q'}$  be the minimal subtree of  $\mathcal{T}[P]$  that contains all the nodes of  $\mathcal{T}[P]$  that are covered by  $Q'$ . Let  $\mathcal{T}' = \mathcal{T}_{Q'} \cup \mathcal{T}_1 \cup \mathcal{T}_2$ . By definition we have that  $\mathcal{T}'$  is a subtree of  $\mathcal{T}[P]$ . Therefore there exist disjoint subtrees  $\mathcal{T}'_1$  and  $\mathcal{T}'_2$  of  $\mathcal{T}'$  such that  $\mathcal{T}_1 \subseteq \mathcal{T}'_1$ ,  $\mathcal{T}_2 \subseteq \mathcal{T}'_2$ , and  $\mathcal{T}' = \mathcal{T}'_1 \cup \mathcal{T}'_2$ . Since  $P$  is basic, it follows by condition (3) of Definition 66 that there exist edge-disjoint subpaths  $Q'_1, Q'_2$  of  $Q'$  such that  $Q'_1$  covers  $\mathcal{T}'_1$  and avoids  $\mathcal{T} \setminus \mathcal{T}'_1$  and  $Q'_2$  covers  $\mathcal{T}'_2$  and avoids  $\mathcal{T} \setminus \mathcal{T}'_2$ . Let  $Q_1 = Q \cap Q'_1$  and  $Q_2 = Q \cap Q'_2$ . It now follows that  $Q_1$  covers  $\mathcal{T}_1$  and avoids  $\mathcal{T} \setminus \mathcal{T}_1$  and  $Q_2$  covers  $\mathcal{T}_2$  and avoids  $\mathcal{T} \setminus \mathcal{T}_2$ , establishing Condition (3) of Definition 66.

It remains to show that  $|P_1|$  is minimal subject to condition (3) of Definition 66. Suppose not. Then there are  $Q, Q' \in P_1$  that are consecutive in  $F$  such that we can replace  $Q$  and  $Q'$  in  $P_1$  by some subpath  $Q''$  that contains  $Q$  and  $Q'$ . If  $Q$  and  $Q'$  are subpaths of the same path in  $P$  then this is a contradiction because by construction there must exist a vertex in

$V(Q'') \setminus (V(Q) \cup V(Q'))$  that  $P_1$  must avoid. Therefore there must exist distinct  $R, R' \in P$  such that  $Q \subseteq R$  and  $Q' \subseteq R'$ . However this means that we can replace  $R$  and  $R'$  in  $P$  by some subpath containing both  $R$  and  $R'$ , contradicting the fact that  $P$  is basic. This establishes that  $|P_1|$  is minimal subject to condition (3) of Definition 66. We have thus obtained that  $Z_1$  is basic. It is also immediate by construction that since  $P$  is elementary,  $Z_1$  is also elementary. By the exact same argument it follows that  $Z_2$  is also basic and elementary.

For any  $i \in \{1, 2\}$  let  $P_i$  be a succinct twin of  $Z_i$  obtained by Lemma 69. Since  $Z_i$  is basic and elementary, it follows that  $P_i$  is also basic and elementary, and thus condition (1) is satisfied. Since  $P_i$  is a twin of  $Z_i$ , it follows that conditions (2) & (3) are satisfied. Since  $E(Z_i) \subset E(P)$  and  $E(P_i) \subseteq E(Z_i)$ , we get  $E(P_i) \subset E(P)$ . Finally, since  $E(Z_1) \cap E(Z_2) = \emptyset$ , we have  $E(P_1) \cap E(P_2) = \emptyset$ , and thus condition (4) is satisfied, which concludes the proof. ◀

► **Definition 71** (*P*-facial restriction). Let  $P = \{Q_1, \dots, Q_m\} \in \mathcal{P}_\infty$  be basic. We define the *P*-facial restriction of  $\mathcal{W}$  the exact same way as in Definition 61. We remark that  $P$  is now a family of paths, while in the planar case  $P$  is a single path; the definition remains the same by replacing the notion of basic path given in Definition 60 by the notion of basic family of paths given in Definition 66.

► **Lemma 72** (Malnič and Mohar [21]). *Let  $S$  be an either orientable or non-orientable surface of Euler genus  $g$ , and let  $x \in S$ . Let  $\mathcal{X}$  be a collection of noncontractible curves. Suppose that at least one of the following holds:*

- (i) *The curves in  $\mathcal{X}$  are disjoint and (freely) nonhomotopic.*
- (ii) *There exist  $x \in S$  such that for every  $C, C' \in \mathcal{X}$ , we have  $C \cap C' = x$ , and the curves in  $\mathcal{X}$  are nonhomotopic (in  $\pi_1(S, x)$ ).*

Then,

$$|\mathcal{X}| \leq \begin{cases} 0 & \text{if } S \text{ is the 2-sphere} \\ 1 & \text{if } S \text{ is the torus or the projective plane} \\ 3(g-1) & \text{otherwise} \end{cases}$$

We recall the following result on the genus of the complete bipartite graph [14].

► **Lemma 73.** *For any  $n, m \geq 1$ , the Euler genus of  $K_{m,n}$  is  $\lceil (m-2)(n-2)/4 \rceil$ .*

► **Lemma 74.** *Let  $P \in \mathcal{P}_\infty$  be elementary and succinct. Then  $|P| \leq 18000g^3$ .*

**Proof.** Let  $\mathcal{T}[P]$  be as in Definition 66. If  $\mathcal{T}[P]$  is trivial, we can find  $Q \in \mathcal{Q}$  such that  $Q$  covers  $\mathcal{T}[P]$  and avoids  $\mathcal{T} \setminus \mathcal{T}[P]$ , and thus  $|P| = 1$  and we are done. Now suppose that  $\mathcal{T}[P]$  is non-trivial. Let  $m = |P|$  and suppose that  $P = \{Q_1, \dots, Q_m\}$  such that  $Q_1, \dots, Q_m$  are subpaths of  $F$  in this order along a traversal of  $F$ . Let  $Z_1, \dots, Z_{m'}$  be a sequence of disjoint subsets of  $P$  such that  $P = \bigcup_{i=1}^{m'} Z_i$ , and each  $Z_i$  is maximal subject to the following condition: Let  $Z'_i$  be the minimal subpath of  $F$  that contains all the paths in  $Z_i$  and does not intersect any other paths in  $P$ ; then  $Z'_i$  avoids all non-trivial tress in  $\mathcal{F} \setminus \mathcal{T}$ . For every  $j \in \{1, \dots, m' - 1\}$ , let  $P_j$  be the subpath between  $Z'_j$  and  $Z'_{j+1}$  and let  $\mathcal{P}' = \{P_1, \dots, P_{m-1}\}$ . Note that since  $P$  is basic, for every two consecutive  $Z'_j$  and  $Z'_{j+1}$ , there exists a non-trivial tree  $\mathcal{T}_j$  such that  $\mathcal{T}_j$  intersects  $P_j$ .

We construct a set  $R$  of non-trivial trees as follows. We initially set  $R = \{\mathcal{T}_1\}$ . In each step  $i > 1$ , if there exists  $\mathcal{T}' \in R$  such that  $\mathcal{T}'$  intersects  $P_i$ , we continue to the next step. Otherwise, we let  $\mathcal{T}'_i$  be some non-trivial tree that intersects  $P_i$  and we add  $\mathcal{T}'_i$  to  $R$  and we continue to the next step. We argue that  $|R| \leq 20g$ . Suppose not. For each  $\mathcal{T}' \in R$  where

$\mathcal{T}'$  intersects some  $P' \in \mathcal{P}'$  at some  $x' \in V(P')$ , we construct a path  $\gamma_{\mathcal{T}'}$  in the surface, with both endpoints on  $\psi(F)$ , and such that after contracting  $\psi(F)$  into single point, the loop resulting from  $\gamma_{\mathcal{T}'}$  is non-contractible, as follows. First suppose that  $\psi(\mathcal{T}')$  contains some non-contractible loop  $\gamma$ . Then let  $\zeta$  be a path in  $\psi(\mathcal{T}')$  between  $x'$  and some point in  $\gamma$ . We set  $\gamma_{\mathcal{T}'}$  to be the path starting at  $x'$ , traversing  $\zeta$ , followed by  $\gamma$ , followed by the reversal of  $\zeta$ , and terminating at  $x'$ . Otherwise, suppose that  $\psi(\mathcal{T}')$  does not contain any non-contractible loops. Since  $\mathcal{T}'$  is non-trivial, it follows that  $\psi(\mathcal{T}')$  contains some path  $\xi$  with both endpoints in  $\psi(F)$  such that after contracting  $\psi(F)$  into a single point, the loop obtained from  $\xi$  is non-contractible. Let  $\xi'$  be some path in  $\psi(\mathcal{T}')$  between  $x'$  and some point in  $\xi$ . By Thomassen's 3-path condition [22] it follows that  $\xi \cup \xi'$  contains some path  $\xi''$  with both endpoints in  $\psi(F)$ , such that one of these endpoints is  $x'$ , and such that after contracting  $\psi(F)$  into a single point, the loop resulting from  $\xi''$  is non-contractible. We set  $\gamma_{\mathcal{T}'}$  to be  $\xi''$ .

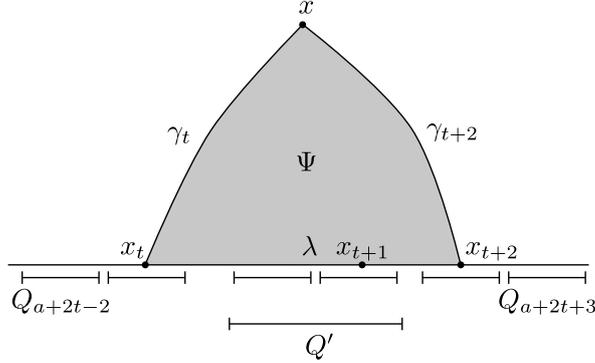
Let  $\mathcal{L}$  be the set of all loops obtained from the paths  $\gamma_{\mathcal{T}'}$  as follows. Pick a point  $x$  in the interior of the disk bounded by  $\psi(F)$ . Connect  $x$  to both endpoints of each  $\gamma_{\mathcal{T}'}$  by paths such that all chosen paths are interior disjoint. During this process each path  $\gamma_{\mathcal{T}'}$  gives rise to a non-contractible loop in  $\mathcal{L}$  such that any two loops in  $\mathcal{L}$  intersect only at  $x$ . Let  $L', L'', L''' \in \mathcal{L}$  be distinct. We show that  $L', L''$  and  $L'''$  can not be all homotopic. Suppose not. Since they are interior-disjoint, by removing them from the surface we obtain three connected components. Since  $\psi(\mathcal{T})$  does not intersect any of  $L', L''$  and  $L'''$ , it has to be inside one of the three connected components completely. We may assume w.l.o.g that  $\mathcal{T}$  is inside the component which is bounded by  $L'$  and  $L''$ . Therefore, there is no path from  $T$  to  $L'''$  without crossing  $L' \cup L''$ , which is a contradiction.

Let  $\mathcal{L}' \subseteq \mathcal{L}$  be a maximal subset such that for all  $L', L'' \in \mathcal{L}'$  we have that  $L'$  and  $L''$  are non-homotopic. Since  $|\mathcal{L}| > 20g$  and for every three loops in  $\mathcal{L}'$  we know that at most two of them are homotopic, we have that  $|\mathcal{L}'| > 10g$ , which contradicts Lemma 72. Therefore, we have that  $|R| \leq 20g$  and thus there exists  $\mathcal{T}_0 \in R$  such that  $\mathcal{T}_0$  intersects at least  $10g = 200g^2/20g$  elements of  $\mathcal{P}'$ . Let  $x_1$  be a point inside the face. Let  $x_2$  be a point in the root of  $\mathcal{T}_0$  and let  $x_3$  be a point in  $\psi(D)$ . There exists  $P'_1, \dots, P'_{10g} \in \mathcal{P}'$  such that for every  $i \in \{1, \dots, 10g\}$ ,  $\mathcal{T}_0$  intersects  $P'_i$ . For every  $i \in \{1, \dots, 10g\}$ , let  $y_i$  be a point on  $P'_i$ . By the construction, for every  $y_i$  we can find non-crossing paths to  $x_1, x_2$  and  $x_3$ . Therefore, we get an embedding of  $K_{3,10g}$  in a surface of genus  $g$  which contradicts Lemma 73. This establishes that  $m' \leq 200g^2$ .

Let  $i \in \{1, \dots, m'\}$ . We next we bound  $|Z_i|$ . Suppose  $Z_i = \{Q_a, Q_{a+1}, \dots, Q_{a+\ell}\}$ . Let  $x$  be an arbitrary point in  $\psi(D)$ . For each  $j \in \{0, \dots, \lfloor (\ell-1)/2 \rfloor\}$  pick an arbitrary point  $x_j \in \psi(Q_{a+2j}) \cap \psi(\mathcal{T})$ ; we define a path in the surface between  $x_j$  and  $x$  as follows: we start from the vertex  $D'$  of  $\mathcal{T}$  containing  $x_j$ ; if  $D'$  is a closed walk then we traverse  $D'$  clockwise until we reach the point that connects  $D'$  to its parent; otherwise we traverse the unique path between  $x_j$  and the point that connects  $D'$  to its parent. We continue in this fashion until we reach  $D$ , and we finally traverse  $D$  clockwise until we reach  $x$ . This completes the definition of the path  $\gamma_j$ . It is immediate that for all  $j \neq j'$ , the paths  $\gamma_j$  and  $\gamma_{j'}$  are non-crossing.

Let  $S'$  be the surface obtained by contracting  $\psi(F)$  into a single point  $y$ , and identifying  $x$  with  $y$  (note that  $S'$  has Euler genus at most  $g+2$ ). For each  $j \in \{0, \dots, \lfloor (\ell-1)/2 \rfloor\}$  let  $\gamma'_j$  be the loop in  $S'$  obtained from  $\gamma_j$  after the above contraction and identification. We argue that there can be at most 4 loops  $\gamma'_j$  that are pairwise homotopic. Suppose for the sake of contradiction that there are at least 5 such loops. It follows that there must exist  $t \in \{0, \dots, \lfloor (\ell-1)/2 \rfloor\}$  such that  $\gamma'_t, \gamma'_{t+1}$ , and  $\gamma'_{t+2}$  are all pairwise homotopic. We are going to obtain a contradiction by arguing that the paths  $Q_{a+2t}$  and  $Q_{a+2t+1}$  violate

the fact that  $P$  is basic. To that end, let  $Q'$  be the minimal subpath of  $F$  that contains both  $Q_{a+2t}$  and  $Q_{a+2t+1}$  and does not intersect any other paths in  $P$ . We will show that  $(P \setminus \{Q_{a+2t}, Q_{a+2t+1}\}) \cup \{Q'\}$  is basic, thus violating the fact that  $|P|$  is minimal subject to condition (3) of definition 66. Let  $\mathcal{T}_1, \mathcal{T}_2$  be disjoint subtrees of  $\mathcal{T}[P]$  such that  $Q'$  covers  $\mathcal{T}_1 \cup \mathcal{T}_2$  and avoids  $\mathcal{T} \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)$ . We need to show that there exist edge-disjoint subpaths  $Q'_1$  and  $Q'_2$  of  $Q'$  such that  $Q' = Q'_1 \cup Q'_2$ ,  $Q'_1$  covers  $\mathcal{T}_1$  and avoids  $\mathcal{T} \setminus \mathcal{T}_1$ , and  $Q'_2$  covers  $\mathcal{T}_2$  and avoids  $\mathcal{T} \setminus \mathcal{T}_2$ . Let  $\lambda$  be the subpath of  $\psi(F)$  in  $S$  between  $x_t$  and  $x_{t+1}$  that contains  $x_{t+1}$ . Since  $\gamma'_t, \gamma'_{t+1}$ , and  $\gamma'_{t+2}$  are homotopic, it follows that  $\gamma_t \cup \lambda \cup \gamma_{t+2}$  bounds a disk  $\Psi$  in  $S$ . Each  $x_s$  is contained in the image of a unique vertex  $D'_s$  of  $\mathcal{T}[P]$ . Let  $B$  be the path in  $\mathcal{T}[P]$  between  $D'_t$  and  $D'_{t+2}$ . For each  $r \in \{1, 2\}$ ,  $\mathcal{T}_r$  intersects  $B$  into some possibly empty subpath  $B_r$ . It follows that there exist disjoint disks  $\Psi_1, \Psi_2 \subset \Psi$  such that for each  $r \in \{1, 2\}$ ,  $\psi(\mathcal{T}_r) \cap \Psi \subset \Psi_r$ . Therefore there exist edge-disjoint subpaths  $Q'_1$  and  $Q'_2$  of  $Q'$  with  $Q' = Q'_1 \cup Q'_2$  such that  $\psi(Q') \cap \Psi_1 \subseteq \psi(Q'_1)$  and  $\psi(Q') \cap \Psi_2 \subseteq \psi(Q'_2)$ . It follows that  $Q'_1$  covers  $\mathcal{T}_1$  and avoids  $\mathcal{T} \setminus \mathcal{T}_1$ , and  $Q'_2$  covers  $\mathcal{T}_2$  and avoids  $\mathcal{T} \setminus \mathcal{T}_2$ . This contradicts the fact that  $P$  is basic, and concludes the proof that there are at most four loop  $\gamma'_t$  that are pairwise homotopic.



Pick  $I \subseteq \{0, \dots, \lfloor (\ell - 1)/2 \rfloor\}$ , with  $|I| \geq \lfloor (\ell - 1)/10 \rfloor$  such that for all  $t \neq t' \in I$ , we have that  $\gamma'_t$  and  $\gamma'_{t'}$  are non-homotopic. Since the paths  $\gamma_t$  are non-crossing, we may assume w.l.o.g. that the paths  $\gamma'_t$  are interior disjoint after an infinitesimal perturbation. By Lemma 72 on  $S'$  it follows that  $|I| \leq 3(g + 1)$ . Since  $|I| \geq \ell/15$ , we get  $\ell \leq 45(g + 1)$ . Therefore  $|Z_i| \leq 45(g + 1)$ .

We conclude that  $m \leq \sum_{i=1}^{m'} |Z_i| \leq m' \cdot \max_{i \in \{1, \dots, m'\}} |Z_i| \leq 9000g^2(g + 1) \leq 18000g^3$ . ◀

► **Lemma 75.** Let  $P_1 = \{Q_1, \dots, Q_m\} \in \mathcal{P}$  be succinct. Let  $P_2 = \{Q'_1, \dots, Q'_l\} \in \mathcal{P}$  such that  $P_1$  and  $P_2$  are twins,  $V(P_1) \subseteq V(P_2)$  and  $E(P_1) \subseteq E(P_2)$ . For every  $i \in \{1, \dots, m\}$ , let  $u_i$  and  $v_i$  be the endpoints of  $Q_i$ . Let  $\mathcal{B}_1 = \bigcup_{i=1}^m (B_{u_i} \cup B_{v_i})$ . Let  $\mathcal{W}_{P_1}$  be the  $P_1$ -facial restriction of  $\mathcal{W}$ . Let  $\Gamma_1 = \bigcup_{\vec{W} \in \mathcal{W}_{P_1}} \vec{W}$ . Let  $C_1$  be the partition of  $\mathcal{B}_1$  that corresponds to the weakly-connected components of  $\Gamma_1$ . For any  $x \in \mathcal{B}_1$  let  $f_1^{\text{in}}(x) = \text{in-degree}_{\mathcal{W}_{P_1}}(x)$  and  $f_1^{\text{out}}(x) = \text{out-degree}_{\mathcal{W}_{P_1}}(x)$ . We similarly define  $C_2, f_2^{\text{in}}, f_2^{\text{out}}$  and  $\mathcal{W}_{P_2}$  for  $P_2$ . Suppose that there exists some  $a_1 \in \mathcal{A} \cup (\mathcal{A} \times \mathcal{A}) \cup \text{nil}$  and  $l_1, r_1, p_1 \in (V(\vec{G}) \cup \text{nil})$  such that the dynamic programming table contains some partial solution  $\mathcal{S}_1$  at location  $(P_1, (C_1, f_1^{\text{in}}, f_1^{\text{out}}, a_1, l_1, r_1, p_1))$ , with  $\text{cost}_{\vec{G}}(\mathcal{S}_1) \leq \text{cost}_{\vec{G}}(\mathcal{W}_{P_1})$ . Then there exists some  $a_2 \in \mathcal{A} \cup (\mathcal{A} \times \mathcal{A}) \cup \text{nil}$  and  $l_2, r_2, p_2 \in (V(\vec{G}) \cup \text{nil})$  such that the dynamic programming table contains some partial solution  $\mathcal{S}_2$  at location  $(P_2, (C_2, f_2^{\text{in}}, f_2^{\text{out}}, a_2, l_2, r_2, p_2))$ , with  $\text{cost}_{\vec{G}}(\mathcal{S}_2) \leq \text{cost}_{\vec{G}}(\mathcal{W}_{P_2})$ .

**Proof.** Let  $E_1 = E(P_2) \setminus E(P_1)$ . For every  $e \in E_1$ , let  $Q_e \in \mathcal{Q}$  be the path containing a single edge  $e$ , and let  $P_e = \{Q_e\}$ . Note that  $P_e$  is an empty basic family of paths and

$|E(P_e)| = 1$ . Therefore for every  $e \in E_1$ , the initialization step of the dynamic programming, finds a partial solution  $\mathcal{S}_e$  for  $P_e$ . By merging  $\mathcal{S}_1$  with all these partial solutions sequentially in an arbitrary order, we get a partial solution  $\mathcal{S}_2$  for  $P_2$ , as desired.  $\blacktriangleleft$

► **Lemma 76.** *Let  $P = \{Q_1 \dots, Q_m\} \in \mathcal{P}$  be basic and trivial (w.r.t.  $\mathcal{W}$ ). For every  $i \in \{1, \dots, m\}$ , let  $u_i$  and  $v_i$  be the endpoints of  $Q_i$ . Let  $\mathcal{B} = \bigcup_{i=1}^m (B_{u_i} \cup B_{v_i})$ . Let  $\mathcal{W}_P$  be the  $P$ -facial restriction of  $\mathcal{W}$ . Let  $\Gamma = \bigcup_{\vec{W} \in \mathcal{W}_P} \vec{W}$ . Let  $\mathcal{C}$  be the partition of  $\mathcal{B}$  that corresponds to the weakly-connected components of  $\Gamma$ . For any  $x \in \mathcal{B}$  let  $f^{\text{in}}(x) = \text{in-degree}_{\mathcal{W}_P}(x)$  and  $f^{\text{out}}(x) = \text{out-degree}_{\mathcal{W}_P}(x)$ . Then there exists some  $a \in \mathcal{A} \cup (\mathcal{A} \times \mathcal{A}) \cup \text{nil}$  and  $l, r, p \in (V(\vec{G}) \cup \text{nil})$  such that the dynamic programming table contains some partial solution  $\mathcal{S}$  at location  $(P, (\mathcal{C}, f^{\text{in}}, f^{\text{out}}, a, l, r, p))$ , with  $\text{cost}_{\vec{G}}(\mathcal{S}) \leq \text{cost}_{\vec{G}}(\mathcal{W}_P)$ .*

**Proof.** Since  $P$  is trivial, we have that  $P \in \mathcal{P}_1$ , and thus the exact same argument as in Lemma 64 applies here.  $\blacktriangleleft$

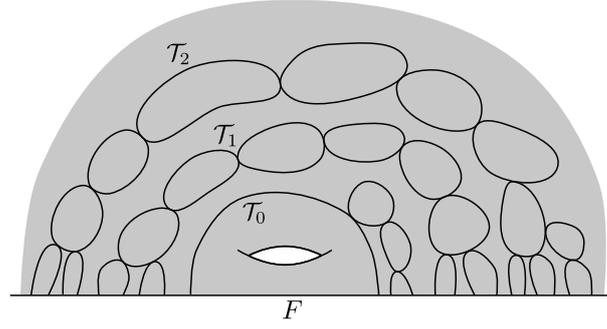
► **Lemma 77.** *Let  $P = \{Q_1 \dots, Q_m\} \in \mathcal{P}$  be succinct and elementary (w.r.t.  $\mathcal{W}$ ). For every  $i \in \{1, \dots, m\}$ , let  $u_i$  and  $v_i$  be the endpoints of  $Q_i$ . Let  $\mathcal{B} = \bigcup_{i=1}^m (B_{u_i} \cup B_{v_i})$ . Let  $\mathcal{W}_P$  be the  $P$ -facial restriction of  $\mathcal{W}$ . Let  $\Gamma = \bigcup_{\vec{W} \in \mathcal{W}_P} \vec{W}$ . Let  $\mathcal{C}$  be the partition of  $\mathcal{B}$  that corresponds to the weakly-connected components of  $\Gamma$ . For any  $x \in \mathcal{B}$  let  $f^{\text{in}}(x) = \text{in-degree}_{\mathcal{W}_P}(x)$  and  $f^{\text{out}}(x) = \text{out-degree}_{\mathcal{W}_P}(x)$ . Then there exists some  $a \in \mathcal{A} \cup (\mathcal{A} \times \mathcal{A}) \cup \text{nil}$  and  $l, r, p \in (V(\vec{G}) \cup \text{nil})$  such that the dynamic programming table contains some partial solution  $\mathcal{S}$  at location  $(P, (\mathcal{C}, f^{\text{in}}, f^{\text{out}}, a, l, r, p))$ , with  $\text{cost}_{\vec{G}}(\mathcal{S}) \leq \text{cost}_{\vec{G}}(\mathcal{W}_P)$ .*

**Proof.** Let  $\mathcal{T}$ ,  $D$ ,  $k$ ,  $D_1, \dots, D_k$  and  $j$  be as in Definition 66. We prove the assertion by induction on  $\mathcal{T}$ . For the base case, where  $D$  is a leaf of  $\mathcal{T}$ , the same argument as in Lemma 64 applies here. Suppose that  $D$  is non-leaf. In this case, we prove the assertion by another induction on  $j$ . For the base case, where  $j = 1$ , the same argument as in Lemma 64 applies here. Now suppose that we have proved the assertion for all  $j' < j$ . Let  $P_1 \in \mathcal{P}_\infty$  such that  $V(P_1) \subseteq V(P)$ ,  $E(P_1) \subseteq E(P)$ ,  $P_1$  covers  $\bigcup_{i=1}^{j-1} \mathcal{T}_{D_i}$  and avoids  $\mathcal{T} \setminus (\bigcup_{i=1}^{j-1} \mathcal{T}_{D_i})$ . Let  $P_2 \in \mathcal{P}_\infty$  such that  $E(P_2) = E(P) \setminus E(P_1)$ . By the construction,  $P_1$  and  $P_2$  are elementary. Therefore, by Lemma 69, there exists elementary and succinct  $P'_1 \in \mathcal{P}_\infty$  and  $P'_2 \in \mathcal{P}_\infty$  for  $P_1$  and  $P_2$  respectively. Moreover, by Lemma 70, we have that  $E(P'_1) \subseteq E(P)$ ,  $E(P'_2) \subseteq E(P)$  and  $E(P'_1) \cap E(P'_2) = \emptyset$ . By Lemma 74 we have that  $P, P'_1, P'_2 \in \mathcal{P}_{18000g^3}$ . We next define some  $P''_1 \in \mathcal{P}_\infty$ . Let  $\Gamma$  be the graph obtained as the union of all paths in  $P$ , that is  $\Gamma = \bigcup_{Q \in P} Q$ . Similarly, let  $\Gamma' = \bigcup_{Q \in P'_2} Q$ , let  $\Gamma'' = \Gamma \setminus E(\Gamma')$ , and let  $\Gamma'''$  be the graph obtained from  $\Gamma''$  by deleting all isolated vertices. We define  $P''_1$  to be the set of connected components in  $\Gamma'''$ . Note that  $\Gamma'''$  has at most  $36000g^3$  connected components, and each such component is a path. Thus  $P''_1 \in \mathcal{P}_{36000g^3}$ , and  $E(P''_1) \subseteq E(P'_1)$ . By the induction hypothesis, there exists a partial solution  $\mathcal{S}'_1$  for  $P''_1$ , and thus by Lemma 75, there exists a partial solution  $\mathcal{S}''_1$  for  $P'_1$ . Also, by the induction hypothesis, there exists a partial solution  $\mathcal{S}'_2$  for  $P'_2$ . By merging  $\mathcal{S}''_1$  and  $\mathcal{S}'_2$ , we get a partial solution  $\mathcal{S}$  for  $P$ , as desired.  $\blacktriangleleft$

We define the following three types of non-trivial trees in  $\mathcal{F}$ :

1. We say that a non-trivial tree  $\mathcal{T}$  is of the *first type*, if there exists a non-leaf  $D \in V(\mathcal{T})$ , such that  $D$  is a closed walk, with  $V(D) \cap V(F) = \emptyset$ , such that  $\psi(D)$  is non-contractible.
2. We say that a non-trivial tree  $\mathcal{T}$  is of the *second type*, if it is not of the first type and there exists a leaf  $D \in V(\mathcal{T})$  such that  $\psi(D) \cup \psi(F)$  is non-contractible.
3. We say that a non-trivial tree  $\mathcal{T}$  is of the *third type*, if it is not of the first type nor of the second type.

We say that a non-trivial tree  $\mathcal{T}$  is *good*, if at least one of the following conditions holds:



■ **Figure 4** Example of good non-trivial trees  $\mathcal{T}_0$ ,  $\mathcal{T}_1$ , and  $\mathcal{T}_2$  that are pairwise friends; note that  $\mathcal{T}_0$  is of the second type while  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are of the third type.

1.  $\mathcal{T}$  is of the second type, and for every  $D_1, D_2 \in V(\mathcal{T})$  where  $\psi(D_1) \cup \psi(F)$  and  $\psi(D_2) \cup \psi(F)$  are non-contractible, we have that the loops obtained from  $\psi(D_1)$  and  $\psi(D_2)$  by contracting  $\psi(F)$  into a single point  $x$  are homotopic in  $\pi_1(S, x)$ . Let  $D \in V(\mathcal{T})$  such that  $\psi(D) \cup \psi(F)$  is non-contractible. We let  $\beta(\mathcal{T})$  to be the homotopy class of the loop  $\psi(D)$  in the surface obtained after contracting  $\psi(F)$  into a single point.
2.  $\mathcal{T}$  is of the third type and the following holds. Let  $X = \psi(F) \cup \bigcup_{D \in V(\mathcal{T})} \psi(D)$  and let  $X'$  be the image of  $X$  after contracting  $\psi(F)$  into a single point  $x$ . Then all non-contractible loops in  $X'$  are homotopic in  $\pi_1(S, x)$ . We let  $\beta(\mathcal{T})$  be the homotopy class in  $\pi_1(S, x)$  of all non-contractible loops in  $X'$ ; note that we may always take a non-contractible loop in  $X'$  that contains the basepoint  $x$  since  $X$  is connected.

Otherwise, we say that  $\mathcal{T}$  is a *bad tree*.

Let  $\mathcal{T}_0$  and  $\mathcal{T}_1$  be non-trivial good trees in  $\mathcal{F}$ . We say that  $\mathcal{T}_0$  is a *friend* of  $\mathcal{T}_1$  if  $\beta(\mathcal{T}_0) = \beta(\mathcal{T}_1)$  (see Figure 4 for an example).

► **Definition 78** (Friendly). Let  $P \in \mathcal{P}_\infty$ . We say that  $P$  is *friendly* if the following holds.

1.  $P$  avoids all non-trivial bad trees.
2. For any two non-trivial good trees  $\mathcal{T}_0$  and  $\mathcal{T}_1$  in  $\mathcal{F}$  that  $P$  covers, we have that  $\beta(\mathcal{T}_0) = \beta(\mathcal{T}_1)$ .
3. If  $P$  covers a non-trivial good tree  $\mathcal{T}_0$ , then  $P$  covers all non-trivial good trees  $\mathcal{T}_1$  with  $\beta(\mathcal{T}_0) = \beta(\mathcal{T}_1)$ .

► **Lemma 79.** *There exists at most 12g bad trees in  $\mathcal{F}$ .*

**Proof.** We partition all bad trees in  $\mathcal{F}$  into three sets:

1.  $F_1 = \{\mathcal{T} \in \mathcal{F} : \mathcal{T} \text{ is a bad tree of the first type}\}$ .
2.  $F_2 = \{\mathcal{T} \in \mathcal{F} : \mathcal{T} \text{ is a bad tree of the second type}\}$ .
3.  $F_3 = \{\mathcal{T} \in \mathcal{F} : \mathcal{T} \text{ is a bad tree of the third type}\}$ .

We first bound  $|F_1|$ . Let  $\mathcal{T} \in F_1$ . We let  $\beta(\mathcal{T})$  to be the homotopy class of some non-leaf  $D \in V(\mathcal{T})$ , such that  $D$  is a closed walk, with  $V(D) \cap V(F) = \emptyset$ , and  $\psi(D)$  is non-contractible. Note that by the definition of trees of the first type, such a non-leaf  $D \in V(\mathcal{T})$  exists. Let  $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \in F_1$  be distinct. We show that  $\beta(\mathcal{T}_1) = \beta(\mathcal{T}_2) = \beta(\mathcal{T}_3)$  cannot happen. Suppose not, and we have that  $\beta(\mathcal{T}_1) = \beta(\mathcal{T}_2) = \beta(\mathcal{T}_3)$ . For any  $i \in \{1, 2, 3\}$ , let  $D_i \in V(\mathcal{T}_i)$  such that  $\beta(\mathcal{T}_i)$  is the homotopy class of  $D_i$ . By removing  $\psi(D_1)$ ,  $\psi(D_2)$  and  $\psi(D_3)$  from the surface we obtain three connected components. We may assume w.l.o.g that  $\psi(D_2)$  is inside an annulus bounded by  $\psi(D_1)$  and  $\psi(D_3)$ . Therefore, there is no path in the surface from

$\psi(D_2)$  to  $\psi(F)$ , that does not intersect  $\psi(D_1 \cup D_3)$ , which is a contradiction. Therefore by Lemma 72 we have that  $|F_1| \leq 6g$ .

Next we bound  $|F_2|$  and  $|F_3|$ . Let  $\mathcal{T} \in F_2$ . By the construction, there exist  $D_1, D_2 \in V(\mathcal{T})$  where  $\psi(D_1) \cup \psi(F)$  and  $\psi(D_2) \cup \psi(F)$  are non-contractible, and the loops obtained from  $\psi(D_1)$  and  $\psi(D_2)$  after contracting  $\psi(F)$  into a single point  $y$  are non-homotopic in  $\pi_1(S, y)$ . Let  $x$  be a point in the interior of the disk bounded by  $\psi(F)$ . For every  $\mathcal{T} \in F_2$ , similarly to Lemma 74, we construct two non-homotopic loops  $\gamma_{\mathcal{T}}$  and  $\gamma'_{\mathcal{T}}$  in the surface, corresponding to  $D_1$  and  $D_2$  respectively, such that they only intersect at  $x$ . Let  $\mathcal{L}$  be the set of all these loops. Let  $L', L'', L''' \in \mathcal{L}$  be distinct. Similarly to Lemma 74, we show that  $L', L''$  and  $L'''$  can not be all homotopic. Suppose not. We may assume w.l.o.g. that  $L''$  is inside the disk bounded by  $L'$  and  $L'''$ . Let  $\mathcal{T}'' \in F_2$  be the tree corresponding to  $L''$ . Now note that  $L''$  is inside the disk bounded by  $L'$  and  $L'''$ , and thus all non-contractible loops in  $\psi(F) \cup \bigcup_{D \in V(\mathcal{T})} \psi(D)$  are in the same homotopy class as  $L''$ . This contradicts the fact that  $\psi(D_1) \cup \psi(F)$  and  $\psi(D_2) \cup \psi(F)$  are non-homotopic. Therefore, by Lemma 72 we have that  $|F_2| \leq 3g$ . Using a similar argument, we can show that  $|F_3| \leq 3g$ .

Therefore, we have that  $|F_1| + |F_2| + |F_3| \leq 12g$ , completing the proof.  $\blacktriangleleft$

► **Lemma 80.** *Let  $P = \{Q_1, \dots, Q_m\} \in \mathcal{P}_\infty$  be friendly, succinct and basic (w.r.t.  $\mathcal{W}$ ). For every  $i \in \{1, \dots, m\}$ , let  $u_i$  and  $v_i$  be the endpoints of  $Q_i$ . Let  $\mathcal{B} = \bigcup_{i=1}^m (B_{u_i} \cup B_{v_i})$ . Let  $\mathcal{W}_P$  be the  $P$ -facial restriction of  $\mathcal{W}$ . Let  $\Gamma = \bigcup_{\vec{W} \in \mathcal{W}_P} \vec{W}$ . Let  $C$  be the partition of  $\mathcal{B}$  that corresponds to the weakly-connected components of  $\Gamma$ . For any  $x \in \mathcal{B}$  let  $f^{\text{in}}(x) = \text{in-degree}_{\mathcal{W}_P}(x)$  and  $f^{\text{out}}(x) = \text{out-degree}_{\mathcal{W}_P}(x)$ . Then there exists some  $a \in \mathcal{A} \cup (\mathcal{A} \times \mathcal{A}) \cup \text{nil}$  and  $l, r, p \in (V(\vec{G}) \cup \text{nil})$  such that the dynamic programming table contains some partial solution  $\mathcal{S}$  at location  $(P, (C, f^{\text{in}}, f^{\text{out}}, a, l, r, p))$ , with  $\text{cost}_{\vec{G}}(\mathcal{S}) \leq \text{cost}_{\vec{G}}(\mathcal{W}_P)$ .*

**Proof.** If  $P$  does not cover any non-trivial trees in  $\mathcal{F}$ , then  $P$  is trivial and thus we have that  $P \in \mathcal{P}_1$  and by Lemma 76 there exists a partial solution  $\mathcal{S}$  for  $P$ . Otherwise suppose that  $P$  covers a non-trivial good tree  $\mathcal{T}$  in  $\mathcal{F}$ . By the definition of friendly, for every non-trivial good tree  $\mathcal{T}'$  in  $\mathcal{F}$  with  $\beta(\mathcal{T}) = \beta(\mathcal{T}')$ , we have that  $P$  covers  $\mathcal{T}'$ . Let  $A = \{\mathcal{T}' \in \mathcal{F} : \beta(\mathcal{T}) = \beta(\mathcal{T}')\}$ . Suppose that  $A = \{\mathcal{T}_0, \dots, \mathcal{T}_m\}$  such that they intersect  $F$  in this order along a traversal of  $F$ . For any  $i \in \{0, \dots, m\}$ , let  $P_i \in \mathcal{P}_\infty$  be elementary and succinct such that  $P_i$  covers  $\mathcal{T}_i$  and avoids any other non-trivial trees in  $\mathcal{F}$ . By Lemma 74 we have that  $P_i \in \mathcal{P}_{18000g^3}$ . Moreover, by Lemma 77 we have that there exists a partial solution  $\mathcal{S}_i$  for  $P_i$ . For any  $i \in \{0, \dots, m\}$ , let  $P'_i \in \mathcal{P}_\infty$  be succinct and basic, such that  $P'_i$  covers  $\bigcup_{j=0}^i \mathcal{T}_j$  and avoids any other non-trivial trees in  $\mathcal{F}$ . By the construction, each  $P'_i$  can be obtained by merging  $P'_{i-1}$  with  $P_i$  and some trivial and empty basic family of paths, and thus we have that  $P'_i \in \mathcal{P}_{36000g^3}$ . Therefore, by induction and Lemma 76, there exists a partial solution  $\mathcal{S}'_i$  for each  $P'_i$ , and thus there exists a partial for solution  $\mathcal{S}'_m$  for  $P'_m$ . Note that  $P'_m = P$ . Therefore, we get a partial solution  $\mathcal{S}$  for  $P$ , as desired.  $\blacktriangleleft$

► **Lemma 81.** *Let  $P = \{F\}$ . Let  $v^\circ \in V(F)$ . Let  $\mathcal{W}_P$  be the  $P$ -facial restriction of  $\mathcal{W}$ . Let  $\Gamma = \bigcup_{\vec{W} \in \mathcal{W}_P} \vec{W}$ . Let  $C$  be the partition of  $B_{v^\circ}$  that corresponds to the weakly-connected components of  $\Gamma$ . For any  $x \in B_{v^\circ}$  let  $f^{\text{in}}(x) = \text{in-degree}_{\mathcal{W}_P}(x)$  and  $f^{\text{out}}(x) = \text{out-degree}_{\mathcal{W}_P}(x)$ . Then there exists some  $a \in \mathcal{A} \cup (\mathcal{A} \times \mathcal{A}) \cup \text{nil}$  and  $l, r, p \in (V(\vec{G}) \cup \text{nil})$  such that the dynamic programming table contains some partial solution  $\mathcal{S}$  at location  $(P, (C, f^{\text{in}}, f^{\text{out}}, a, l, r, p))$ , with  $\text{cost}_{\vec{G}}(\mathcal{S}) \leq \text{cost}_{\vec{G}}(\mathcal{W}_P)$ .*

**Proof.** We first partition  $F$  to the sets of basic families of paths as follows:

1.  $F_1 = \{P' \in \mathcal{P}_\infty : P' \text{ is elementary and succinct, } P' \text{ covers some bad tree } \mathcal{T}\}$ .
2.  $F_2 = \{P' \in \mathcal{P}_\infty : P' \text{ is basic, succinct and friendly}\}$ .

3.  $F_3 = \{P' \in \mathcal{P}_\infty : P' \text{ is basic, succinct and trivial}\}$ .
4.  $F_4 = \{P' \in \mathcal{P}_\infty : P' \text{ is basic, succinct and empty}\}$ .

For every bad tree  $\mathcal{T}$  in  $\mathcal{F}$ , let  $P_1 \in F_1$  such that  $P_1$  covers  $\mathcal{T}$  and avoids any other non-trivial tree in  $\mathcal{F}$ . Since  $P_1$  is elementary and succinct, by Lemma 74 we have that  $|P_1| \leq 18000g^3$ . Moreover by Lemma 77 there exists a partial solution  $\mathcal{S}_1$  for  $P_1$ . Also by Lemma 79, there exist at most  $12g$  bad trees in  $\mathcal{F}$ , and thus  $|F_1| \leq 12g$ . Therefore there exists  $P'_1 \in \mathcal{P}_{216000g^4}$ , such that for any  $\mathcal{T}' \in \mathcal{F}$ ,  $P'_1$  covers (resp. avoids)  $\mathcal{T}'$  if and only if there exists  $P_1 \in F_1$  such that  $P_1$  covers (resp. avoids)  $\mathcal{T}'$ . Moreover the dynamic programming table contains a partial solution  $\mathcal{S}'_1$  for  $P'_1$ . Now we will show that  $|F_2| \leq 3g$ . For every  $P_2 \in F_2$ , let  $\mathcal{T}_2$  be some non-trivial good tree that  $P_2$  covers. Let  $x$  be a point in the interior of the disk bounded by  $\psi(F)$ . Similar to Lemma 74 we construct a non-contractible loop  $\gamma_{P_2}$  that contains  $x$ , corresponding to  $\mathcal{T}_2$ . Let  $L$  be the set of all these interior-disjoint loops. For every  $P', P'' \in F_2$ , since  $P'$  and  $P''$  are succinct and friendly,  $\gamma_{P'}$  and  $\gamma_{P''}$  are not homotopic. Therefore by Lemma 72 we have that  $|L| \leq 3g$ , and thus  $|F_2| \leq 3g$ . Furthermore for every  $P_2 \in F_2$ , by Lemma 80, we have that  $P_2 \in \mathcal{P}_{36000g^3}$ , and there exists a partial solution  $\mathcal{S}_2$  for  $P_2$ . Therefore there exists  $P'_2 \in \mathcal{P}_{108000g^4}$ , such that for any  $\mathcal{T}' \in \mathcal{F}$ ,  $P'_2$  covers (resp. avoids)  $\mathcal{T}'$  if and only if there exists  $P_2 \in F_2$  such that  $P_2$  covers (resp. avoids)  $\mathcal{T}'$ . Moreover the dynamic programming table contains a partial solution  $\mathcal{S}'_2$  for  $P'_2$ . Let  $P'_{1,2} \in \mathcal{P}_{324000g^4}$  such that for any  $\mathcal{T}' \in \mathcal{F}$ ,  $P'_{1,2}$  covers (resp. avoids)  $\mathcal{T}'$  if and only if there exists  $P' \in F_1 \cup F_2$  such that  $P'$  covers (resp. avoids)  $\mathcal{T}'$ . By merging  $\mathcal{S}'_1$  and  $\mathcal{S}'_2$  we get a partial solution  $\mathcal{S}'_{1,2}$  for  $P'_{1,2}$ .

Let

$$F_5 = \{\{Q\} \in F_3 \cup F_4 : Q \text{ is maximal subject to } E(Q) \cap E(P'_{1,2}) = \emptyset\}.$$

For every  $P_3 \in F_3$ , by Lemma 76 we have that  $P_3 \in \mathcal{P}_1$  and there exists a partial solution  $\mathcal{S}_3$  for  $P_3$ . Finally, for every  $P_4 \in F_4$ , we have that  $P_4 \in \mathcal{P}_1$  and also the dynamic programming table contains a partial solution  $\mathcal{S}_4$  for  $P_4$ . Therefore for every  $\{Q\} \in F_5$ , the dynamic programming table contains a partial solution for  $\{Q\}$ . By merging these partial solutions with  $P'_{1,2}$  in an arbitrary order, we get a partial solution  $\mathcal{S}$ , as desired. We remark that after merging the current partial solution with the partial solution for some  $\{Q\} \in F_5$ , we obtain a new partial solution for some  $P \in P_\infty$  with fewer paths. Therefore for all intermediate  $P \in P_\infty$  we have  $P \in \mathcal{P}_{324000g^4}$ , concluding the proof.  $\blacktriangleleft$

► **Theorem 82.** *Let  $\vec{G}$  be an  $n$ -vertex  $(0, g, 1, p)$ -nearly embeddable graph and let  $\vec{H}$  be the vortex of  $\vec{G}$ . Then there exists an algorithm which computes a walk  $\vec{W}$  visiting all vertices in  $V(\vec{H})$  of total length  $\text{OPT}_{\vec{G}}(V(\vec{H}))$  in time  $n^{O(pg^4)}$ .*

**Proof.** Let  $P = \{F\}$ . By Lemma 81, there exists a partial solution for  $P$ . Now the exact same argument as in Theorem 65 applies here.  $\blacktriangleleft$

## 14 The algorithm for traversing a vortex in a nearly-embeddable graph

In this Section we obtain an exact algorithm for computing a closed walk that traverses all the vertices in the single vortex of a nearly-embeddable graph.

**Proof of Theorem 6.** The algorithm is as follows. Let  $A \subset V(\vec{G})$  be the set of apices and let  $\vec{F}$  be the face on which the vortex is attached. For each  $A' \subseteq A$  we construct a  $(0, g, 1, p+a)$ -nearly embeddable graph  $\vec{G}_{A'}$  as follows: Initially we set  $\vec{G}_{A'} = \vec{G} \setminus A$ . We then add  $A'$  to

$V(\vec{G}_{A'})$  and for every  $u \in A'$  and every  $v \in V(\vec{F})$  we add edges  $(u, v)$  and  $(v, u)$  of length  $d_{\vec{G}}(u, v)$  and  $d_{\vec{G}}(v, u)$  respectively; let  $E_{A'}$  be the set of all these new edges. We consider  $A'$  as being part of the vortex and we modify the path decomposition of the vortex by adding  $A'$  to each one of its bubbles; it is immediate that the result is a path decomposition of width at most  $a + p$ . We then run the algorithm of Theorem 82 on  $\vec{G}_{A'}$  and find an optimal closed walk visiting all vertices in  $V(\vec{H}) \cup A'$ . Let  $\vec{W}_{A'}$  be the resulting walk in  $\vec{G}_{A'}$ . We obtain a walk  $\vec{W}'_{A'}$  in  $\vec{G}$  visiting all vertices in  $V(\vec{H})$  with  $\text{cost}_{\vec{G}}(\vec{W}'_{A'}) = \text{cost}_{\vec{G}_{A'}}(\vec{W}_{A'})$  by replacing every edge in  $(u, v) \in E_{A'}$  by a shortest path from  $u$  to  $v$  in  $\vec{G}$ . After considering all subsets  $A' \subseteq A$ , we output the walk  $\vec{W}'_{A'}$  of minimum cost that we find. This completes the description of the algorithm.

It is immediate that the running time is  $O(2^a n^{(a+p)g^4}) = O(n^{(a+p)g^4})$ . It remains to show that the algorithm computes an optimum walk. Let  $\vec{R}$  be the walk computed by the algorithm. Let  $\vec{W}_{\text{OPT}}$  be a walk of minimum cost in  $\vec{G}$  visiting all vertices in  $V(\vec{H})$ . Let  $A^*$  be the set of apices visited by  $\vec{W}_{\text{OPT}}$ , that is  $A^* = A \cap V(\vec{W}_{\text{OPT}})$ . Let  $\vec{G}'$  be the genus- $g$  piece of  $\vec{G}$ . We can construct a walk  $\vec{Z}$  in  $\vec{G}_{A^*}$  that visits all the vertices in  $V(\vec{H}) \cup A^*$  of cost  $\text{cost}_{\vec{G}}(\vec{W}_{\text{OPT}})$  as follows: we replace every sub-walk of  $\vec{W}_{\text{OPT}}$  that is contained in  $\vec{G}'$ , has endpoints  $u, v \in V(\vec{F})$ , and visits some apex  $w \in A^*$ , by the path  $u, w, v$  in  $\vec{G}_{A^*}$ . It is immediate that the resulting walk does not traverse any apices and therefore it is contained in  $\vec{G}_{A^*}$ . Thus we have  $\text{cost}_{\vec{G}}(\vec{R}) \leq \text{cost}_{\vec{G}}(\vec{W}'_{A^*}) = \text{cost}_{\vec{G}_{A^*}}(\vec{W}_{A^*}) \leq \text{cost}_{\vec{G}}(\vec{W}_{\text{OPT}}) \leq \text{OPT}_{\vec{G}}$ , which concludes the proof.  $\blacktriangleleft$

## 15 The lower bound for graphs of bounded pathwidth

In this section we present the proof of Theorem 2. This is done via the following chain of reductions:



### 15.1 Edge Balancing

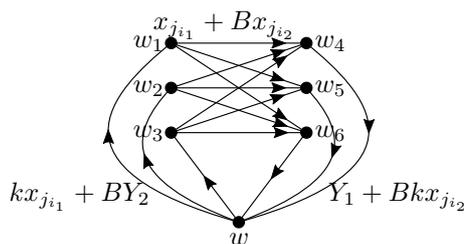
Let  $D$  be a directed graph and let  $\chi : E(D) \rightarrow \mathbb{Z}^+$  be an assignment of integers to the edges. We can extend  $\chi$  to a set  $F \subseteq E(D)$  of edges in the obvious way by defining  $\chi(F) = \sum_{e \in F} \chi(e)$ . Let  $\delta_D^+(v)$  and  $\delta_D^-(v)$  be the set of outgoing and incoming edges of  $v$ , respectively. We say that  $\chi$  is *balanced* at  $v \in V(D)$  if  $\sum_{e \in \delta_D^+(v)} \chi(e) = \sum_{e \in \delta_D^-(v)} \chi(e)$  holds and we say that  $\chi$  is *balanced* if it is balanced at every  $v \in V(D)$ .

EDGE BALANCING: Given a directed graph  $D$  and a set  $X_e$  of positive integers for every edge  $e \in E(D)$ , the task is to select a  $\chi(e) \in X_e$  for every edge  $e \in E(D)$  such that  $\chi$  is balanced.

An easy counting argument shows that it is sufficient to require that  $\chi$  balanced at all but one vertex:

► **Proposition 83.** *If  $\chi$  is balanced at every vertex of  $V(D) \setminus \{v\}$ , then it is also balanced at  $v$ .*

We give a lower bound for EDGE BALANCING by a parameterized reduction from MULTICOLORED BICLIQUE.



■ **Figure 5** The instance of EDGE BALANCING constructed in the proof of Lemma 86. The values on the edges indicate the value of  $\chi$  corresponding to a solution  $(v_{i,j_i})_{i=1,\dots,2k}$  of the MULTICOLORED BICLIQUE instance (we have  $Y_1 = \sum_{1 \leq i \leq k} x_{j_i}$  and  $Y_2 = \sum_{k+1 \leq i \leq 2k} x_{j_i}$ ).

MULTICOLORED BICLIQUE: Given a graph  $G$  with a partition  $(V_1, \dots, V_{2k})$  of vertices, find a vertex  $v_i \in V_i$  for every  $1 \leq i \leq 2k$  such that  $v_{i_1}$  and  $v_{i_2}$  are adjacent for every  $1 \leq i_1 \leq k$  and  $k + 1 \leq i_2 \leq 2k$ .

There is a very simple folklore reduction from CLIQUE to MULTICOLORED BICLIQUE (see, e.g., [6]) where the output parameter equals the input one, hence the lower bound of Chen et al. [5] for CLIQUE can be transferred to MULTICOLORED BICLIQUE.

► **Theorem 84.** *Assuming ETH, MULTICOLORED BICLIQUE cannot be solved in time  $f(k)n^{o(k)}$  for any computable function  $f$ .*

The reduction from MULTICOLORED BICLIQUE to EDGE BALANCING uses the technique of  $k$ -non-averaging sets to encode vertices [10, 17]. We say that a set  $X$  of positive integers is  $k$ -non-averaging if for every choice of  $k$  (not necessarily distinct) integers  $x_1, \dots, x_k \in X$ , their average is in  $X$  only if  $x_1 = x_2 = \dots = x_k$ . For example,  $X = \{(k + 1)^i \mid 1 \leq i \leq n\}$  is certainly a  $k$ -non-averaging set of size  $n$ . However, somewhat surprisingly, it is possible to construct much denser  $k$ -non-averaging sets where each integer is polynomially bounded in  $k$  and the size  $n$  of the set.

► **Lemma 85** (Jansen et al. [17]). *For every  $k$  and  $n$  there exists a  $k$ -non-averaging set  $X$  of  $n$  positive integers such that the largest element of  $X$  has value at most  $32k^2n^2$ . Furthermore,  $X$  can be constructed in time  $O(k^2n^3)$  time.*

► **Lemma 86.** *Assuming ETH, EDGE BALANCING has no  $f(k)n^{o(k)}$  time algorithm for any computable function  $f$ , where  $k$  is the number of vertices of  $D$ .*

**Proof.** The proof is by reduction from MULTICOLORED BICLIQUE. Let  $G$  be an undirected graph with a partition  $V_1, \dots, V_{2k}$  of the vertices in  $V(G)$ . By padding the instance with isolated vertices, we may assume without loss of generality that each  $V_i$  has the same number  $n$  of vertices; let  $v_{i,j}$  be the  $j$ -th vertex of  $V_i$ . We construct an instance of EDGE BALANCING on a directed graph  $D$  having  $2k + 1$  vertices  $w, w_1, w_2, \dots, w_{2k}$ . Let us use Lemma 85 to construct a  $k$ -non-averaging set  $X = \{x_1, \dots, x_n\}$  of  $n$  positive integers such that the maximum value in  $X$  is  $M = O(k^2n^2)$ . Let  $B = 2kM$ .

The edges of  $D$  and the sets of integers on them are constructed the following way (see Figure 5). For every  $1 \leq i_1 \leq k$  and  $k + 1 \leq i_2 \leq 2k$ , we introduce the edge  $(w_{i_1}, w_{i_2})$  into  $D$  and we define the set  $X_{(w_{i_1}, w_{i_2})}$  the following way: for every edge  $v_{i_1, j_1} v_{i_2, j_2}$  between  $V_{i_1}$  and  $V_{i_2}$ , let us introduce the positive integer  $x_{j_1} + Bx_{j_2}$  into  $X_{(w_{i_1}, w_{i_2})}$ . Next, for every  $1 \leq i \leq k$ , we introduce the edge  $(w, w_i)$  and let  $X_{(w, w_i)} = \{kx + By \mid x \in X, 1 \leq y \leq kM\}$ . Finally, for every  $k + 1 \leq i \leq 2k$ , we introduce the edge  $(w_i, w)$  and let  $X_{(w_i, w)} = \{y + Bkx \mid x \in X, 1 \leq y \leq kM\}$ . This completes the description of the reduction.

**Biclique  $\Rightarrow$  balanced assignment  $\chi$ .** Suppose that  $v_{i,j_i} \in V_i$  for  $1 \leq i \leq 2k$  form a solution of MULTICOLORED BICLIQUE. We define  $\chi : E(D) \rightarrow \mathbb{Z}^+$  the following way. For every  $1 \leq i_1 \leq k$  and  $k+1 \leq i_2 \leq 2k$ , we set (see the values on the edges in Figure 5)

- $\chi((w_{i_1}, w_{i_2})) = x_{j_{i_1}} + Bx_{j_{i_2}}$ ,
- $\chi((w, w_{i_1})) = kx_{j_{i_1}} + BY_2$ , where  $Y_2 = \sum_{k+1 \leq i \leq 2k} x_{j_i}$ , and
- $\chi((w_{i_1}, w)) = Y_1 + Bkx_{j_{i_2}}$ , where  $Y_1 = \sum_{1 \leq i \leq k} x_{j_i}$ .

Note that  $\chi(e) \in X_e$  holds for every edge  $e \in E(D)$ . Let us verify that  $\chi$  is balanced. For any  $1 \leq i_1 \leq k$ , we have  $\chi(\delta_D^+(w_{i_1})) = \sum_{k+1 \leq i_2 \leq 2k} (x_{j_{i_1}} + Bx_{j_{i_2}}) = kx_{j_{i_1}} + BY_2 = \chi((w, w_{i_1})) = \chi(\delta_D^-(w_{i_1}))$ , as required. Similarly, for any  $k+1 \leq i_2 \leq 2k$ , we have  $\chi(\delta_D^-(w_{i_2})) = \sum_{1 \leq i_1 \leq k} (x_{j_{i_1}} + Bx_{j_{i_2}}) = Y_1 + Bkx_{j_{i_2}} = \chi((w_{i_2}, w)) = \chi(\delta_D^+(w_{i_2}))$ . Thus we have shown that  $\chi$  is balanced at  $w_1, \dots, w_{2k}$  and it follows by Proposition 83 that  $\chi$  is balanced also at  $w$ .

**Balanced assignment  $\chi \Rightarrow$  biclique.** For the reverse direction of the equivalence, suppose that  $\chi : E(D) \rightarrow \mathbb{Z}^+$  is a balanced assignment with  $\chi(e) \in X_e$  for every  $e \in E(D)$ . For every  $1 \leq i_1 \leq k$ , the definition of  $X_{(w, w_{i_1})}$  implies that  $\chi((w, w_{i_1}))$  is of the form  $kx_{j_{i_1}} + By_{i_1}$  where  $x_{j_{i_1}} \in X$  for some  $1 \leq j_{i_1} \leq n$  and  $y_{i_1}$  is a positive integer. As  $kx_{j_{i_1}} \leq kM < B$  follows from  $x_{j_{i_1}} \in X$ , the value of  $\chi((w, w_{i_1}))$  uniquely determines  $j_{i_1}$  and  $y_{i_1}$ . Similarly, for every  $k+1 \leq i_2 \leq 2k$ , we have that  $\chi((w_{i_2}, w))$  is of the form  $y_{i_2} + Bkx_{j_{i_2}}$  for uniquely determined positive integers  $1 \leq j_{i_2} \leq n$  and  $y_{i_2}$ .

Having defined the values  $j_1, \dots, j_{2k}$ , we show that  $\chi((w_{i_1}, w_{i_2})) = x_{j_{i_1}} + Bx_{j_{i_2}}$  for every  $1 \leq i_1 \leq k$  and  $k+1 \leq i_2 \leq 2k$ . If this is true, then the vertices  $v_{i,j_i} \in V_i$  for  $1 \leq i \leq 2k$  form a solution of MULTICOLORED BICLIQUE: the fact that  $x_{j_{i_1}} + Bx_{j_{i_2}}$  was introduced into  $X_{(w_{i_1}, w_{i_2})}$  implies that there is an edge between  $v_{i_1, j_{i_1}} \in V_{i_1}$  and  $v_{i_2, j_{i_2}} \in V_{i_2}$ .

As the balance requirement holds at vertex  $w_{i_1}$ , it also holds if we count modulo  $B$ . We have that  $\chi((w, w_{i_1}))$  modulo  $B$  is exactly  $kx_{j_{i_1}} < B$  (here we use that  $kM < B$ ). For every  $k+1 \leq i_2 \leq 2k$ , the value  $\chi((w_{i_1}, w_{i_2}))$  modulo  $B$  is an integer from  $X$  and the value  $\chi(\delta_D^+(w_{i_1}))$  modulo  $B$  is exactly the sum of these  $k$  integers from  $X$  (as again by  $kM < B$ , this sum cannot reach  $B$ ). Therefore, we have that the sum of  $k$  integers from  $X$  is exactly  $kx_{j_{i_1}}$ . Since  $X$  is a  $k$ -non-averaging set, this is only possible if these  $k$  integers are all equal to  $x_{j_{i_1}}$ . Thus we have shown that  $\chi((w_{i_1}, w_{i_2})) = x_{j_{i_1}}$  modulo  $B$  for every  $1 \leq i_1 \leq k$  and  $k+1 \leq i_2 \leq 2k$ .

Let us consider now a vertex  $w_{i_2}$  for  $k+1 \leq i_2 \leq 2k$ . The balance requirement in particular implies that  $\lfloor \chi(\delta_D^+(w_{i_2}))/B \rfloor = \lfloor \chi(\delta_D^-(w_{i_2}))/B \rfloor$ . First, we have  $\lfloor \chi(\delta_D^+(w_{i_2}))/B \rfloor = \lfloor \chi((w_{i_2}, w))/B \rfloor = kx_{j_{i_2}}$ . Therefore, the fact that  $\chi$  is balanced at  $w_{i_2}$  implies  $kx_{j_{i_2}} = \lfloor \chi(\delta_D^-(w_{i_2}))/B \rfloor = \left\lfloor \sum_{1 \leq i_1 \leq k} \chi((w_{i_1}, w_{i_2}))/B \right\rfloor = \sum_{1 \leq i_1 \leq k} \lfloor \chi((w_{i_1}, w_{i_2}))/B \rfloor$ , where the third equality holds because we have  $\chi((w_{i_1}, w_{i_2}))/B - \lfloor \chi((w_{i_1}, w_{i_2}))/B \rfloor \leq M/B < 1/k$ . The definition of  $X_{(w_{i_1}, w_{i_2})}$  implies that  $\lfloor \chi((w_{i_1}, w_{i_2}))/B \rfloor$  is an integer from  $X$ . Therefore, the equation above states the the sum of  $k$  integers from  $X$  is exactly  $kx_{j_{i_2}}$ . Since  $X$  is a  $k$ -non-averaging set, this is only possible if these  $k$  integers are all equal to  $x_{j_{i_2}}$ . Therefore, we have shown that  $\chi((w_{i_1}, w_{i_2}))$  modulo  $B$  is exactly  $x_{j_{i_1}}$  and  $\lfloor \chi((w_{i_1}, w_{i_2}))/B \rfloor$  is exactly  $x_{j_{i_2}}$ , proving that  $\chi((w_{i_1}, w_{i_2})) = x_{j_{i_1}} + Bx_{j_{i_2}}$  indeed holds.  $\blacktriangleleft$

## 15.2 Constrained Closed Walk and ATSP

To make the hardness proof for ATSP cleaner, we first prove hardness for the variant of the problem, where instead of optimizing the length of the tour, the only constraint is that certain vertices cannot be visited more than once.

**CONSTRAINED CLOSED WALK:** Given an unweighted directed graph  $G$  and set  $U \subseteq V(G)$  of vertices, find a closed walk (of any length) that visits each vertex at least once and visits each vertex in  $U$  exactly once.

There is a simple reduction from CONSTRAINED CLOSED WALK to ATSP that preserves treewidth.

► **Lemma 87.** *An instance of CONSTRAINED CLOSED WALK on an unweighted directed graph  $D$  can be reduced in polynomial time to an instance of ATSP with polynomially bounded positive integer weights on an edge-weighted version  $D^*$  of  $D$ .*

**Proof.** It is easy to see that if we assign weight 1 to every edge  $(u, v)$  with  $v \in U$  and weight 0 to every other edge, then the CONSTRAINED CLOSED WALK instance has a solution if and only if the resulting weighted graph has closed walk of length at most  $|U|$  (or, equivalently, less than  $|U| + 1$ ) visiting every vertex. To ensure that every weight is positive, let us replace every weight 0 with weight  $\epsilon := 1/(2n^2)$ . As a minimum solution of ATSP contains at most  $n^2$  edges, this modification increases the minimum cost by at most  $1/2$ . Thus it remains true that the CONSTRAINED CLOSED WALK instance has a solution if and only if there is a closed walk of length less than  $|U| + 1$  visiting every vertex. Finally, to ensure that every cost is integer, we multiply each of them by  $2n^2$ . ◀

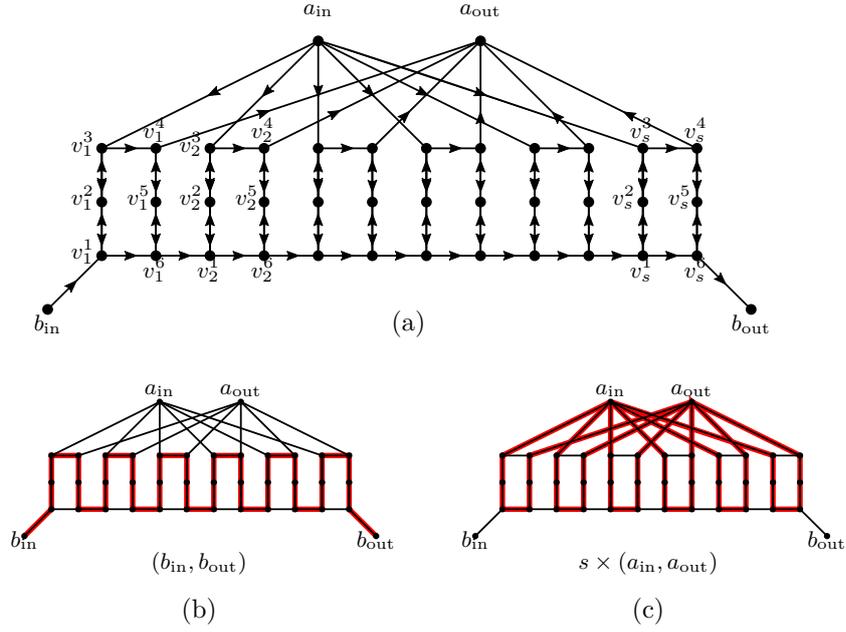
The rest of the section is devoted to giving a lower bound for CONSTRAINED CLOSED WALK. The lower bound proof uses certain gadgets in the construction of the instances. Formally, we define a *gadget* to be a graph with a set of distinguished vertices called *external vertices*; every other vertex is *internal*. To avoid degenerate situations, we always require that the external vertices of a gadget are independent and each external vertex has either indegree 0 or outdegree 0 in the gadget; in particular, this implies that a path between two external vertices contains no other external vertex. Also, this implies that there is no closed walk containing an external vertex.

We say that a set  $\mathcal{P}$  of paths of the gadget *satisfies* a gadget if (1) both endpoints of each path are external vertices and (2) every internal vertex of the gadget is visited by exactly one path in  $\mathcal{P}$ . If a path  $P \in \mathcal{P}$  connects two external vertices of a gadget, then we define the *type* of  $P$  to be the (ordered) pair of its endpoints. If  $\mathcal{P}$  satisfies the gadget, then we define the *type* of  $\mathcal{P}$  to be the multiset of the types of the paths in  $\mathcal{P}$ . For brevity, we use notation such as  $a \times (v_1, v_2) + b \times (v_3, v_4)$  to denote the type that contains  $a$  times the pair  $(v_1, v_2)$  and  $b$  times the pair  $(v_3, v_4)$ . For a gadget  $H$ , we let the set  $\mathcal{T}(H)$  contain every possible type of a set  $\mathcal{P}$  of paths satisfying  $H$ .

We construct gadgets where we can exactly tell the type of the collections of paths that can satisfy the gadget, that is, the set  $\mathcal{T}(H)$  is of a certain form. In the first gadget, we have a simple choice between one path or a specified number of paths.

► **Lemma 88.** *For every  $s \geq 1$ , we can construct in time polynomial in  $n$  a gadget  $H_s$  with the following properties:*

1.  $H_s$  has four external vertices  $a_{\text{in}}$ ,  $a_{\text{out}}$ ,  $b_{\text{in}}$ , and  $b_{\text{out}}$ .
2.  $H_s$  minus its external vertices has constant pathwidth.
3.  $\mathcal{T}(H_s)$  contains exactly two types: the type  $(b_{\text{in}}, b_{\text{out}})$  and the type  $s \times (a_{\text{in}}, a_{\text{out}})$  (in other words, the gadget can be satisfied by a path from  $b_{\text{in}}$  to  $b_{\text{out}}$ , can be satisfied by a collection of  $s$  paths from  $a_{\text{in}}$  to  $a_{\text{out}}$ , but cannot be satisfied by any other type of collection of paths).



■ **Figure 6** The gadget of Lemma 88 with two collections of paths satisfying it.

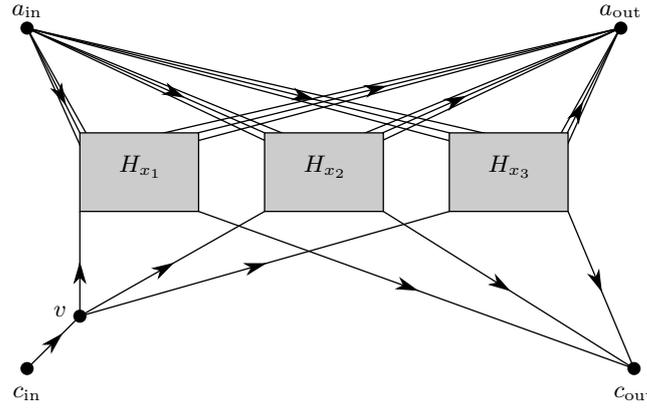
**Proof.** The gadget  $H_s$  has  $6s$  internal vertices  $v_j^i$  ( $1 \leq i \leq 6$ ,  $1 \leq j \leq s$ ) connected as shown in Figure 6(a). Additionally, we introduce the edges  $(b_{in}, v_1^1)$ ,  $(v_s^6, b_{out})$ , and for every  $1 \leq j \leq s$ , the edges  $(a_{in}, v_j^3)$  and  $(v_j^4, a_{out})$ . It is clear that statement (2) holds:  $H_s$  minus its vertices is a graph with constant pathwidth.

This gadget can be satisfied by a path from  $b_{in}$  to  $b_{out}$  (see Figure 6(b)) and also by a collection of  $s$  paths where the  $j$ -th path is  $a_{in}, v_j^3, v_j^4, v_j^5, v_j^6, v_j^1, v_j^2, v_j^3, a_{out}$  (see Figure 6(c)). To complete the proof of statement (3), we need to show that if  $\mathcal{P}$  satisfies  $H_s$ , then  $\mathcal{P}$  is one of these two types. The basic observation is that if a path in  $\mathcal{P}$  contains  $v_j^2$ , then it has to contain  $v_j^1$  and  $v_j^3$  as well (as each internal vertex is visited exactly once), hence the three vertices  $v_j^1, v_j^2, v_j^3$  have to appear on the same path of  $\mathcal{P}$ . The same is true for the vertices  $v_j^4, v_j^5, v_j^6$ . Suppose that  $\mathcal{P}$  contains a path  $P$  starting at  $b_{in}$ . Then its next vertex is  $v_1^1$ , which should be followed by  $v_1^2$  and  $v_1^3$  by the argument above. The next vertex is  $v_1^4$  (the only outneighbor of  $v_1^3$  not yet visited), which is followed by  $v_1^5$  and  $v_1^6$ . Now the next vertex is  $v_2^1$ , the only outneighbor of  $v_1^6$  not yet visited. With similar arguments, we can show that  $P$  is exactly of the form shown in Figure 6(b), hence  $\mathcal{P}$  contains only this path, and  $\mathcal{P}$  is of type  $\{(b_{in}, b_{out})\}$ .

Suppose now that  $\mathcal{P}$  does not contain a path starting at  $b_{in}$ . Then the only way to reach vertex  $v_1^1$  is with a path starting as  $a_{in}, v_1^3, v_1^4, v_1^5, v_1^6$ . This has to be followed by the unique outneighbor  $v_1^1$  of  $v_1^6$  that was not yet visited. This means that the path contains also  $v_1^2$  and  $v_1^3$ , which has to be followed by  $a_{out}$ . Then with similar arguments, we can show for every  $j \geq 2$  that  $v_j^1$  is visited by the path  $a_{in}, v_j^3, v_j^4, v_j^5, v_j^6, v_j^1, v_j^2, v_j^3, a_{out}$ . This means that  $|\mathcal{P}| = s$  and the type of  $\mathcal{P}$  is  $s \times (a_{in}, a_{out})$ . ◀

► **Lemma 89.** Let  $X$  be a set of  $n$  positive integers, each at most  $M$  and let  $S = \sum_{x \in X} x$ . In time polynomial in  $n$  and  $M$ , we can construct a gadget  $H_X$  with the following properties:

1.  $H_X$  has four external vertices  $a_{in}$ ,  $a_{out}$ ,  $c_{in}$ , and  $c_{out}$ .
2.  $H_X$  minus its external vertices has constant pathwidth.



■ **Figure 7** The gadget  $H_X$  of Lemma 89 for a set  $X = \{x_1, x_2, x_3\}$  of three integers. The gray rectangles represent the *internal* vertices of the three gadgets  $H_{x_1}$ ,  $H_{x_2}$ , and  $H_{x_3}$ .

3.  $\mathcal{T}(H_s)$  contains exactly  $|X|$  types: for every  $x \in X$ , it contains the type  $(c_{\text{in}}, c_{\text{out}}) + (S - x) \times (a_{\text{in}}, a_{\text{out}})$ .

**Proof.** The gadget  $H_X$  is constructed the following way (see Figure 7). Let us introduce an internal vertex  $v$  and the edge  $(c_{\text{in}}, v)$ . For every  $x \in X$ , let us introduce a copy of  $H_x$  defined by Lemma 88 where  $a_{\text{in}}, a_{\text{out}}, v, c_{\text{out}}$  of  $H_X$  play the role of  $a_{\text{in}}, a_{\text{out}}, b_{\text{in}}, b_{\text{out}}$ , respectively. If we remove the four external vertices of  $H_X$ , then we get a graph with constant pathwidth: if we remove one more vertex,  $v$ , then we get the disjoint union of internal vertices of the gadgets  $H_x$ 's, which have constant pathwidth by Lemma 88.

For every  $x \in X$ , the gadget can be satisfied by the following collection of paths. The copy of  $H_x$  in  $H_X$  can be satisfied by a path from  $v$  to  $c_{\text{out}}$ , which can be extended with the edge  $(c_{\text{in}}, v)$  to a path from  $c_{\text{in}}$  to  $c_{\text{out}}$ . For every  $x' \in X, x' \neq x$ , we can satisfy the copy of  $H_{x'}$  in  $H_X$  by a collection of  $x'$  paths from  $a_{\text{in}}$  to  $a_{\text{out}}$ . This way, we constructed a collection  $\mathcal{P}$  of paths satisfying  $H_X$  that consists of a single path of type  $(c_{\text{in}}, c_{\text{out}})$  and exactly  $\sum_{x' \in X \setminus \{x\}} x' = S - x$  paths of type  $(a_{\text{in}}, a_{\text{out}})$ .

To complete the proof of statement (3), consider a collection  $\mathcal{P}$  of paths satisfying  $H_X$ . Let  $P$  be the unique path of  $\mathcal{P}$  visiting vertex  $v$ . The vertex of  $P$  after  $v$  has to be an internal vertex of the copy of  $H_x$  for some  $x \in X$  (here we use that the external vertices of the gadget  $H_x$  are independent, hence  $v$  cannot be followed by any of  $a_{\text{in}}, a_{\text{out}}$ , and  $c_{\text{out}}$ ). As  $v$  was identified with vertex  $b_{\text{in}}$  of  $H_x$ , Lemma 88 implies that  $P$  visits every internal vertex of this copy of  $H_x$  and leaves  $H_x$  at its vertex  $b_{\text{out}}$ , which was identified with  $c_{\text{out}}$ . Consider now some  $x' \in X$  with  $x' \neq x$ . Vertex  $b_{\text{in}}$  of  $H_{x'}$  was identified with  $v$ , path  $P$  is the only path of  $\mathcal{P}$  visiting  $v$ , and  $P$  does not visit any internal vertex of  $H_{x'}$ . Therefore, by Lemma 88, the internal vertices of  $H_{x'}$  are visited by exactly  $x'$  paths of type  $(a_{\text{in}}, a_{\text{out}})$ . Thus  $\mathcal{P}$  contains one path of type  $(c_{\text{in}}, c_{\text{out}})$  and exactly  $\sum_{x' \in X \setminus \{x\}} x' = S - x$  paths of type  $(a_{\text{in}}, a_{\text{out}})$ . ◀

► **Lemma 90.** *Assuming ETH, there is no  $f(p)n^{o(p)}$  time algorithm for CONSTRAINED CLOSED WALK on graphs of pathwidth at most  $p$  for any computable function  $f$ .*

**Proof.** The proof is by reduction from EDGE BALANCING on a directed graph  $D$  with  $k$  vertices  $w_1, \dots, w_k$ . We construct a CONSTRAINED CLOSED WALK instance on a directed graph  $D^*$  the following way. First, let us introduce the vertices  $w_1, \dots, w_k$  into  $D^*$ , as well as two auxiliary vertices  $c_{\text{in}}$  and  $c_{\text{out}}$ . For every edge  $e = (w_{i_1}, w_{i_2}) \in E(D)$  with a set  $X_e$

of integers associated to it in the EDGE BALANCING instance, we construct a copy of the gadget  $H_{X_e}$  defined by Lemma 89 and identify external vertices  $a_{\text{in}}, a_{\text{out}}, c_{\text{in}}, c_{\text{out}}$  of the gadget  $H_{X_e}$  with vertices  $w_{i_1}, w_{i_2}, c_{\text{in}}, c_{\text{out}}$  of  $D^*$ , respectively. Let  $S_e = \sum_{x \in X_e} x$  for every edge  $e \in V(D)$ , let  $S_i^+ = \sum_{e \in \delta_D^+(w_i)} S_e$ , let  $S_i^- = \sum_{e \in \delta_D^-(w_i)} S_e$ , and let  $S^* = \sum_{e \in E(D)} X_e = \sum_{i=1}^k S_i^+ = \sum_{i=1}^k S_i^-$ . We further extend  $D^*$  the following way.

1. For every  $1 \leq i \leq k$ , we introduce a set  $\mathcal{P}_i^+$  of  $S_i^+$  paths of length two from  $c_{\text{in}}$  to  $w_i$  (that is, each of these paths consists of vertex  $c_{\text{in}}$ , vertex  $w_i$ , and one extra newly introduced vertex).
2. For every  $1 \leq i \leq k$ , we introduce a set  $\mathcal{P}_i^-$  of  $S_i^-$  paths of length two from  $w_i$  to  $c_{\text{out}}$ .
3. We introduce a set  $\mathcal{P}^*$  of  $S^* + |E(D)|$  paths of length two from  $c_{\text{out}}$  to  $c_{\text{in}}$ .

Let  $Z := \{w_1, \dots, w_k, c_{\text{in}}, c_{\text{out}}\}$ ; note that  $Z$  form an independent set in  $G^*$  (as the external vertices of each gadget are independent). We define  $U := V(D^*) \setminus Z$  to be the set of vertices that have to be visited exactly once. This completes the description of the reduction.

Observe that if we remove  $Z$  from  $D^*$ , then what remains is the disjoint union of the internal vertices of the gadgets  $H_{X_e}$ , which have constant pathwidth by Lemma 89. As removing a vertex can decrease pathwidth at most by one, it follows that  $D^*$  has pathwidth  $|Z| + O(1) = O(k)$ . Thus if we are able to show that the constructed instance  $D^*$  of CONSTRAINED CLOSED WALK is a yes-instance if and only if  $D$  is a yes-instance of EDGE BALANCING, then this implies that an  $f(p)n^{o(p)}$  time algorithm for CONSTRAINED CLOSED WALK on graphs of pathwidth  $p$  can be used to solve EDGE BALANCING on  $k$  vertex graphs in time  $(k)n^{o(k)}$ , which would contradict ETH by Lemma 86.

**Balanced assignment  $\chi \Rightarrow$  closed walk.** Suppose that balanced assignment  $\chi : E(D) \rightarrow \mathbb{Z}^+$  is a solution to the EDGE BALANCING instance. For every  $e = (w_{i_1}, w_{i_2}) \in E(D)$ , the construction of the gadget  $H_{X_e}$  implies that  $H_{X_e}$  can be satisfied by a collection  $\mathcal{P}_e$  of paths having type  $(c_{\text{in}}, c_{\text{out}}) + (S_e - \chi(e)) \times (w_{i_1}, w_{i_2})$ . Let  $\mathcal{P}$  be a collection of paths that is the union of the set  $\mathcal{P}^*$ , the sets  $\mathcal{P}_i^+$  and  $\mathcal{P}_i^-$  for  $1 \leq i \leq k$ , and the set  $\mathcal{P}_e$  for  $e \in E(G)$ . Observe that every vertex of  $U$  is contained in exactly one path in  $\mathcal{P}$  and the paths in  $\mathcal{P}$  are edge disjoint. Let  $H^*$  be the subgraph of  $D^*$  formed by the union of every path in  $\mathcal{P}$ . It is easy to see that  $H^*$  is connected: every path in  $\mathcal{P}$  has endpoints in  $Z$  and the paths in  $\mathcal{P}^*, \mathcal{P}_i^-, \mathcal{P}_i^+$  ensure that every vertex of  $Z$  is in the same component of  $H^*$ . It is also clear that every vertex of  $U$  has indegree and outdegree exactly 1, as each vertex in  $U$  is visited by exactly one path in  $\mathcal{P}$ . We show below that every vertex of  $Z$  is balanced in  $H^*$  (its indegree equals its outdegree). If this is true, then  $H^*$  has a closed Eulerian walk, which gives a closed walk in  $G^*$  visiting every vertex at least once and every vertex in  $U$  exactly once, what we had to show.

The endpoints of every path in  $\mathcal{P}$  are in  $Z$ , hence every vertex of  $U$  is balanced in  $H^*$  (in particular has indegree and outdegree exactly 1). Consider now a vertex  $w_i$ .

- For every  $e \in \delta_D^+(w_i)$ , the set  $\mathcal{P}_e$  contains  $S_e - \chi(e)$  paths starting at  $w_i$ .
- For every  $e \in \delta_D^-(w_i)$ , the set  $\mathcal{P}_e$  contains  $S_e - \chi(e)$  paths ending at  $w_i$ .
- The set  $\mathcal{P}_i^+$  contains  $S_i^+$  paths ending at  $w_i$ .
- The set  $\mathcal{P}_i^-$  contains  $S_i^-$  paths starting at  $w_i$ .

As these paths are edge disjoint, the difference between the outdegree and the indegree of  $w_i$  in  $H^*$  is  $(S_i^- + \sum_{e \in \delta_D^+(w_i)} (S_e - \chi(e))) - (S_i^+ + \sum_{e \in \delta_D^-(w_i)} (S_e - \chi(e))) = (S_i^- + S_i^+ - \chi(\delta_D^+(w_i))) - (S_i^+ + S_i^- - \chi(\delta_D^-(w_i))) = 0$ , since  $\chi$  is balanced at  $w_i$ . Consider now vertex  $c_{\text{in}}$ .

- For every  $1 \leq i \leq k$ , the set  $\mathcal{P}_i^+$  contains  $S_i^+$  paths starting at  $c_{\text{in}}$ .
- For every  $e \in E(D)$ , the set  $\mathcal{P}_e$  contains one path starting at  $c_{\text{in}}$ .
- The set  $\mathcal{P}^*$  contains  $S^* + |E(D)|$  paths ending at  $c_{\text{in}}$ .

It follows that  $c_{\text{in}}$  is balanced in  $H^*$  with indegree and outdegree exactly  $S^* + |E(D)| = \sum_{i=1}^k S_i^+ + |E(D)|$  and a similar argument shows the same for  $c_{\text{out}}$ . Thus we have shown that the CONstrained CLOSED WALK instance has a solution.

**Closed walk  $\Rightarrow$  balanced assignment  $\chi$ .** For the reverse direction, suppose that the constructed CONstrained CLOSED WALK instance has a solution (a closed walk  $W$ ). The closed walk can be split into a collection  $\mathcal{P}$  of walks with endpoints in  $Z$  and every internal vertex in  $U$ . In fact, these walks are paths: (1) as each vertex of  $U$  is visited only once, the internal vertices of each walk are distinct, (2) the walk cannot be a cycle, since we have stated earlier that no gadget has a cycle through an external vertex. When defining the sets  $\mathcal{P}^*$ ,  $\mathcal{P}_i^+$ ,  $\mathcal{P}_i^-$ , we introduced a large number of vertices into  $D^*$  with indegree and outdegree 1. The fact that these vertices are visited implies that  $\mathcal{P}$  has to contain the set  $\mathcal{P}^*$  and the sets  $\mathcal{P}_i^+$  and  $\mathcal{P}_i^-$  for every  $1 \leq i \leq k$ . Moreover, every path of  $\mathcal{P}$  not in these sets contains an internal vertex of some gadget  $H_{X_e}$  (here we use that  $Z$  is independent) and a path of  $\mathcal{P}$  cannot contain the internal vertices of two gadgets (as this would imply that it has an internal vertex in  $Z$ ). Therefore, the remaining paths can be partitioned into sets  $\mathcal{P}_e$  for  $e \in E(D)$  such that the internal vertices of  $H_{X_e}$  are used only by the paths in  $\mathcal{P}_e$ . This means that the set  $\mathcal{P}_e$  satisfies gadget  $H_{X_e}$ . If  $e = (w_{i_1}, w_{i_2})$ , then it follows by Lemma 89 that  $\mathcal{P}_e$  has type  $(c_{\text{in}}, c_{\text{out}}) + (S_e - \chi(e))(w_{i_1}, w_{i_2})$  for some integer  $\chi(e) \in X_e$ . In particular, this means that  $\mathcal{P}_e$  contains  $S_e - \chi(e)$  paths starting at  $w_{i_1}$  and the same number of paths ending at  $w_{i_2}$ .

We claim that  $\chi$  form a solution of the EDGE BALANCING problem. Consider a vertex  $w_i$ . Taking into account the contribution of the paths in  $\mathcal{P}_i^-$  and  $\mathcal{P}_e$  for  $e \in \delta_D^+(w_i)$ , we have that the outdegree of  $w_i$  in the walk  $W$  is exactly  $S_i^- + \sum_{e \in \delta_D^+(w_i)} (S_e - \chi(e)) = S_i^+ + S_i^- - \chi(\delta_D^+(w_i))$ . Taking into account the contribution of the paths in  $\mathcal{P}_i^+$  and  $\mathcal{P}_e$  for  $e \in \delta_D^-(w_i)$ , we have that the indegree of  $w_i$  in the walk  $W$  is exactly  $S_i^+ + \sum_{e \in \delta_D^-(w_i)} (S_e - \chi(e)) = S_i^- + S_i^+ - \chi(\delta_D^-(w_i))$ . As the indegree of  $w_i$  in  $W$  is clearly the same as its outdegree, these two values have to be equal. This is only possible if  $\chi(\delta_D^+(w_i)) = \chi(\delta_D^-(w_i))$ , that is  $\chi$  is balanced at  $w_i$ . As this is true for every  $1 \leq i \leq k$ , it follows that the EDGE BALANCING instance has a solution.  $\blacktriangleleft$

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