Packing Cycles through Prescribed Vertices

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Abstract

The well-known theorem of Erdős and Pósa says that a graph G has either k vertex-disjoint cycles or a vertex set X of order at most $f(k)$ such that $G \setminus X$ is a forest. Starting with this result, there are many results concerning packing and covering cycles in graph theory and combinatorial optimization.

In this paper, we generalize Erdős-Pósa's result to cycles that are required to go through a set S of vertices. Given an integer k and a vertex subset S (possibly unbounded number of vertices) in a given graph G, we prove that either G has k vertex-disjoint cycles, each of which contains at least one vertex of S, or G has a vertex set X of order at most $f(k) = 40k^2 \log_2 k$ such that $G \setminus X$ has no cycle that intersects S.

1 Introduction

Packing and covering vertex-disjoint cycles are one of the central areas in both graph theory and theoretical computer science. The starting point of this research area goes back to the following well-known theorem due to Erdős and Pósa [1] in early 1960's.

Theorem 1.1 (Erdős and Pósa [1]) *For any integer k and any graph G, either G contains k vertex-disjoint cycles or a vertex set X of order at most $c \cdot k \log k$ (for some constant c) such that $G \setminus X$ is a forest.*

In fact, Theorem 1.1 gives rise to the well-known Erdős-Pósa property. A family $\mathcal{F}$ of graphs is said to have the *Erdős-Pósa property*, if for every integer k there is an integer $f(k, \mathcal{F})$ such that every graph G contains either k vertex-disjoint subgraphs each isomorphic to a graph in $\mathcal{F}$ or a set C of at most $f(k, \mathcal{F})$ vertices such that $G \setminus C$ has no subgraph isomorphic to a graph in $\mathcal{F}$. The term *Erdős-Pósa property* arose because of Theorem 1.1 which proves that the family of cycles has

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this property. Other families of graphs having Erdős-Pósa property are the one of even cycles [9] and the one of directed cycles in a digraph [5]. Furthermore, the family of the minors of a fixed planar graph has Erdős-Pósa property [6].

Theorem 1.1 is about both "packing," i.e., k vertex-disjoint cycles and "covering," i.e., at most $f(k)$ vertices that hit all the cycles in G. Starting with this result, there is a host of results in this direction. Packing appears almost everywhere in extremal graph theory, while covering leads to the well-known concept "feedback set" in theoretical computer science. Also, the cycle packing problem, which asks whether or not there are k vertex-disjoint cycles in an input graph G, is a well-known problem, e.g., [3].

In addition to the feedback set problem, a natural generalization of the cycle packing problem has been studied extensively in theoretical computer science. The problem called "S-cycle packing" is that we are given a graph G and a subset S of its vertices, and the goal is to find among the cycles that intersect S a maximum number of vertex-disjoint (or edge-disjoint) ones. See [3] for more details. As pointed out there, this problem is rather close to the well-known "disjoint paths" problem [7], and approximation algorithms to find an S-cycle packing have been studied extensively. On the other hand, it seems that the Erdős-Pósa property for S-cycles has not been explored yet; our main result is generalizing Theorem 1.1 to the "subset" version and thereby proving that Erdős-Pósa property indeed holds for the S-cycles.

Let us formally define the S-cycle packing. Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E. For $S \subseteq V$, an *S-cycle* is a cycle which has a vertex in S. We denote by $\nu_S(G)$ the maximum k such that G has k S-cycles that are pairwise vertex-disjoint. The minimum size of a vertex subset that meets all S-cycles is denoted by $\tau_S(G)$. Our main result is the following:

Theorem 1.2 *Let k be a positive integer. Then any graph $G = (V, E)$ with $S \subseteq V$ satisfies $\nu_S(G) \geq k$ or $\tau_S(G) \leq 40k^2 \log_2 k$.*

Very recently, Pontecorvi and Wollan [4] improved the bound to $O(k \log k)$. Their bound is tight: for the case where S coincides with V, it is known that there exists a graph with $\tau_S(G) = \Omega(k \log k)$.

In the next section, we give some lemmas needed for the proof of Theorem 1.2. Our main proof follows in Section 3.

2 Packing Paths through Prescribed Vertices

Let $G = (V, E)$ be a graph, and let $A, B \subseteq V$. A *separation* in G is an ordered pair (X, Y) of subsets of V with $X \cup Y = V$ so that G has no edges between $X \setminus Y$ and $Y \setminus X$. Its *order* is $|X \cap Y|$.

For $S, T \subseteq V$ with $S \cap T = \emptyset$, an *S-path with respect to T* is a path with end vertices in T such that it has at least one vertex of S. The end vertices of an S-path are called the *terminals*. We obtain the following theorem, which says that the family of S-paths has the Erdős-Pósa property. This follows from the odd path theorem by Geelen et al. [2].

Theorem 2.1 *Let $G = (V, E)$ be a graph, and $S, T \subseteq V$ with $S \cap T = \emptyset$. Then, if G has no k vertex-disjoint S-paths with respect to T, then there exists $Z \subseteq V$ with $|Z| \leq 2k - 2$ that intersects every S-path with respect to T.*

Theorem 2.2 (Geelen et al. [2]) *Let $G = (V, E)$ be a graph with $T \subseteq V$. Then, if G has no k vertex-disjoint paths each of which has an odd number of edges and its end points in T, then there exists $Z \subseteq V$ with $|Z| \leq 2k - 2$ that intersects every such path.*

Proof of Theorem 2.1: We construct a graph G' from G as follows. We first subdivide every edge of G with a new vertex. Moreover, for every vertex s in S, we add new edges between s and all its original neighbors. Then, if a path connecting two vertices of T in G' is odd, then the corresponding path in G has to contain a vertex of S (otherwise it uses only the subdivided edges and hence its length is even). Moreover, an S-path with respect to T in G gives rise to an odd path connecting two vertices of T in G'. To see this, consider a path P in G of length ℓ that goes through a vertex $s \in S$. Using the subdivided edges, there is a corresponding path of length 2ℓ in G'. We can make this path one edge shorter by using one of the edges that connect s with its neighbor in P. Therefore, G' has k vertex-disjoint odd paths with end vertices in T if and only if G has k vertex-disjoint S-paths with respect to T. Thus Theorem 2.1 follows from Theorem 2.2. □

3 Erdős-Pósa Property for Cycles through Prescribed Vertices

In this section, we shall prove Theorem 1.2. We first show in Lemma 3.1 below that if a long S-cycle C has many vertex-disjoint S'-paths, where $S' = S \setminus V(C)$, then a graph has k vertex-disjoint S-cycles.

Lemma 3.1 *Let $G = (V, E)$ be a graph with $S \subseteq V$. Let k be a positive integer with $k \geq 2$, and define $K = 4k \log_2(k+10)$. Assume that G has a cycle C of length at least $2K$ and let $S' = S \setminus V(C)$. If G has K vertex-disjoint S'-paths with respect to $V(C)$, then there exist k vertex-disjoint S-cycles.*

Proof: Consider the subgraph G' of G formed by C and by the K vertex-disjoint paths. It is sufficient to show that G' has k vertex-disjoint cycles. Indeed, since C is the only cycle of G' which may not be an S-cycle and C intersects every other cycle in G', every cycle in a collection of k vertex-disjoint cycles is an S-cycle. Clearly, G' has $2K$ vertices of degree 3 and every other vertex is of degree 2. Therefore, by a result of Simonovits [8], G' has at least $\lfloor \frac{1}{4}(2K)/\log_2(2K) \rfloor$ vertex-disjoint cycles. It can be checked that $2K \leq (k+10)^2$ for every $k \geq 1$, thus $\lfloor \frac{1}{4}(2K)/\log_2(2K) \rfloor \geq \lfloor K/(2\log_2(k+10)^2) \rfloor \geq k$, that is, there are k vertex-disjoint cycles in G'. □

We prove Theorem 1.2 by induction on k. If $k = 1$, $\nu_S(G) < 1$ implies $\tau_S(G) = 0$, and we are done. We henceforth suppose that, for $\ell < k$, any graph G satisfies either $\nu_S(G) \geq \ell$ or $\tau_S(G) \leq 40 \cdot \ell^2 \log_2 \ell$.

To prove the statement for k, assume to the contrary that there exists a graph G with $\nu_S(G) < k$ and $\tau_S(G) > 40k^2 \log_2 k$. Let C be an S-cycle that contains as few vertices of S as possible. We denote $S' = S \setminus V(C)$.

Let $K = 4k \log_2(k+10)$. Note that $K \leq 15k \log_2 k$, which follows from $\log_2(k+10) = \log_2 k + \log_2(1 + \frac{10}{k})$ and $\log_2(1 + \frac{10}{k}) \leq \log_2 6 \log_2 k$ for $k \geq 2$. First suppose that C has length less than $2K$. Since $\nu_S(G \setminus V(C)) < k-1$ by $\nu_S(G) < k$, the induction hypothesis implies that $\tau_S(G \setminus V(C)) \leq 40(k-1)^2 \log_2(k-1)$. Therefore, $\tau_S(G) \leq 2K + \tau_S(G \setminus V(C)) \leq 30k \log_2 k + 40(k-1)^2 \log_2(k-1) \leq 40k^2 \log_2 k$, which is a contradiction. Thus C has length at least $2K$.

Since G has no k vertex-disjoint S-cycles, it follows from Lemma 3.1 that G has no K vertex-disjoint S'-paths with respect to $V(C)$. By Theorem 2.1 and $S' \cap V(C) = \emptyset$, there is a vertex subset $Z \subseteq V$ of size $\leq 2K-2$ such that $G \setminus Z$ has no S'-path with respect to $T = V(C) \setminus Z$. Note that T is nonempty by $|V(C)| > |Z|$.

Let $Z' = Z \cup \{s\}$ for an arbitrary vertex s of $T \cap S$ and let $Z' = Z$ if $T \cap S = \emptyset$. Since $|Z'| \leq |Z| + 1 \leq 2K - 1 < \tau_S(G)$, the graph $G \setminus Z'$ has an S-cycle D. By the minimality of C, the cycle D has a vertex v of S'. Otherwise the nonempty set $D \cap S$ would be a subset of $C \cap S$, and as

Z' contains an element of $C \cap S$, we would have $D \cap S \subset C \cap S$. Since $G \setminus Z$ has no two internally disjoint paths from v to T, it follows from Menger's theorem that $G \setminus Z$ has a separation (X, Y) with $|X \cap Y| \leq 1$, $T \subseteq X$, $v \in Y$, and $V(D) \subseteq Y$. By letting $A = X \cup Z$ and $B = Y \cup Z$, the graph G has a separation (A, B) of order $\leq 2K - 1$ such that both sides of the separation have S-cycles C and D, respectively. Note that since G has such two disjoint S-cycles, we may assume $k \geq 3$.

Since $\nu_S(G) < k$, the existence of C and D implies $\nu_S(G \setminus A), \nu_S(G \setminus B) < k - 1$. More precisely, by $\nu_S(G \setminus A) + \nu_S(G \setminus B) < k$, we have $\nu_S(G \setminus A) < i$ and $\nu_S(G \setminus B) < k - i + 1$ for some $i \in \{2, \ldots, k-1\}$. Hence the induction hypothesis implies that $\tau_S(G \setminus A) \leq 40 \cdot i^2 \log_2 i$ and $\tau_S(G \setminus B) \leq 40(k - i + 1)^2 \log_2(k - i + 1)$. Since every S-cycle that is not a cycle of $G \setminus A$ or $G \setminus B$ meets $A \cap B$, we have

$$\tau_S(G) \leq \tau_S(G \setminus A) + \tau_S(G \setminus B) + |A \cap B| \leq 40 \left(i^2 \log_2 i + (k-i+1)^2 \log_2(k-i+1)\right) + 2K - 1.$$

Let $g(i) = i^2 \log_2 i + (k - i + 1)^2 \log_2(k - i + 1)$. Since g is a convex function over $2 \leq i \leq k - 1$, we have $g(i) \leq \max\{g(2), g(k-1)\} = (k-1)^2 \log_2(k-1) + 4$. Therefore, by $K \leq 15k \log_2 k$ for $k \geq 2$, we have

$$\begin{aligned} \tau_S(G) &\leq 40(k-1)^2 \log_2(k-1) + 160 + 30k \log_2 k \\ &\leq 40k^2 \log_2 k + (-50k + 40) \log_2 k + 160. \end{aligned}$$

Hence for $k \geq 3$ we obtain $\tau_S(G) \leq 40k^2 \log_2 k$. This completes the proof of Theorem 1.2.

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