A short proof of the NP-completeness of minimum sum interval coloring

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Abstract

In the minimum sum coloring problem we have to assign positive integers to the vertices of a graph in such a way that neighbors receive different numbers and the sum of the numbers is minimized. Szkalicki [9] has shown that minimum sum coloring is NP-hard for interval graphs. Here we present a simpler proof of this result.

Keywords: computational complexity, graph coloring, minimum sum coloring, chromatic sum, interval graph

1 Introduction

Kubicka [5] and Supowit [8] independently introduced the concept of chromatic sum. If the vertices of a graph are properly colored using positive integers, then the *sum* of the coloring is the sum of the numbers assigned to the vertices. The *chromatic sum* of a graph is the smallest sum that a proper coloring can have. In the *minimum sum coloring* problem we have to find a coloring that minimizes the sum. Chromatic sum has important applications in VLSI routing [9, 8] and scheduling [1].

The combinatorial properties of the chromatic sum and the complexity of minimum sum coloring received a lot of attention in the literature. It was shown by Kubicka and Schwenk [6] that minimum sum coloring is NP-hard in general, but can be solved in polynomial time for trees. The problem remains NP-hard when restricted to bipartite graphs [2], planar graphs [4, 7] and interval graphs [9]. The proof in [9] for the NP-hardness of minimum sum coloring on interval graphs is quite involved, the aim of this note is to give a simpler proof of this result.

2 The reduction

A graph G is a *circular arc graph* if it is the intersection graph of arcs on a circle, that is, the vertices of G can be placed in one-to-one correspondence with a set of arcs in such a way that two vertices in G are adjacent if and only if the corresponding two arcs intersect each other. Our reduction is from circular arc coloring, whose NP-hardness was established in [3].

Theorem 2.1. *Minimum sum coloring restricted to interval graphs is* NP*-hard.*

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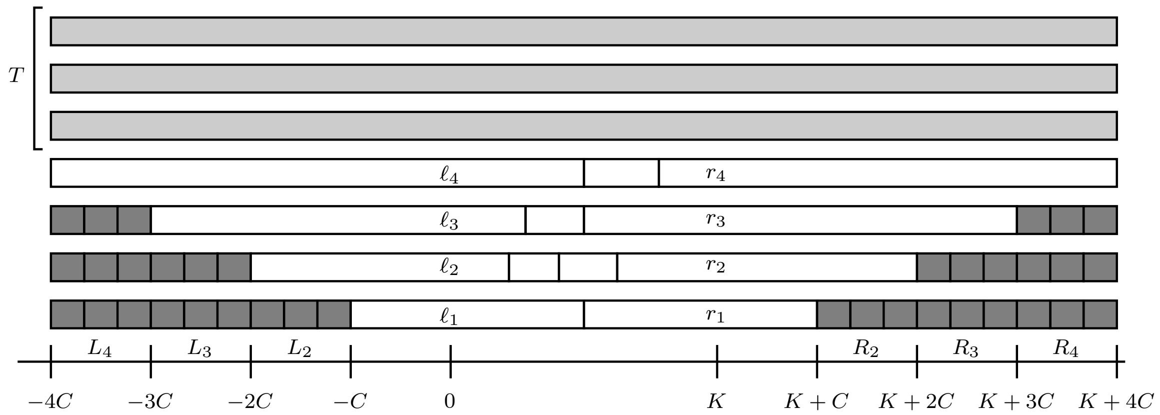

Figure 1: The interval graph G'' for $k = 4$ and $C = 3$. The white rectangles show the $n + k$ intervals of G', the light gray rectangles are the intervals in T, the dark gray rectangles are the intervals in L_i, R_i.

Proof. Given a circular arc graph G and an integer k, it has to be decided whether G can be colored with k colors. A circular arc representation of G can be found in polynomial time [10]. It can be assumed that the arcs are open: two arcs that share only an end point do not intersect each other. Let x be an arbitrary point on the circle that is not the end point of any of the arcs. It can be assumed that x is contained in exactly k arcs: if x is contained only in the arcs $a_1, \dots, a_{k'}$, then we can add $k - k'$ sufficiently small arcs that intersect only $a_1, \dots, a_{k'}$. Clearly, this cannot increase the chromatic sum above k. The number of arcs in the circular arc graph will be denoted by n. Let $C := 2k(n + k)$.

Let $a_1, \dots, a_k$ be the arcs that contain x. Split each arc a_i into two parts ℓ_i and r_i at point x. Let x be the clockwise (resp. counter-clockwise) end point of ℓ_i (resp. r_i). Now this graph G' is an interval graph, since x is contained in neither interval. Therefore G' has an interval representation where the left end point of each interval ℓ_i is 0, the right end point of each r_i is K, and no interval extends to the left of 0 or to the right of K. We can modify the left end point of ℓ_i to $-iC$ and the right end point of r_i to $K + iC$, this does not change the interval graph. It is clear that if G is k-colorable, then G' is k-colorable as well. The converse it not necessarily true: G is k-colorable only if G' has a k-coloring where ℓ_i and r_i receive the same color for every $1 \le i \le k$.

We add new intervals as follows (see Fig. 1). For every $2 \le i \le k$, add a set L_i of $C(i - 1)$ intervals containing $i - 1$ copies of the intervals $(-iC + j, -iC + j + 1)$ where $0 \le j \le C - 1$. Similarly, the set R_i contains $i - 1$ copies of the intervals $(K + (i - 1)C + j, K + (i - 1)C + j + 1)$ where $0 \le j \le C - 1$. Moreover, add a set T that contains C copies of the interval $(-kC, K + kC)$. We claim that the resulting interval graph G'' has chromatic sum less than

$$B := 2\sum_{i=2}^{k} Ci(i-1)/2 + \sum_{i=k+1}^{k+C} i + C$$

if and only if the original circular arc graph G can be colored with k colors.

Assume that G can be colored with k colors, then it has a coloring where arc a_i receives color i. Thus the interval graph G' has a k-coloring where ℓ_i and r_i receive color i. We show that this coloring of G' can be extended to a coloring of G'' with sum less than B. The $n + k$ intervals of G' use only colors not greater than k, hence their total sum is at most $k(n + k) < C$. For every $2 \le i \le k$, the intervals in L_i can be colored using the first $i - 1$ colors, since these colors are not used by the intervals ℓ_i, ℓ_{i+1}, $\dots$, ℓ_k. Therefore the sum of the intervals in L_i is $C\sum_{j=1}^{i-1} j = Ci(i - 1)/2$. The situation is similar with the intervals in R_i. The C intervals in T can be colored using colors $k+1$, $k+2$, ..., $k+C$, hence their sum is $\sum_{i=k+1}^{k+C} i$. Thus the total sum is less than B, as required.

Now assume that G'' has a coloring with sum less than B. It can be assumed that the intervals in T use the last C colors: if color c_1 is used by an interval in T, and a color $c_2 > c_1$ is used by one or more intervals not in T, then exchanging colors c_1 and c_2 does not increase the sum (notice that c_1 cannot be used by more than one interval in T). Since $L_k \cup \{\ell_k\}$ needs at least k colors, thus T uses only colors above k.

We claim that the intervals outside T use only the first k colors. If they use at least $k+1$ colors, then the intervals in T can use only colors above $k+1$, hence their sum is at least $\sum_{i=k+2}^{k+C+1} i = C + \sum_{i=k+1}^{k+C} i$. The total sum of the intervals in L_i (and similarly, in R_i) is at least $Ci(i-1)/2$ in any coloring. Therefore the sum is at least B, a contradiction.

The sum of the intervals in T is at least $\sum_{i=k+1}^{k+C} i$ if they use only colors above k, and the sum of each L_i and R_i is at least $Ci(i-1)/2$. Therefore the total sum of the intervals in L_2, ..., L_k, R_2, ..., R_k, T is at least $B - C$ in every coloring. Furthermore, this means that if L_i has sum at least $Ci(i-1)/2 + C$ (i.e., the sum exceeds the optimum of L_i by at least C), then the total sum is at least B. Therefore L_i has to use the first $i-1$ colors, it is easy to see that if L_i skips a color $c \leq i-1$, then the sum of L_i increases by at least C. This means that L_k forces ℓ_k to use color k. Now ℓ_{k-1} cannot use color k (because of ℓ_k) and cannot use a color below $k-1$ (because of L_{k-1}), hence it has color $k-1$. Continuing for $i = k-2,\ k-3, \ldots, 1$, it follows that ℓ_i has color i. By a similar argument, interval r_i also has color i. Thus the color of ℓ_i and r_i is the same, implying that the circular arc graph G has a k-coloring, what we had to show. ■

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