On the Parameterized Complexity of Finding Separators with Non-Hereditary Properties*

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Abstract. We study the problem of finding small s–t separators that induce graphs having certain properties. It is known that finding a minimum clique s–t separator is polynomial-time solvable (Tarjan 1985), while for example the problems of finding a minimum s–t separator that is a connected graph or an independent set are fixed-parameter tractable (Marx, O'Sullivan and Razgon, manuscript). We extend these results the following way:

(1) Finding a minimum c-connected s–t separator is FPT for $c = 2$ and $W[1]$-hard for any $c \geq 3$.
(2) Finding a minimum s–t separator with diameter at most d is $W[1]$-hard for any $d \geq 2$.
(3) Finding a minimum r-regular s–t separator is $W[1]$-hard for any $r \geq 1$.
(4) For any decidable graph property, finding a minimum s–t separator with this property is FPT parameterized jointly by the size of the separator and the maximum degree.

We also show that finding a connected s–t separator of minimum size does not have a polynomial kernel, even when restricted to graphs of maximum degree at most 3, unless NP $\subseteq$ coNP/poly.

1 Introduction

One of the classic topics in combinatorial optimization and algorithmic graph theory deals with finding cuts and separators in graphs. Recently, the study of this type of problems from a parameterized complexity point of view has attracted a large amount of interest [5, 6, 11, 14–21]. Given a graph G and two vertices s and t of G, a subset of vertices $S \subseteq V(G) \setminus \{s, t\}$ is an s–t *separator* if s and t appear in different connected components of the graph $G - S$. In separation

* This work is supported by the Research Council of Norway and by the European Research Council (ERC) grant 280152.

problems, we are typically looking for *small* separators S. A natural extension of the problem is to demand $G[S]$, i.e., the subgraph induced by S, to satisfy a certain property. (For convenience, when the graph $G[S]$ has a certain property, we will say that the set S itself also has this property; for example, we say that a set $S \subseteq V(G)$ is 2-connected if $G[S]$ is 2-connected.) A classical result in this direction by Tarjan [22] shows that finding small *clique* separators is polynomial-time solvable. To our knowledge, this is the only known polynomial-time solvable problem of this type. Therefore, we explore here the problem from the viewpoint of parameterized complexity.

Parameterized complexity associates with every instance of a problem a non-negative integer k, called the *parameter*. As is common in the parameterized study of separator problems, the parameter k in this paper will always be the size of the separator we are looking for. We use n and m to denote the number of vertices and edges, respectively, in the input graph. A parameterized problem is *fixed-parameter tractable* (or FPT) if it can be solved in time $f(k) \cdot n^{O(1)}$ for some function f that only depends on k [9]. By showing that a parameterized problem is W[1]-hard, we can give strong evidence that it is unlikely to be FPT; we refer to [9] for more background on parameterized complexity.

For any graph class $\mathcal{G}$, let us consider the following parameterized problem.

$\mathcal{G}$-Separator
Input: A graph G, two vertices s and t of G, and an integer k.
Parameter: k.
Question: Does G have an s–t separator S of size at most k such that $G[S] \in \mathcal{G}$?

If $\mathcal{G}$ is the class of all complete graphs, then $\mathcal{G}$-Separator is polynomial-time solvable by the result of Tarjan [22]. Furthermore, Marx et al. [18, 19] showed that the problem is fixed-parameter tractable for many natural classes $\mathcal{G}$. We say that $\mathcal{G}$ is *hereditary* if, for every graph in $\mathcal{G}$, each of its induced subgraphs also belongs to $\mathcal{G}$.

Theorem 1 ([18, 19]). *For any decidable and hereditary graph class $\mathcal{G}$, the $\mathcal{G}$-Separator problem can be solved in time $f_{\mathcal{G}}(k) \cdot (n + m)$.*

For example, by letting $\mathcal{G}$ be the class of all graphs without edges, Theorem 1 shows that finding an independent set of size at most k separating s and t is FPT. The proof is based on a combinatorial statement called Treewidth Reduction Theorem, which shows (roughly speaking) that all the inclusionwise minimal s–t separators lie in a bounded-treewidth part of the graph and hence they can be found efficiently. Note that if $\mathcal{G}$ is hereditary, then we can always assume that the separator is inclusionwise minimal (otherwise we can remove vertices from it without leaving $\mathcal{G}$).

Theorem 1 naturally raises the question what the parameterized complexity of the $\mathcal{G}$-Separator problem is for graph classes $\mathcal{G}$ that are *not* hereditary.

Perhaps the most natural candidate is the class of *connected* graphs. The Connected Separator problem of deciding whether a graph G has a connected s–t separator of size at most k has been studied by Marx et al. [19]. Although it is not immediately clear how to apply the Treewidth Reduction Theorem to this problem, Marx et al. [19] managed to extend their framework from [18] to prove the following result.

Theorem 2 ([19]). *The* Connected Separator *problem can be solved in time $f(k) \cdot (n + m)$.*

Our results. Motivated by the results in [18, 19], we study the problem of finding small s–t separators satisfying different non-hereditary properties. Let us focus on the three tractable classes mentioned above (connected graphs, cliques, independent sets) and try to investigate further related classes.

As Connected Separator is FPT, it is natural to explore what happens if we require higher-order connectivity. It turns out that, somewhat surprisingly, finding a c-connected s–t separator of size at most k remains FPT also for $c = 2$, but becomes $W[1]$-hard for any $c \geq 3$. In order to prove this, we show that the natural c-connected generalization of Steiner Tree is FPT for $c = 2$ and $W[1]$-hard for any $c \geq 3$. This result could be of independent interest.

We can generalize the class of cliques by considering the class of graphs with diameter at most d. We show that the problem of finding an s–t separator of size at most k that induces a graph with diameter d in G is $W[1]$-hard for any $d \geq 2$. This is in stark contrast with the case $d = 1$, as the problem of finding a *clique* separator of size at most k is known to be solvable in polynomial time [22].

Independent sets can be thought of as 0-regular graphs. This motivates exploring the problem of finding an r-regular s–t separator. We show that, unlike the $r = 0$ case which is FPT by Theorem 1, for any $r \geq 1$, it is $W[1]$-hard to decide if a graph G has an r-regular s–t separator of size at most k.

All the above results are on general graphs, i.e., graph G can be arbitrary. It comes as no surprise that the problem is much easier restricted to bounded-degree graphs. In particular, finding a small connected separator is FPT due to the fact that a bounded-degree graph contains only a bounded number of small connected sets. More interestingly, we show in Section 4 that for *every* (not necessarily hereditary) decidable graph class $\mathcal{G}$, the $\mathcal{G}$-Separator problem can be can be solved in time $h_{\mathcal{G}}(k, \Delta) \cdot m \log n$ on graphs of maximum degree at most Δ. We prove this by showing that the following problem can be solved in time $f(|V(H)|, \Delta) \cdot m \log n$ on graphs of maximum degree at most Δ: Given two graphs G and H and two vertices s and t of G, decide whether G has an s–t separator S such that $G[S]$ is isomorphic to H. This means that we can solve the $\mathcal{G}$-Separator problem by simply trying all members H of $\mathcal{G}$ having k vertices.

Finally, we investigate the existence of polynomial kernels for the problem of finding small s–t separators. A parameterized problem is said to admit a *kernel* if there is a polynomial-time algorithm that transforms each instance of the problem into an *equivalent* instance whose size and parameter value are bounded from above by $g(k)$ for some (possibly exponential) function g. It is known that a parameterized problem is FPT if and only if it is decidable and admits a kernel [9].

In the desirable case that $g(k)$ is a polynomial in k, we say that the problem admits a *polynomial kernel.* Many problems have been shown to admit polynomial kernels, including classes of problems that are covered by some kernelization meta-theorems [3, 12]. Recently developed methods for proving non-existence of polynomial kernels, up to some complexity theoretical assumptions [2, 4, 13], significantly contributed to the establishment of kernelization as an important and rapidly growing subfield of parameterized complexity.

Although the Connected Separator problem is FPT by Theorem 2 and therefore admits a kernel [9], we show in Section 5 that this problem does not have a *polynomial* kernel, even when restricted to input graphs of maximum degree at most 3, unless NP $\subseteq$ coNP/poly. This means that techniques other than kernelization (e.g., treewidth reduction) seem to be essential for the efficient solution of the problem *even on bounded-degree graphs.*

2 Finding s–t Separators with Higher Connectivity

Theorem 2 states that the problem of finding a connected s–t separator of size at most k is FPT. In this section, we study the parameterized complexity of finding s–t separators of higher connectivity. A graph $G = (V, E)$ is *c-connected* if $|V| > c$ and $G - X$ is connected for every $X \subseteq V$ with $|X| < c$. Menger's Theorem provides an equivalent definition (see [8]): a graph is c-connected if any two of its vertices can be joined by c internally vertex-disjoint paths. For any integer $c \geq 1$, the c-Connected Separator problem takes as input a graph G, two vertices s and t of G, and an integer k (the parameter), and asks whether there is an s–t separator of size at most k that induces a c-connected graph. Theorem 2 states that this problem is FPT when $c = 1$. Interestingly, it turns out that the problem remains FPT for $c = 2$, but becomes $W[1]$-hard for any $c \geq 3$.

The algorithm in [19] for finding a minimum connected s–t separator uses an FPT algorithm for Steiner Tree as a subroutine. For our purposes, we need to define the following natural c-connected generalization of the Steiner Tree problem. For any integer $c \geq 1$, the c-Connected Steiner problem takes as input a graph G, a set $T \subseteq V(G)$ of terminals and an integer k (the parameter). The objective is to decide whether G has a c-connected subgraph H on at most k vertices such that H contains all the terminals. Such a graph H is called a *solution.* A solution H is *minimal* if no proper subgraph of H is a solution, and H is *minimum* if there is no solution H' with $|V(H')| < |V(H)|$. When $c = 1$, this problem is equivalent to the well-known Steiner Tree problem, which is known to be FPT when parameterized by k [10]. We show below that the c-Connected Steiner problem remains FPT when $c = 2$, but becomes $W[1]$-hard for higher values of c.

A different way of generalizing Steiner Tree would be to require the weaker condition saying that H contains c internally vertex-disjoint paths between any two terminals. The following lemma shows that for $c = 2$ this is almost the same problem, as any minimal solution satisfying the weaker requirement satisfies the stronger requirement as well:

Lemma 1. (★)[1] *Let H be a graph and $T \subseteq V(H)$ a set of vertices such that there are two internally vertex-disjoint paths between any $t_1, t_2 \in T$. If H has no proper subgraph (containing T) having this property, then H is 2-connected.*

We note that for $c \geq 3$, the analog of Lemma 1 is not true. Thus the weaker requirement would result in a different problem, but we do not investigate it further in the current paper.

Our algorithm for 2-Connected Steiner crucially depends on the following structural property of any minimal solution:

Lemma 2. *Let (G, T, k) be an instance of the* 2-Connected Steiner *problem. If H is a minimal solution, then $H - T$ is a forest.*

Proof. Since the lemma trivially holds when $|T| \leq 1$, we assume that $|T| \geq 2$. Suppose H is a minimal solution. We show that every cycle in H contains at least one vertex of T, which implies that $H - T$ is a forest. For contradiction, let C be a cycle in H that contains none of the terminals. We will identify an edge e of C such that it remains true in $H - e$ that there are two internally vertex-disjoint paths between any two terminals. Then by Lemma 1, $H - e$ has a 2-connected subgraph which is a solution, contradicting the minimality of H.

We define a partition T_1, T_2 of the terminals as follows. A terminal $t \in T$ belongs to T_2 if there is another terminal $t' \in T$ such that for every pair P_1, P_2 of internally vertex-disjoint paths between t and t' in H, both P_1 and P_2 use at least one vertex of C, i.e., if t and t' belong to different connected components of $H - V(C)$. Note that in such a case t' is also in T_2. We define $T_1 = T \setminus T_2$.

Let $t \in T_1$. By definition, for any $t' \in T \setminus \{t\}$, there exist two internally vertex-disjoint paths in H between t and t' such that at least one of them does not use any vertex of C. Let H' be the graph obtained from H by deleting any edge of C. Then H' still contains two internally vertex-disjoint paths between t and any $t' \in T \setminus \{t\}$, as any path between t and t' that used the deleted edge can be rerouted on the cycle. Hence, if T_2 is empty, we can delete any edge from C and obtain a new solution, contradicting the minimality of H.

Now suppose $T_2 \neq \emptyset$. Let us define a *shortcut* of C to be a path P of length at least 2 between two vertices a and b of C, such that none of the internal vertices of P are in C. It follows from the definition of T_2 that for each $t \in T_2$, there are two distinct vertices a, b on C such that there are two internally vertex-disjoint paths P_a, P_b from t to a and b, respectively, whose internal vertices are not in C. In other words, for every $t \in T_2$, there is a shortcut of C that contains t. Let M be a shortest subpath of C such that there is a shortcut P^* of C between the endpoints a and b of M. Let $\overline{M}$ be the other path between a and b on the cycle C. Let a' be the neighbor of a on M (possibly $a' = b$). We claim that after removing the edge aa' from H, the obtained graph $H - aa'$ still contains two internally vertex-disjoint paths between each pair of terminals in T_2.

[1] Proofs marked with a star have been omitted due to page restrictions.

By the well-known properties of the 2-connected components of graphs, the relation "being in the same 2-connected component" (or equivalently, the relation "there is a cycle containing both edges") defined on the edges of $H - aa'$ is an equivalence relation. Every edge of $\overline{M}$ is in the same equivalence class of this relation: $\overline{M}$ together with P^* forms a cycle containing all these edges. We claim that every $t \in T_2$ is also in this 2-connected component. As observed above, there is a shortcut P_t going through t. Let M_t be the subpath of the cycle C between the endpoints of P_t avoiding aa'. The paths P_t and M_t together form a cycle. This cycle contains at least one edge of $\overline{M}$, since M_t cannot be a proper subpath of M by the minimality of M. Thus the edges of this cycle are in the same 2-connected component as the edges of $\overline{M}$. We have shown that every $t \in T_2$ is in this 2-connected component of $H - aa'$. Consequently, there are two internally vertex-disjoint paths in $H - aa'$ between any two terminals $t_1, t_2 \in T$, yielding the desired contradiction to the assumption that H is a minimal solution.

We conclude that every cycle in H contains at least one vertex of T, which implies that $H - T$ is a forest. □

Lemma 2 tells us that we have to find an appropriate forest that connects to the terminals in an appropriate way. Fixed-parameter tractability results for finding trees (or more generally, bounded-treewidth graphs) under various technical constraints can usually be obtained using standard application of dynamic programming. Here we need the following variant:

Lemma 3. (★) *Let F be a forest, G an undirected graph, and $c : V(F) \times V(G) \to \mathbb{Z}^+$ a cost function. In time $f(|V(F)|) \cdot n^{O(1)}$, one can find a mapping $\phi : V(F) \to V(G)$ such that $\phi(u)\phi(v) \in E(G)$ for every $uv \in E(F)$ and the total cost $\sum_{v \in V(F)} c(v, \phi(v))$ is minimized.*

The structural observation of Lemma 2 and the algorithm of Lemma 3 allow us to establish the fixed-parameter tractability of the 2-Connected Steiner problem, which could be interesting in its own right. Furthermore, it will be used as a subroutine in our FPT-algorithm for finding a 2-connected s–t separator of size at most k.

Theorem 3. *The* 2-Connected Steiner *problem is FPT.*

Proof. Let (G, T, k) be a yes-instance of the 2-Connected Steiner problem and let H be a minimal solution. By Lemma 2, $H - T$ is a forest. We try all graphs H on at most k vertices that are candidates for being isomorphic to the solution H: that is, H is 2-connected, $T \subseteq V(H)$, and $H - T$ is a forest. The number of such graphs is a function of k only. For each such H, we define a cost function c such that for $x \in V(H - T)$ and $y \in V(G)$, we have $c(x, y) = 0$ if $N_H(x) \cap T \subseteq N_G(y) \cap T$ and $c(x, y) = \infty$ otherwise. In other words, we allow mapping x to y only if every terminal neighbor of x is also a neighbor of y. Let us use the algorithm of Lemma 3 to find a mapping ϕ of $H - T$ into G minimizing the cost. If the cost of ϕ is 0, then ϕ can be extended to a mapping of H into G, showing that H is a subgraph of G, which gives us a solution. Otherwise, we proceed with the next candidate H. If the algorithm finds no solution after processing all candidates, we can safely return "no". $\square$

In order to prove that 2-Connected Separator is FPT, we will make use of the Treewidth Reduction Theorem due to Marx, O'Sullivan and Razgon [18, 19]. In fact, instead of using the Treewidth Reduction Theorem itself, we use a lemma (a slight reformulation of Lemma 2.8 in [18]) that forms its crucial ingredient. In order to state it, we need an additional definition. Let G be a graph and $C \subseteq V(G)$. The graph *torso*(G, C) has vertex set C, and vertices $a, b \in C$ are connected by an edge if $ab \in E(G)$ or if there is a path in G connecting a and b whose internal vertices are not in C.

Lemma 4 ([18]). *Let s and t be two vertices of a graph G, let k be an integer, and let C be the union of all minimal s–t separators in G of size at most k. Then there is an $f(k) \cdot (n+m)$ time algorithm that returns a set $C' \supseteq C \cup \{s,t\}$, such that the treewidth of torso(G, C') is at most $g(k)$.*

Note that even if G has a 2-connected s–t separator S of size at most k, G might not have a *minimal* s–t separator of size at most k that is 2-connected, since 2-connectivity is not a hereditary property. However, G does contain a minimal s–t separator that can be extended to a 2-connected set of size at most k. We call a set $S' \subseteq V(G)$ *k-biconnectable* if there is a 2-connected set $S \subseteq V(G)$ of size at most k such that $S' \subseteq S$.

Observation 4 *Let G be a graph. A set $S' \subseteq V(G)$ is k-biconnectable if and only if (G, S', k) is a yes-instance of the* 2-Connected Steiner *problem.*

The set C' in Lemma 4 contains every minimal s–t separator S' that is k-biconnectable, but there is no guarantee that S' can be extended to a 2-connected set within C'. The next lemma shows that we can extend C' to a larger set C'' such that every k-biconnectable s–t separator $S' \subseteq C'$ can be extended to a 2-connected s–t separator $S \subseteq C''$ of size at most k.

Lemma 5. *Let s and t be two vertices of a graph G, and let k be an integer. There is a set $C'' \subseteq V(G)$ such that the treewidth of torso(G, C'') is bounded by a constant depending only on k and the following holds: if G has a 2-connected s–t separator of size at most k, then G also has a 2-connected s–t separator S of size at most k such that $S \subseteq C''$. Moreover, such a set C'' can be found in time $h(k) \cdot n^{O(1)}$.*

Proof. Let $C' \subseteq V(G)$ be the set of Lemma 4 that contains every minimal s–t separator of G of size at most k, such that the treewidth of $\mathrm{torso}(G, C')$ is bounded by a function of k. Let $K_1, \ldots, K_q$ be the connected components of $G - C'$, and let N_i be the neighborhood of K_i in C' for $1 \leq i \leq q$. By the definition of torso, each N_i forms a clique in $\mathrm{torso}(G, C')$. Since each clique of a graph must appear in a single bag of any tree decomposition of that graph, we have $|N_i| \leq \mathrm{tw}(\mathrm{torso}(G, C')) + 1$, so the size of each N_i is bounded by a function of k only.

Our algorithm for constructing C'' iterates over all $i \in \{1, \ldots, q\}$, all non-empty subsets $X \subseteq N_i$, all graphs $F_{i,X}$ on at most $k - |X|$ vertices, and all possible ways in which the vertices of $F_{i,X}$ can be made adjacent to the vertices of X. For each of those choices, let $G_{i,X}$ be the graph obtained from $G[V(K_i) \cup X]$ and $F_{i,X}$ by adding edges between these two graphs in the way that we chose earlier. We then run the algorithm of Theorem 3 to check if there is a solution $H_{i,X}$ for the 2-Connected Steiner problem with instance $(G_{i,X}, V(F_{i,X}) \cup X, k - |V(F_{i,X})| - |X|)$. If so, we take $H_{i,X}$ to be the minimum such solution; otherwise we let $H_{i,X} = \emptyset$. For each $H_{i,X}$, we mark all the vertices of $H_{i,X}$ that belong to K_i. Finally, we define C'' to be the set consisting of all the vertices of C' plus all the vertices that were marked during this entire process.

In order to prove the correctness of this algorithm, let us consider a 2-connected s–t separator S of size at most k in G such that $|S \setminus C''|$ is as small as possible. We need to show that $|S \setminus C''| = 0$. For contradiction, we assume that $|S \setminus C''| \geq 1$. Let K_i be a connected component of $G - C'$ such that K_i contains a vertex of $S \setminus C''$, let $S_i = S \setminus V(K_i)$, and let $X = S \cap N_i$. Note that $X \neq \emptyset$. Also note that S_i is a k-biconnectable set in the graph $G[V(K_i) \cup S_i]$. Hence, by Observation 4, $(G[V(K_i) \cup S_i], S_i, k)$ is a yes-instance of 2-Connected Steiner. Since $X \neq \emptyset$, in some iteration of the algorithm, we considered a graph $G_{i,X}$ that is isomorphic to $G[V(K_i) \cup S_i]$ and hence found a minimum solution $H_{i,X}$ of 2-Connected Steiner for exactly the instance $(G[V(K_i) \cup S_i], S_i, k)$. Let $S' = S_i \cup V(H_{i,X})$. By construction, S' is 2-connected. Note that $S \cap C'$ is an s–t separator, since otherwise there would be a minimal s–t separator of size at most k in G that contains a vertex outside C', contradicting Lemma 4. Since $S \cap C' \subseteq S'$, S' is an s–t separator. It is clear that $S' \subseteq C''$, which means that $|S' \setminus C''| = 0$. Hence $|S' \setminus C''| < |S \setminus C''|$, contradicting the minimality of S.

For each $i \in \{1, \ldots, q\}$, C'' contains at most $k|N_i|^k$ vertices of K_i, and hence the treewidth of $\text{torso}(K_i, C'' \cap V(K_i))$ is bounded by a constant depending only on k. It follows that the difference between the treewidth of $\text{torso}(G, C'')$ and the treewidth of $\text{torso}(G, C')$ is a constant depending on k (see also Lemma 2.9 in [19]), implying that the treewidth of $\text{torso}(G, C'')$ is bounded by a function of k. Finding the set C' can be done in time $f(k) \cdot (m + n)$ by Lemma 4. For each choice of i and X, the possible number of different graphs $G_{i,X}$, and consequently the number of instances of 2-Connected Steiner we have to solve, is bounded by some function of k. Since 2-Connected Steiner is FPT by Theorem 3, the overall running time of the algorithm is $h(k) \cdot n^{O(1)}$ for some function h that depends only on k. □

Theorem 5. *The* 2-Connected Separator *problem is FPT.*

Proof. Let (G, s, t, k) be an instance of 2-Connected Separator. We start by constructing the set $C'' \subseteq V(G)$ of Lemma 4. Let $G^* = \text{torso}(G, C'')$. We assign a color to each edge uv in G^*: we color uv black if uv is also an edge in G, and we color uv red otherwise. By Lemma 5, G contains a 2-connected s–t separator S of size at most k if and only if G^* contains an s–t separator S^* of size at most k such that deleting the red edges from $G^*[S^*]$ results in a 2-connected graph.

The theorem now follows from Courcelle's Theorem [7] and the fact that this problem can be expressed in monadic second-order logic (see [19]). □

We now show that the c-Connected Steiner problem becomes hard when the connectivity of the solution is required to be at least 3.

Theorem 6. (★) *c-Connected Steiner is $W[1]$-hard for any $c \geq 3$.*

Since we can transform an instance of c-Connected Steiner into an instance of c-Connected Separator by making two new vertices s and t adjacent to each of the terminals, Theorem 6 readily implies the following result.

Theorem 7. *c-Connected Separator is $W[1]$-hard for any $c \geq 3$.*

3 More $W[1]$-Hardness Results on General Graphs

We say that a graph G is *r-regular* if the degree of every vertex in G is exactly r. For every $r \geq 0$, let r-Regular Separator denote the problem of deciding whether an input graph G has an s–t separator S of size at most k such that $G[S]$ is r-regular. Since the class of 0-regular graphs is hereditary, Theorem 1 implies that 0-Regular Separator, i.e., the problem of finding an s–t separator that is an independent set of size at most k, is FPT. We show that r-Regular Separator is $W[1]$-hard for every $r \geq 1$ when parameterized by k. Note that the class of r-regular graphs is not hereditary for any $r \geq 1$.

Theorem 8. (★) *r-Regular Separator is $W[1]$-hard for any $r \geq 1$.*

The *diameter* of a graph G is the maximum distance between any two vertices in G, where the *distance* between two vertices u and v is defined as the number of edges in a shortest path from u to v. The problem of finding an s–t separator that forms a *clique* is well-known to be solvable in polynomial time [22]. Since cliques induce subgraphs of diameter 1, it is natural to consider the problem of finding an s–t separator that induces a graph of diameter 2, or, more generally, of any fixed diameter $d \geq 2$. Note that for any $d \geq 2$, the class of graphs with diameter (at most) d is not hereditary; consider for example a chordless cycle on $2d + 1$ vertices. The class of graphs with diameter 1, however, *is* hereditary. d-Diameter Separator is the problem of deciding if an input graph G has an s–t separator S of size at most k such that $G[S]$ has diameter d.

Theorem 9. (★) *d-Diameter Separator is $W[1]$-hard for any $d \geq 2$.*

4 Finding s–t Separators in Graphs of Bounded Degree

Theorem 1 states that $\mathcal{G}$-Separator is FPT for any decidable and hereditary graph class $\mathcal{G}$. In the previous sections, we identified several non-hereditary graph classes $\mathcal{G}$ for which $\mathcal{G}$-Separator is $W[1]$-hard on general graphs. In this section, we prove that for any decidable (but not necessarily hereditary) graph class $\mathcal{G}$, the $\mathcal{G}$-Separator problem is FPT on graphs of bounded degree. We do this by showing that the following problem is FPT on graphs of bounded degree.

PATTERN SEPARATOR *Input:* Two graphs G and H, and two vertices s and t of G. *Parameter:* $k = \|V(H)\|$. *Question:* Does G have an s–t separator S such that $G[S]$ is isomorphic to H?

We use a variant of the color coding technique of Alon, Yuster and Zwick [1] to reduce the PATTERN SEPARATOR problem on bounded degree graphs to the problem of finding an s–t separator of size at most k that has a certain hereditary property, which enables us to use Theorem 1.

For the remainder of this section, let G and H be two graphs, let s and t be two vertices of G, and let $H_1, \ldots, H_c$ be the connected components of H. We use n and m to denote the number of vertices and edges in G, respectively, and k to denote the number of vertices in H. Let ψ be a (not necessarily proper) coloring of a graph G. A subset of vertices $V' \subseteq V(G)$ is *colorful* if ψ colors no two vertices of V' with the same color. For any subset C' of colors, we say that $V' \subseteq V(G)$ is C'-*colorful* if $|V'| = |C'|$ and every vertex in V' receives a different color from C'.

Definition 1. *Let $\psi : V(G) \to \{1, 2, \ldots, c, c+1\}$ be a $(c+1)$-coloring of G. We say that ψ is H-*good *if G has an s–t separator S satisfying the following properties:*

(i) each connected component of $G[S]$ is colored monochromatically with a color from $\{1, \ldots, c\}$;
(ii) no two connected components of $G[S]$ receive the same color;
(iii) the connected component of $G[S]$ with color i is isomorphic to H_i;
(iv) every vertex in $N_G(S)$ receives color $c+1$.

It immediately follows from Definition 1 that (G, H, s, t) is a yes-instance of PATTERN SEPARATOR if and only if G has an H-good coloring. The main idea of our algorithm is that finding a separator S satisfying these requirements essentially boils down to finding a separator that is a colorful independent set, which is fixed-parameter tractable by the results of [18, 19]. The following lemma plays a crucial role in our FPT algorithm.

Lemma 6. (★) *Given a $(c+1)$-coloring ψ of G, we can decide in $g(k) \cdot (n+m)$ time whether ψ is H-good.*

Let (G, H, s, t) be an instance of PATTERN SEPARATOR, where the graph G has maximum degree at most Δ. Suppose (G, H, s, t) is a yes-instance, and let S be an s–t separator of G such that $G[S]$ is isomorphic to H. Since $|S| = |V(H)| = k$, and every vertex in S has at most Δ neighbors, $|N_G[S]| \le (\Delta+1)k$. Using the notion of a *k-perfect family of hash functions*, we can construct in time $((\Delta+1)k)! \cdot 2^{O((\Delta+1)k)} \cdot \log n$ a family Φ of $(c+1)$-colorings of G such that (G, H, s, t) is a yes-instance if and only if Φ contains an H-good coloring, where the size of Φ is bounded by $((\Delta+1)k)! \cdot 2^{O((\Delta+1)k)} \cdot \log n$ (see for example [1]).

By Lemma 6, we can check for each coloring in Φ whether or not it is H-good in $g(k) \cdot (n+m)$ time, for some function g that does not depend on n. This yields the following result.

Theorem 10. (★) Pattern Separator *can be solved in* $f(k, \Delta) \cdot m \log n$ *time on graphs of maximum degree at most* Δ.

We can solve $\mathcal{G}$-Separator by using our algorithm for Pattern Separator to try every member of $\mathcal{G}$ having size at most k:

Theorem 11. (★) *For any decidable class* $\mathcal{G}$*, the* $\mathcal{G}$-Separator *problem can be solved in time* $h_{\mathcal{G}}(k, \Delta) \cdot m \log n$ *on graphs of maximum degree at most* Δ.

5 No Polynomial Kernel for Connected Separator

In this section, we show that the Connected Separator problem does not admit a polynomial kernel, even when restricted to graphs with maximum degree at most 3, unless NP $\subseteq$ coNP/poly. The Connected Separator problem is easily seen to be NP-complete by a simple polynomial-time reduction from Steiner Tree. The following result shows that the problem remains NP-complete on graphs of maximum degree at most 3.

Theorem 12. (★) *The* Connected Separator *problem is NP-complete on graphs of maximum degree at most* 3*, in which the vertices* s *and* t *have degree* 2.

An *or-composition algorithm* for a parameterized problem $Q \subseteq \Sigma^* \times \mathbb{N}$ is an algorithm that receives as input a sequence $((x_1, k), \ldots, (x_r, k))$, with $(x_i, k) \in \Sigma^* \times \mathbb{N}^+$ for each $1 \leq i \leq r$, and outputs a pair (x', k'), such that

- the algorithm uses time polynomial in $\sum_{i=1}^{r} |x_i| + k$;
- k' is bounded by a polynomial in k; and
- $(x', k') \in Q$ if and only if there exists an $i \in \{1, \ldots, r\}$ with $(x_i, k) \in Q$.

A parameterized problem Q is said to be *or-compositional* if there exists an or-composition algorithm for Q.

Theorem 13. (★) *The* Connected Separator *problem, restricted to graphs with maximum degree at most* 3*, is or-compositional.*

Combining results of Bodlaender et al. [2] and Fortnow and Santhanam [13] on the non-existence of polynomial kernels, together with Theorems 12 and 13, yields the following result.

Theorem 14. (★) *The* Connected Separator *problem, restricted to graphs of maximum degree at most 3, has no polynomial kernel, unless NP* $\subseteq$ *coNP/poly.*

Acknowledgements. We would like to thank Pål Grønås Drange for fruitful discussions.

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