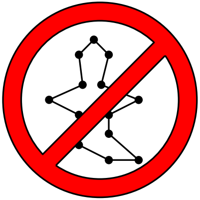
Decomposition theorems for graphs excluding structures



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Budapest, Hungary

EuroComb 2013
September 13, 2013
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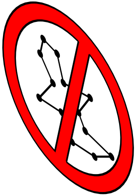

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Classes of graphs

Classes of graphs can be described by

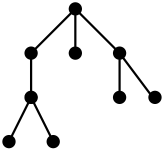
1. what they do not have,
   (excluded structures)
2. how they look like
   (constructions and decompositions).

In general, the second description is more useful for algorithmic purposes.

Classes of graphs

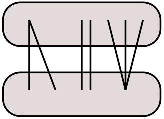
Example: Trees

1. Do not contain cycles (and connected)
2. Have a tree structure.



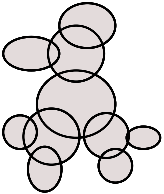
Example: Bipartite graphs

1. Do not contain odd cycles,
2. Edges going only between two classes.



Example: Chordal graphs

1. Do not contain induced cycles,
2. Clique-tree decomposition and simplicial ordering.



Main message

In many cases, we can obtain statements of the following form:

If a graph excludes X, then it can be built from components that obviously exclude (larger versions of) X.

Main message

Consequence:

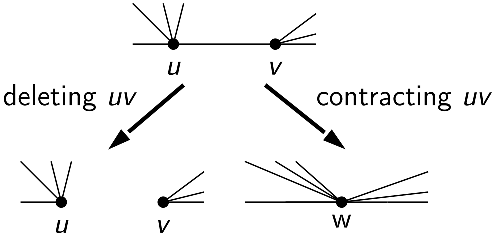
- *If we exclude simpler objects, then the building blocks are simpler and more constrained.*
- *If we exclude more complicated objects, then the building blocks are more complicated and more general.*

Excluding minors

The monumental work of Robertson and Seymour developed a deep theory of graphs excluding a fixed minor H.

Definition

Graph H is a **minor** of G ($H \leq G$) if H can be obtained from G by deleting edges, deleting vertices, and contracting edges.

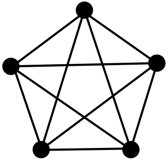


Example: $K_3 \leq G$ if and only if G has a cycle.

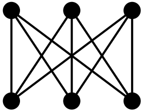
Excluding minors

Theorem [Wagner 1937]

A graph is planar if and only if it excludes K_5 and $K_{3,3}$ as a minor.



K_5



$K_{3,3}$

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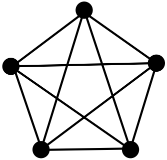

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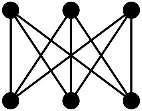


$K_{3,3}$

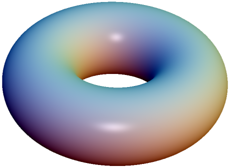
- How do graphs excluding H (or $H_1, \ldots, H_k$) look like?
- What other classes can be defined this way?

The work of Robertson and Seymour gives some kind of combinatorial answer to that and provides tools for the related algorithmic questions.

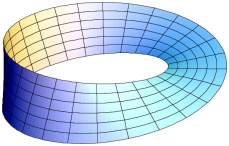
Graphs on surfaces

The notion of planar graphs can be generalized to graphs drawn on other surfaces.

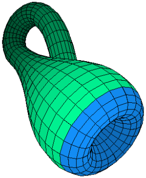
torus



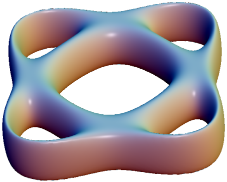
Möbius strip



Klein bottle



genus 5



Excluding minors

Graphs drawn on a fixed surface Σ form a class of graphs excluding a minor:

Fact

For every surface Σ, there is a $k_\Sigma \geq 1$ such that graphs drawn on Σ do not contain K_{k_Σ} as a minor.

- Can we describe somehow H-minor-free graphs using graphs drawn on surfaces?
- Is it true for every H that H-minor-free graphs can be drawn on some fixed surface?

Excluding minors

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NO (clique sums), **NO** (apices), **NO** (vortices)

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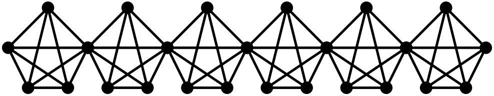
- Is it true for every H that H-minor-free graphs can be drawn on some fixed surface?

NO (clique sums), **NO** (apices), **NO** (vortices)

YES (in a sense — Robertson-Seymour Structure Theorem)

Excluding minors

Graphs of the following form do not have K_6-minors, but their genus can be arbitrary large:



Connecting bounded-genus graphs can increase genus without creating a clique minor.

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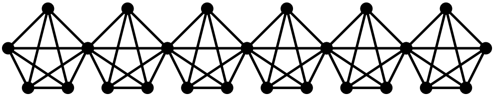


Connecting bounded-genus graphs can increase genus without creating a clique minor.

We need to introduce an operation of connecting graphs in a way that does not create large clique minors.

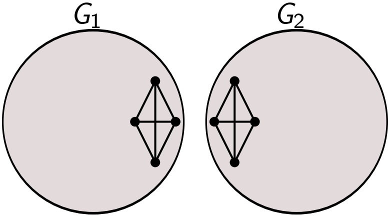
Two ways of explaining this operation:

- clique sums and
- torsos of tree decompositions.

Clique sums

Definition

Let G_1 and G_2 be two graphs with two cliques $K_1 \subseteq V(G_1)$ and $K_2 \subseteq V(G_2)$ of the same size. Graph G is a **clique sum** of G_1 and G_2 if it can be obtained by identifying K_1 and K_2, and then removing some of the edges of the clique.



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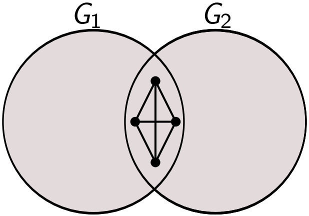

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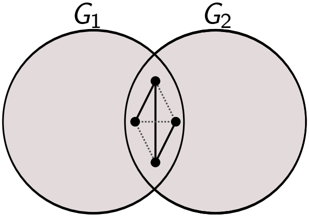

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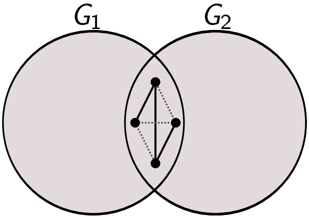


Observation

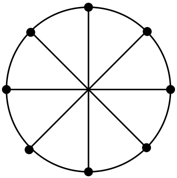
If $K_k \not\preceq G_1, G_2$ and G is a clique sum of G_1 and G_2, then $K_k \not\preceq G$.

Thus we can build K_k-minor-free graphs by repeated clique sums.

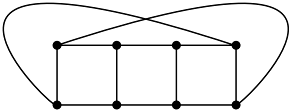
Excluding K_5

Theorem [Wagner 1937]

A graph is K_5-minor-free if and only if it can be built from planar graphs and V_8 by repeated clique sums.



V_8

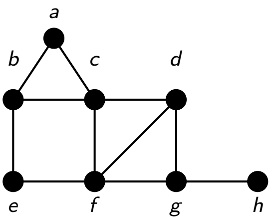


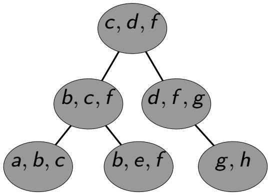
V_8

Tree decompositions

Tree decomposition: Vertices are arranged in a tree structure satisfying the following properties:

1. If u and v are neighbors, then there is a bag containing both of them.
2. For every v, the bags containing v form a connected subtree.





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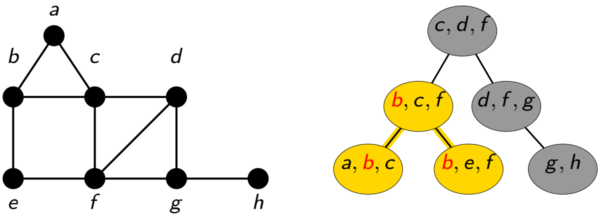

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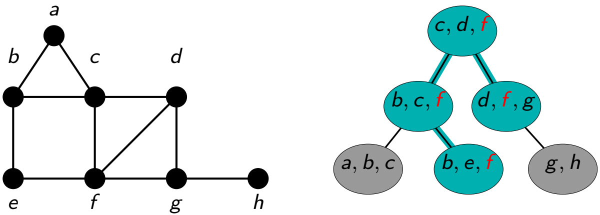
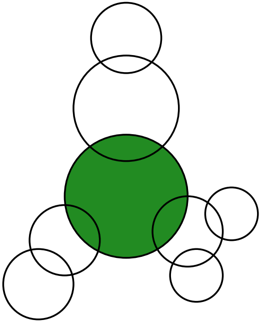


Torso

Torso of a bag: we make the intersections with the adjacent bags cliques.



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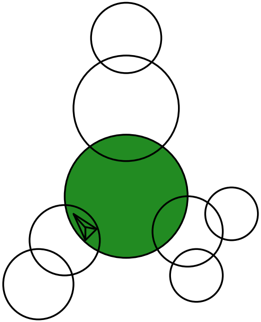

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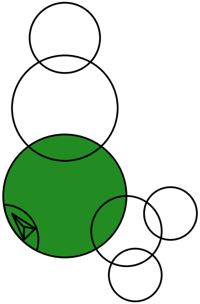

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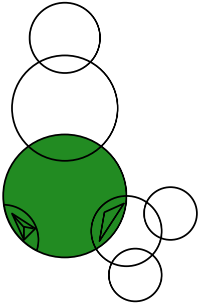

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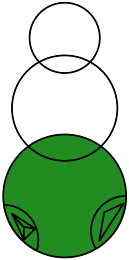

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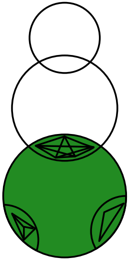

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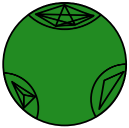


Excluding K_5 — restated

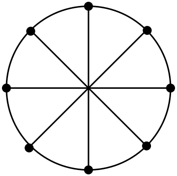
Theorem [Wagner 1937]

A graph is K_5-minor-free if and only if it can be built from planar graphs and from V_8 by repeated clique sums.

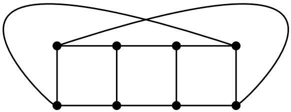
Equivalently:

Theorem [Wagner 1937]

A graph is K_5-minor-free if and only if it has a tree decomposition where every torso is either a planar graph or the graph V_8.



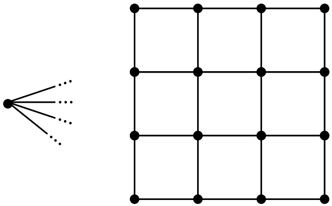
V_8



V_8

Apex vertices

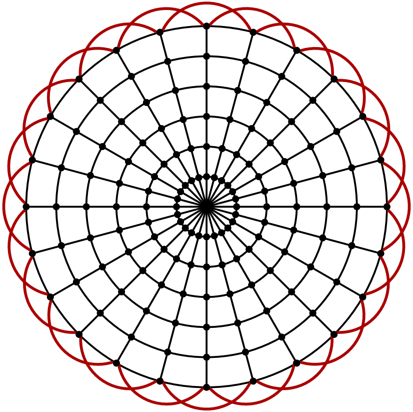
The graph formed from a grid by attaching a universal vertex is K_6-minor-free, but has large genus.



- A planar graph + k extra vertices has no K_{k+5}-minor.
- Instead of bounded genus graphs, our building blocks should be "bounded genus graphs + a bounded number of apex vertices connected arbitrarily."

Vortices

One can show that the following graph has large genus, but cannot have a K_8-minor.



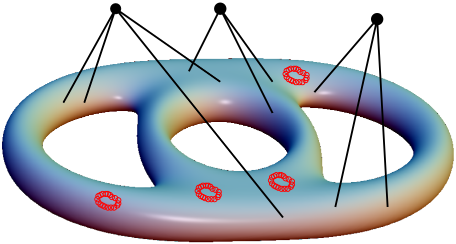
We define a notion of "vortex of width k" for structures like this (details omitted).

k-almost embeddable

Definition

Graph G is k-**almost embeddable** in surface Σ if

- there is a set X of at most k apex vertices and
- a graph G_0 embedded in Σ, such that
- $G \setminus X$ can be obtained from G_0 by attaching vortices of width k on disjoint disks $D_1, \ldots, D_k$.



Graph Structure Theorem

Decomposing H-minor-free graphs into almost embeddable parts:

Theorem [Robertson-Seymour]

For every graph H, there is an integer k and a surface Σ such that every H-minor-free graph

- can be built by clique sums from graphs that are k-almost embeddable in Σ,

(or equivalently)

- has a tree decomposition where every torso is k-almost embeddable in Σ.

Originally stated only combinatorially, algorithmic versions are known.

Excluding cliques

A k-almost embeddable graph on Σ cannot have a clique minor larger than $f(k, \Sigma)$.

The decomposition approximately characterizes graphs excluding a clique as a minor:

No K_k-minor	$\Longrightarrow$	tree decomposition with torsos k'-almost embeddable in Σ
tree decomposition with torsos k'-almost embeddable in Σ	$\Longrightarrow$	no $K_{k''}$-minor

Algorithmic applications

General message: if something works for planar graphs, then we might generalize it to bounded genus graphs and H-minor-free graphs.

- Approximation schemes: $2^{O(1/\epsilon)} \cdot n^{O(1)}$ time algorithm for Maximum Independent Set on H-minor-free graphs.
- Parameterized algorithms and bidimensionality: $2^{O(\sqrt{k})} \cdot n^{O(1)}$ time algorithm for Maximum Independent Set on H-minor-free graphs.

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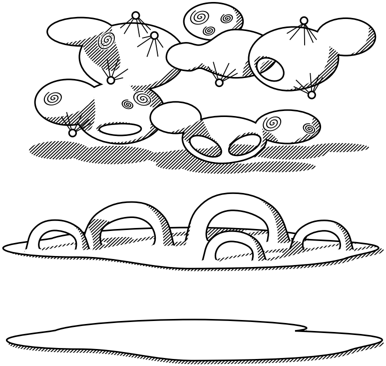
- Parameterized algorithms and bidimensionality: $2^{O(\sqrt{k})} \cdot n^{O(1)}$ time algorithm for Maximum Independent Set on H-minor-free graphs.

The understanding of graphs excluding minors is essential for finding minors:

Theorem [Robertson and Seymour]

H-minor testing can be solved in time $f(H) \cdot n^3$.

Algorithmic applications relying on (variants of) minor testing, e.g., k-Disjoint Paths.



H-Minor-Free

$\cup$

Bounded Genus

$\cup$

Planar

[figure by Felix Reidl]

Excluding planar graphs

If we exclude simpler H, we expect the building blocks to be simpler.

Theorem [Robertson and Seymour]

For every planar graph H, there is a constant k_H such that every H-minor-free graph

- can be built from graphs of size at most k_H by clique sums,

  (or equivalently)
- has a tree decomposition where every bag has size at most k_H.

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In a different language:

Width of a tree decomposition:
maximum bag size (minus one).

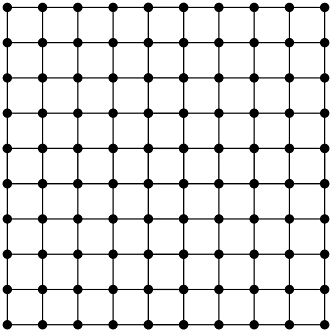
Treewidth of a graph:
minimum width of a decomposition.

Excluding a planar minor implies bounded treewidth.

Excluded Grid Theorem

Excluded Grid Theorem [Diestel et al. 1999]

If the treewidth of G is at least $k^{4k^2(k+2)}$, then G has a $k \times k$ grid minor.



(A $k^{O(1)}$ bound was just announced [Chekuri and Chuznoy 2013]!)

Excluded Grid Theorem

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If the treewidth of G is at least $k^{4k^2(k+2)}$, then G has a $k \times k$ grid minor.

A large grid minor is a "witness" that treewidth is large, but the relation is approximate:

No $k \times k$ grid minor	$\Longrightarrow$	tree decomposition of width $< f(k)$
tree decomposition of width $< f(k)$	$\Longrightarrow$	no $f(k) \times f(k)$ grid minor

Excluding trees

As every forest is planar, the following holds for every forest F:

no F-minor	$\Longrightarrow$	tree decomposition of width $< f(F)$
tree decomposition of width $< f(F)$	$\Longrightarrow$	Does not exclude any tree as minor!

This is not a good (approximate) structure theorem.

Excluding trees

Path decomposition: the tree of bags is a path.
Pathwidth: defined analogously to treewidth.
Example: A complete binary tree on k levels has pathwidth $k - 1$.

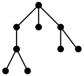
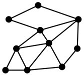
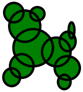
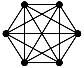
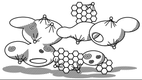
Theorem [Diestel 1995]

If F is a forest, then every F-minor-free graph has pathwidth at most $|V(F)| - 2$.

no F-minor	$\Longrightarrow$	path decomposition of width $< f(F)$
path decomposition of width $< f(F)$	$\Longrightarrow$	No $(f(F) + 1)$-level complete binary tree

Excluding minors

We have seen that a graph excluding a fixed minor can be built from simple building blocks:

Excluding a tree 	$\Longrightarrow$	small blocks, in a pathlike way 
Excluding a planar graph 	$\Longrightarrow$	small blocks, in a treelike way 
Excluding a clique 	$\Longrightarrow$	k-almost embeddable blocks, in a treelike way 

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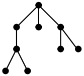
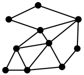


⟹

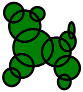
small blocks, in a pathlike way

Excluding a planar graph



⟹

small blocks, in a treelike way



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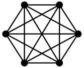
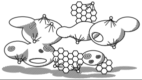


⟹

k-almost embeddable blocks, in a treelike way



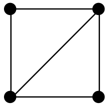
Next: Notions of containment stricter than minors.

Topological subgraphs

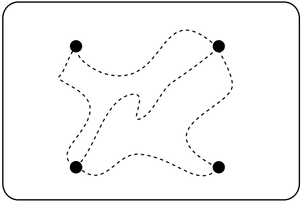
Definition

Subdivision of a graph: replacing each edge by a path of length 1 or more.

Graph H is a **topological subgraph** of G (or **topological minor** of G, or $H \leq_T G$) if a subdivision of H is a subgraph of G.



$\leq_T$



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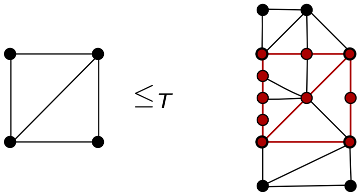

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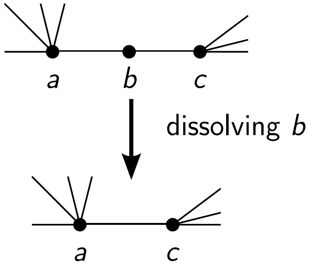
Equivalently, $H \leq_T G$ means that H can be obtained from G by removing vertices, removing edges, and dissolving degree-two vertices.



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Simple observations:

- $H \leq_T G$ implies $H \leq G$.
- The converse is not true: a 3-regular graph excludes $K_{1,4}$ as a subdivision, but can contain large clique minors.

Topological subgraphs

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Subdivision of a graph: replacing each edge by a path of length 1 or more.

Graph H is a **topological subgraph** of G (or **topological minor** of G, or $H \leq_T G$) if a subdivision of H is a subgraph of G.

Finding subdivisions:

Theorem [Robertson and Seymour]

We can decide in time $n^{f(H)}$ if $H \leq_T G$.

Theorem [Grohe, Kawarabayashi, M., Wollan 2011]

We can decide in time $f(H) \cdot n^3$ if $H \leq_T G$.

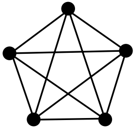
A classical result

Theorem [Kuratowski 1930]

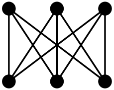
A graph G is planar if and only if $K_5 \not\preceq_T G$ and $K_{3,3} \not\preceq_T G$.

Theorem [Wagner 1937]

A graph G is planar if and only if $K_5 \not\preceq G$ and $K_{3,3} \not\preceq G$.



K_5



$K_{3,3}$

Remarkable coincidence!

Structure theorems for excluding subdivisions

We can build H-subdivision-free graphs from two types of blocks:

Theorem [Grohe and M. 2012]

For every H, there is an integer $k \geq 1$ such that every H-subdivision-free graph has a tree decomposition where the torso of every bag is either

- K_k-minor-free or
- has degree at most k with the exception of at most k vertices ("almost bounded degree").

Note: there is an $f(H) \cdot n^{O(1)}$ time algorithm for computing such a decomposition.

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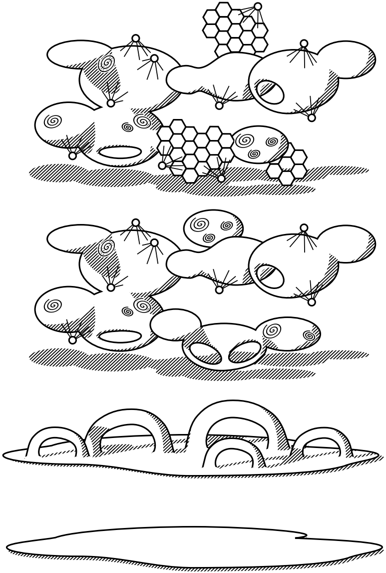

H-Topological- Minor-Free

$\cup$

H-Minor-Free

$\cup$

Bounded Genus

$\cup$

Planar

[figure by Felix Reidl]

Algorithmic applications

Theorem [Grohe and M. 2012]

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General message:

If a problem can be solved both

- on (almost-) embeddable graphs and
- on (almost-) bounded degree graphs,

then these results can be raised to

- H-subdivision-free graphs

without too much extra effort.

Graph Isomorphism

Theorem [Luks 1982] [Babai, Luks 1983]

For every fixed d, GRAPH ISOMORPHISM can be solved in polynomial time on graphs with maximum degree d.

Theorem [Ponomarenko 1988]

For every fixed H, GRAPH ISOMORPHISM can be solved in polynomial time on H-minor-free graphs.

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Theorem [Grohe and M. 2012]

For every fixed H, GRAPH ISOMORPHISM can be solved in polynomial-time on H-subdivision-free graphs.

Note:

- Requires a more general "invariant acyclic tree-like decomposition."
- Running time is $n^{f(H)}$.

Containment notions

Excluding H as a minor

almost embeddable parts

↓

Excluding H as a subdivision

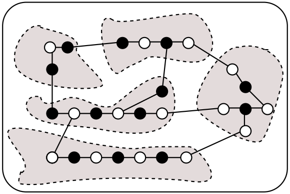
almost embeddable and
almost bounded-degree parts

Odd minors

Definition

Graph H is an **odd minor** of G ($H \leq_{odd} G$) if G has a 2-coloring and there is a mapping ϕ that maps each vertex of H to a tree of G such that

- $\phi(u)$ and $\phi(v)$ are disjoint if $u \neq v$,
- every edge of $\phi(u)$ is bichromatic,
- if $uv \in E(H)$, then there is a monochromatic edge between $\phi(u)$ and $\phi(v)$.



Example: K_3 is an odd minor of G if and only if G is not bipartite.

Odd minors

Finding odd minors:

Theorem [Kawarabayashi, Reed, Wollan 2011]

There is an $f(H) \cdot n^{O(1)}$ time algorithm for finding an odd H-minor.

Structure theorem:

Theorem [Demaine, Hajiaghayi, Kawarabayashi 2010]

For every H, there is a $k \geq 1$ such that every odd H-minor-free graph has a tree decomposition where the torso of every bag is

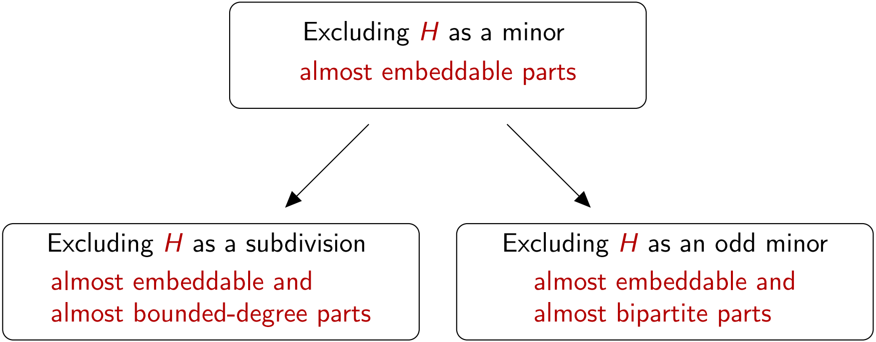
- k-almost embeddable in a surface of genus at most k or
- bipartite after deleting at most k vertices ("almost bipartite").

Consequence:

Theorem [Demaine, Hajiaghayi, Kawarabayashi 2010]

For every fixed H, there is a polynomial-time 2-approximation algorithm for chromatic number on odd H-minor-free graphs.

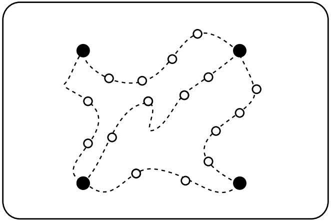
Containment notions



Odd subdivisions

Definition

Odd subdivision of a graph: replacing each edge by a path of odd length (1 or more).



If G contains an odd H-subdivision, then $H \leq_T G$ and $H \leq_{odd} G$.

Odd subdivisions

A structure theorem for excluding an odd H-subdivision should be more general than

- the structure theorem for excluded subdivisions (k-almost embeddable, almost bounded degree) and
- the structure theorem for excluded odd minors (k-almost embeddable, almost bipartite).

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Theorem [Kawarabayashi 2013]

For every H, there is an integer $k \geq 1$ such that every odd H-subdivision-free graph has a tree decomposition where the torso of every bag is either

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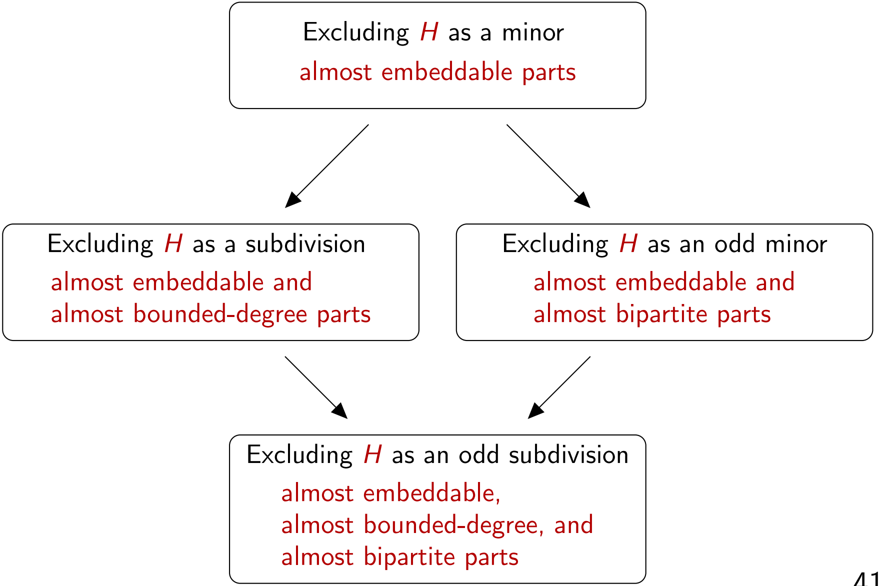
- bipartite after deleting at most k vertices ("almost bipartite").

Theorem [Kawarabayashi 2013]

For every H, there is a polynomial-time algorithm that, given an odd H-subdivision-free graph G, finds a coloring of G with $2\chi(G) + 6(V(H) - 1)$ colors.

Containment notions

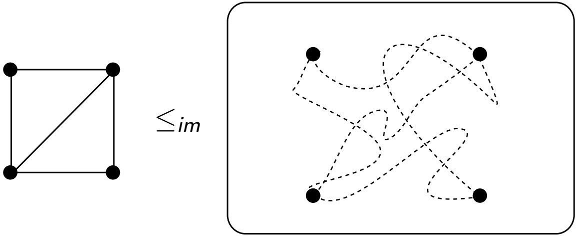


Immersions

Definition

Graph H has an **immersion** in G ($H \leq_{im} G$) if there is a mapping ϕ such that

- For every $v \in V(H)$, $\phi(v)$ is a distinct vertex in G.
- For every $xy \in E(H)$, $\phi(xy)$ is a path between $\phi(x)$ and $\phi(y)$, and all these paths are edge disjoint.



Note: $H \leq_T G$ implies $H \leq_{im} G$.

Excluding immersions

As excluding K_k-immersions implies excluding K_k-subdivisions, we get:

Theorem [Grohe and M. 2012]

For every H, there is an integer $k \geq 1$ such that every H-immersion-free graph has a tree decomposition where the torso of every bag is either

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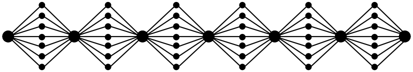
However, embeddability does not seem to be relevant for immersions: the following graph has large clique immersions.



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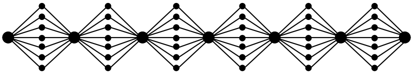


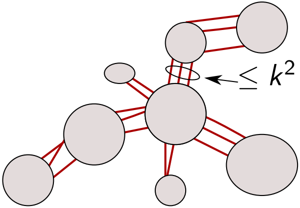
Can we omit the first case?

Excluding immersions

Theorem [Wollan]

If K_k has no immersion in G, then G has a "tree-cut decomposition" of adhesion at most k^2 such that each "torso" has at most k vertices of degree at least k^2.

Tree cut decomposition: a partition of the vertex set in tree-like way.



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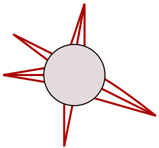

Summary

- General form of statements:

  If a graph excludes X, then it can be built from components that obviously exclude (larger versions of) X.

- Trade-ff between the excluded object and the simplicity of the building blocks:

  If we exclude more complicated objects, then the building blocks are more complicated and more general.

- The building blocks were small, planar, almost embeddable, almost bounded-degree, almost bipartite.
- The algorithmic applications depend on how simple the building blocks are.