Every graph is easy or hard: dichotomy theorems for graph problems

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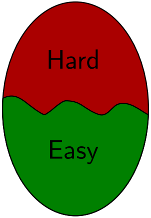
Dichotomy theorems

We survey results where we can precisely tell which graphs make the problem easy and which graphs make the problem hard.



Focus will be on

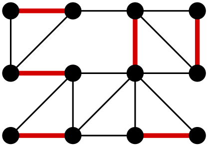
- how to formulate questions that lead to such results and
- what results of this type are known,

but less on how to prove such results.

Factor problems

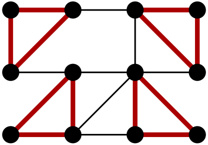
Perfect Matching
Input: graph G.
Task: find $|V(G)|/2$ vertex-disjoint edges.

Polynomial-time solvable [Edmonds 1961].



Triangle Factor
Input: graph G.
Task: find $|V(G)|/3$ vertex-disjoint triangles.

NP-complete [Karp 1975]



Factor problems

H-FACTOR
Input: graph G.
Task: find $|V(G)|/|V(H)|$ vertex-disjoint copies of H in G.

Polynomial-time solvable for $H = K_2$ and NP-hard for $H = K_3$.

Which graphs H make H-FACTOR easy and which graphs make it hard?

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Which graphs H make H-FACTOR easy and which graphs make it hard?

Theorem [Kirkpatrick and Hell 1978]

H-FACTOR is NP-hard for every connected graph H with at least 3 vertices.

Factor problems

Instead of publishing

Kirkpatrick and Hell: NP-completeness of packing cycles. 1978.
Kirkpatrick and Hell: NP-completeness of packing trees. 1979.
Kirkpatrick and Hell: NP-completeness of packing stars. 1980.
Kirkpatrick and Hell: NP-completeness of packing wheels. 1981.
Kirkpatrick and Hell: NP-completeness of packing Petersen graphs. 1982.
Kirkpatrick and Hell: NP-completeness of packing Starfish graphs. 1983.
Kirkpatrick and Hell: NP-completeness of packing Jaws. 1984.

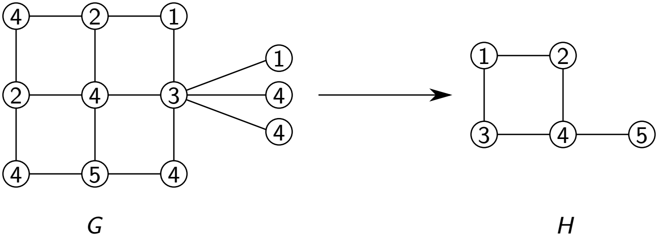
$\vdots$

they only published

Kirkpatrick and Hell: On the Completeness of a Generalized Matching Problem. 1978

H-coloring

A **homomorphism** from G to H is a mapping $f\colon V(G) \to V(H)$ such that if ab is an edge of G, then $f(a)f(b)$ is an edge of H.



H-COLORING
Input: graph G.
Task: Find a homomorphism from G to H.

- If $H = K_r$, then equivalent to r-COLORING.
- If H is bipartite, then the problem is equivalent to G being bipartite.

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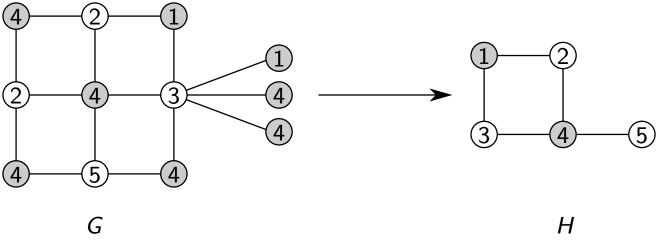

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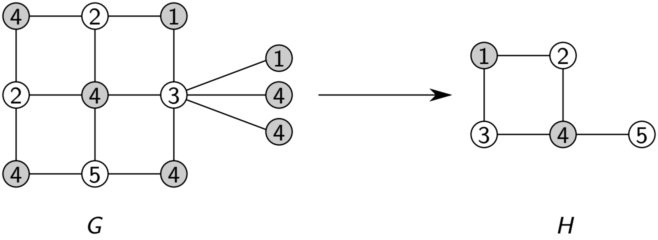

> H-COLORING
> **Input:** graph G.
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Theorem [Hell and Nešetřil 1990]

For every simple graph H, H-COLORING is polynomial-time solvable if H is bipartite and NP-complete if H is not bipartite.

Dichotomy theorems

Dichotomy theorem: classifying every member of a family of problems as easy or hard.

Why are such theorems surprising?

1. The characterization of easy/hard is a simple combinatorial property.

   So far, we have seen:

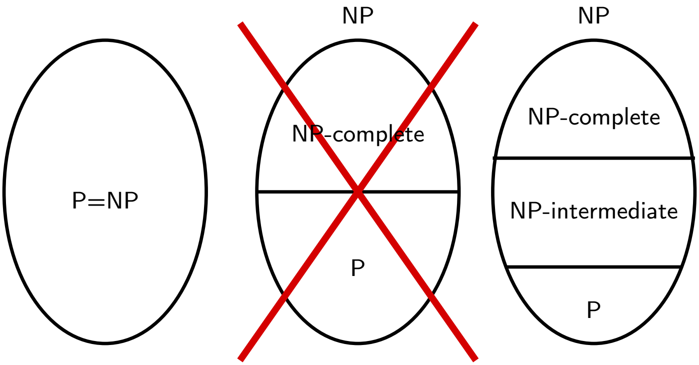
   - at least 3 vertices,
   - nonbipartite.

Dichotomy theorems

2. Every problem is either in P or NP-complete, there are no NP-intermediate problems in the family.

Theorem [Ladner 1973]

If $P \neq NP$, then there is a language $L \in NP \setminus P$ that is not NP-complete.



Dichotomy theorems

- Dichotomy theorems give goods research programs: easy to formulate, but can be hard to complete.
- The search for dichotomy theorems may uncover algorithmic results that no one has thought of.
- Proving dichotomy theorems may require good command of both algorithmic and hardness proof techniques.

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So far:
Each problem in the family was defined by fixing a graph H.

Next:
Each problem is defined by fixing a **class** of graph $\mathcal{H}$.

Hereditary deletion problems

$\mathcal{H}$-Deletion
Input: a graph G and an integer k.
Task: find a set S of k vertices such that $G - S \in \mathcal{H}$

Examples:

- $\mathcal{H}$ is the set of all graphs without edges: Vertex Cover.
- $\mathcal{H}$ is the set of all acyclic graphs: Feedback Vertex Set.

$\mathcal{H}$ is **hereditary** if it is closed under taking induced subgraphs.

Hereditary:

- planar
- chordal
- interval
- bipartite

Not hereditary:

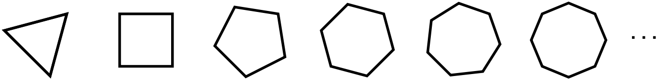
- connected
- 3-regular
- Hamiltonian
- nonbipartite

Hereditary deletion problems

Theorem [Yannakakis 1978]

For every hereditary class $\mathcal{H}$, the $\mathcal{H}$-DELETION problem is NP-complete.

Hereditary class $\mathcal{H}$ can be characterized by a (finite or infinite) list of minimal forbidden induced subgraphs.

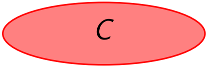


Hereditary deletion problems

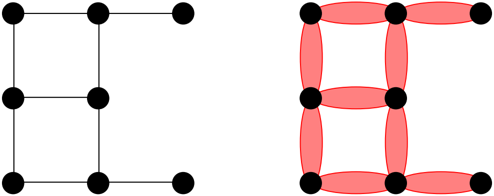
Theorem [Yannakakis 1978]

For every hereditary class $\mathcal{H}$, the $\mathcal{H}$-DELETION problem is NP-complete.

Simpler case: suppose that every minimal forbidden induced subgraph is 2-connected and let C be the smallest forbidden induced subgraph.



Reduction from VERTEX COVER:



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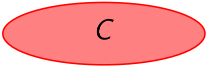

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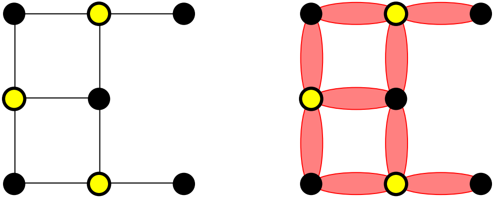

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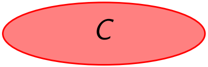

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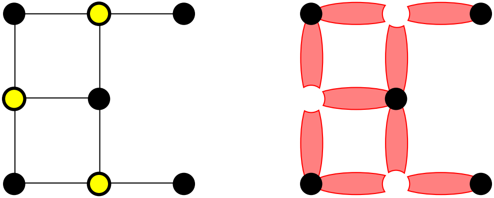
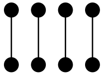


Finding subgraphs

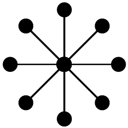
$\textsc{Sub}(\mathcal{H})$
Input: a graph $H \in \mathcal{H}$ and an arbitrary graph G.
Task: decide if H is a subgraph of G.

Some classes for which $\textsc{Sub}(\mathcal{H})$ is polynomial-time solvable:

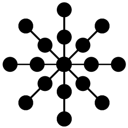
- $\mathcal{H}$ is the class of all matchings
- $\mathcal{H}$ is the class of all stars
- $\mathcal{H}$ is the class of all stars, each edge subdivided once
- $\mathcal{H}$ is the class of all windmills



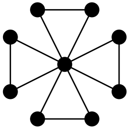
matching



star



subdivided star

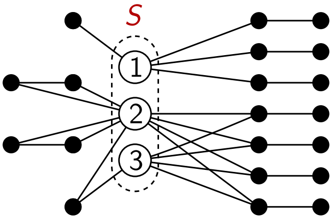


windmill

Finding subgraphs

Definition

Class $\mathcal{H}$ is **matching splittable** if there is a constant c such that every $H \in \mathcal{H}$ has a set S of at most c vertices such that every component of $H - S$ has size at most 2.



Theorem [Jansen and M. 2015]

Let $\mathcal{H}$ be a hereditary class of graphs. If $\mathcal{H}$ is matching splittable, then $\textsc{Sub}(\mathcal{H})$ is randomized polynomial-time solvable and NP-hard otherwise.

Finding subgraphs (algorithm)

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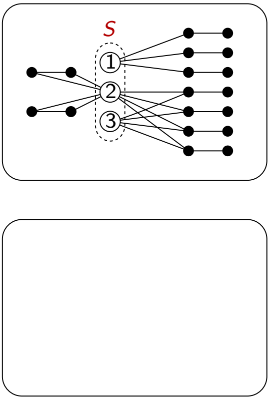
H

G



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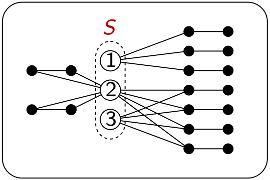
Theorem [Jansen and M. 2015]

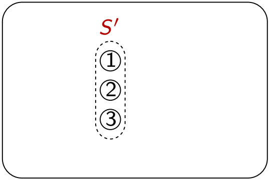
If hereditary class $\mathcal{H}$ is matching splittable, then $\textsc{Sub}(\mathcal{H})$ is randomized polynomial-time solvable.

- Guess the image S' of S in G.

H



G



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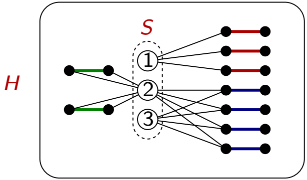
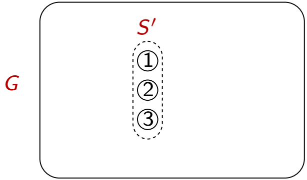
- Guess the image S' of S in G.
- Classify the edges of $H - S$ according to their neighborhoods in S (at most 2^{2c} colors).





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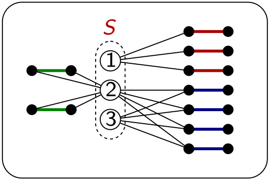
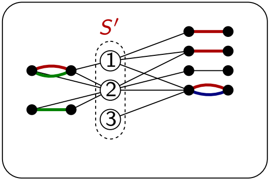
- Guess the image S' of S in G.
- Classify the edges of $H - S$ according to their neighborhoods in S (at most 2^{2c} colors).
- Classify the edges of $G - S'$ according to which edge of $H - S$ can be mapped into it (use parallel edges if needed).
- Task is to find a matching in $G - S'$ with a certain number of edges of each color.

H



G



Finding subgraphs (algorithm)

Theorem [Mulmuley, Vazirani, Vazirani 1987]

There is a randomized polynomial-time algorithm that, given a graph G with red and blue edges and integer k, decides if there is a perfect matching with exactly k red edges.

More generally:

Theorem

Given a graph G with edges colored with c colors and c integers k_1, $\ldots$, k_c, we can decide in randomized time $n^{O(c)}$ if there is a matching with exactly k_i edges of color i.

This is precisely what we need to complete the algorithm for $\textsc{Sub}(\mathcal{H})$ for matching splittable $\mathcal{H}$.

Finding subgraphs (hardness proof)

Lemma

Let $\mathcal{H}$ be a hereditary class of graphs that is not matching splittable. Then at least one of the following is true.

- $\mathcal{H}$ contains every clique.
- $\mathcal{H}$ contains every biclique.
- For every $n \geq 1$, $\mathcal{H}$ contains $n \cdot K_3$.
- For every $n \geq 1$, $\mathcal{H}$ contains $n \cdot P_3$ (where P_3 is the path on 3 vertices).

In each case, $\textsc{Sub}(\mathcal{H})$ is NP-hard (recall that P_3-FACTOR and K_3-FACTOR are NP-hard).

Finding subgraphs (hardness proof)

Recall: Class $\mathcal{H}$ is **matching splittable** if there is a constant c such that every $H \in \mathcal{H}$ has a set S of at most c vertices such that every component of $H - S$ has size at most 2.

Equivalently: in every $H \in \mathcal{H}$, we can cover every 3-vertex connected set (i.e., every K_3 and P_3) by c vertices.

Observation: either

- there are r vertex disjoint K_3, or
- there are r vertex disjoint P_3, or
- we can cover every K_3 and every P_3 by $6r$ vertices.

Ramsey's Theorem: There is a monochromatic r-clique in every c-coloring of the edges of a clique with at least c^{cr} vertices.

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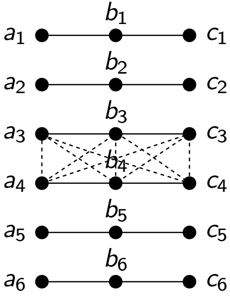
- Consider many vertex-disjoint P_3's.
- For every $i < j$, there are 2^9 possibilities between $\{a_i, b_i, c_i\}$ and $\{a_j, b_j, c_j\}$.
- There is a homogeneous set of many P_3's with respect to these 2^9 possibilities.
- In each of the 2^9 cases, we find many disjoint P_3's, a clique, or a biclique.



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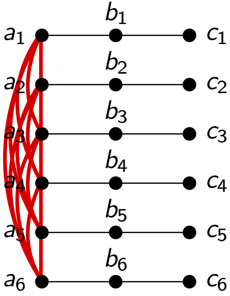

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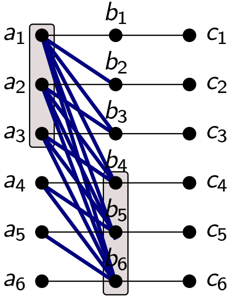

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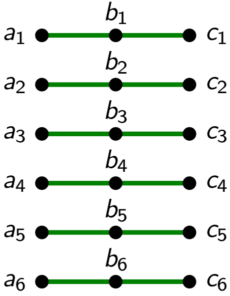

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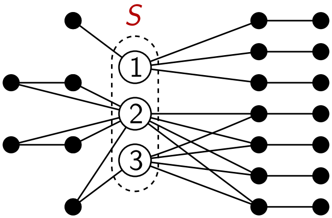


Theorem [Jansen and M. 2015]

Let $\mathcal{H}$ be a hereditary class of graphs. If $\mathcal{H}$ is matching splittable, then $\textsc{Sub}(\mathcal{H})$ is randomized polynomial-time solvable and NP-hard otherwise.

Fixed-parameter tractability

More refined analysis of the running time: we express the running time as a function of input size n and a parameter k.

Definition

A problem is **fixed-parameter tractable (FPT)** parameterized by k if it can be solved in time $f(k) \cdot n^{O(1)}$ for some computable function f.

Examples of FPT problems (having $2^{O(k)} \cdot n^{O(1)}$ time algorithms):

- Finding a vertex cover of size k.
- Finding a feedback vertex set of size k.
- Finding a path of length k.
- Finding k vertex-disjoint triangles.
- ...

W[1]-hardness

Negative evidence similar to NP-completeness. If a problem is **W[1]-hard,** then the problem is not FPT, unless FPT = W[1].

Some W[1]-hard problems:

- Finding a clique/independent set of size k.
- Finding a dominating set of size k.
- Finding k pairwise disjoint sets.
- ...

For these problems, the exponent of n has to depend on k (the running time is typically $n^{O(k)}$).

Finding subgraphs

Ideally, we would like to classify $\textsc{Sub}(\mathcal{H})$ problems into three categories:

1. (Randomized) polynomial-time solvable

   Example: matchings, matching-splittable graphs

2. No polytime algorithm, but FPT parameterized by $|V(H)|$ (solvable in time $f(|V(H)|)n^{O(1)}$)

   Example: paths, disjoint triangles, low-treewidth graphs

3. Not FPT parameterized by $|V(H)|$.

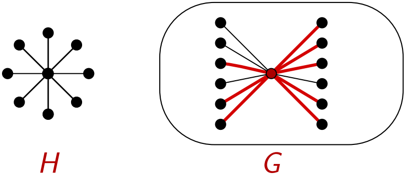
   Example: cliques, complete bipartite graphs

No such classification is known yet!

Counting subgraphs

$\#\text{SUB}(\mathcal{H})$
Input: a graph $H \in \mathcal{H}$ and an arbitrary graph G.
Task: calculate the number of copies of H in G.

If $\mathcal{H}$ is the class of all stars, then $\#\text{SUB}(\mathcal{H})$ is easy: for each placement of the center of the star, calculate the number of possible different assignments of the leaves.



Counting subgraphs

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Input: a graph $H \in \mathcal{H}$ and an arbitrary graph G.
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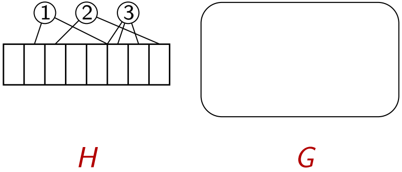
Theorem

If every graph in $\mathcal{H}$ has vertex cover number at most c, then $\#\text{SUB}(\mathcal{H})$ is polynomial-time solvable.



Running time is $n^{2^{O(c)}}$, better algorithms known [Vassilevska Williams and Williams], [Kowaluk, Lingas, and Lundell].

Counting subgraphs

#Sub($\mathcal{H}$)
Input: a graph $H \in \mathcal{H}$ and an arbitrary graph G.
Task: calculate the number of copies of H in G.

Theorem

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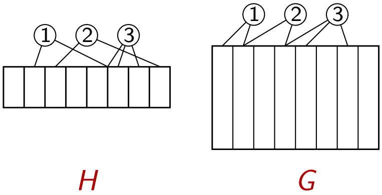

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Who are the bad guys now?

Theorem [Flum and Grohe 2002]

If $\mathcal{H}$ is the set of all paths, then $\#\textsc{Sub}(\mathcal{H})$ is #W[1]-hard.

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Let $\mathcal{H}$ be a recursively enumerable class of graphs. If $\mathcal{H}$ has unbounded vertex cover number, then $\#\textsc{Sub}(\mathcal{H})$ is #W[1]-hard.

($\nu(G) \leq \tau(G) \leq 2\nu(G)$, hence "unbounded vertex cover number" and "unbounded matching number" are the same.)

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($\nu(G) \le \tau(G) \le 2\nu(G)$, hence "unbounded vertex cover number" and "unbounded matching number" are the same.)

There is a simple proof if $\mathcal{H}$ is hereditary, but the general case is more difficult.

Counting subgraphs

Observation

At least one of the following holds for every hereditary class $\mathcal{H}$ with unbounded vertex cover number:

- $\mathcal{H}$ contains every matching.
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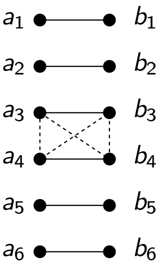

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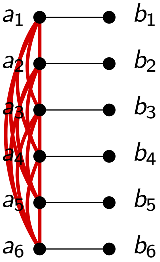

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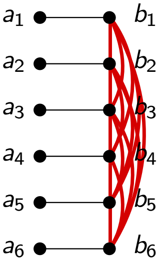

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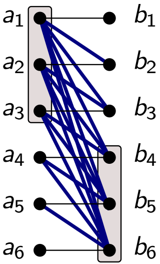

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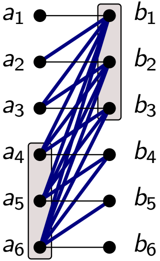

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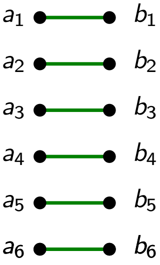

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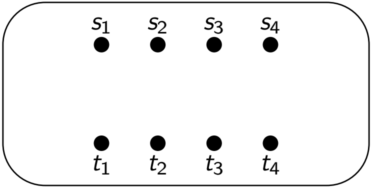
- If $\mathcal{H}$ has unbounded vertex cover number, then $\#\textsc{Sub}(\mathcal{H})$ is #W[1]-hard (parameterized by $|V(H)|$).

Fixed-parameter tractability does not give us any extra power here!

Disjoint paths

k-DISJOINT PATHS
Input: graph G and pairs of vertices $(s_1, t_1), \dots, (s_k, t_k)$.
Task: find pairwise vertex-disjoint paths $P_1, \dots, P_k$ such that P_i connects s_i and t_i.



NP-hard, but FPT parameterized by k:

Theorem [Robertson and Seymour]

The k-DISJOINT PATHS problem can be solved in time $f(k)n^3$.

We consider now a maximization version of the problem.

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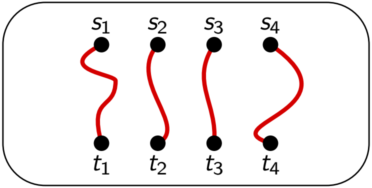

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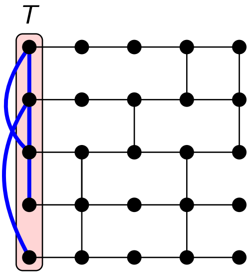

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Maximum Disjoint $\mathcal{H}$-Paths: special case when H restricted to be a member of $\mathcal{H}$.

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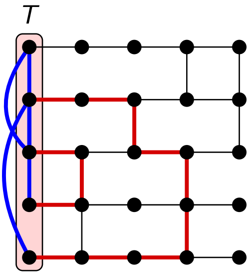

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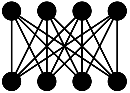
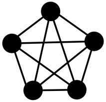
Maximum Disjoint $\mathcal{H}$-Paths



bicliques:
in P



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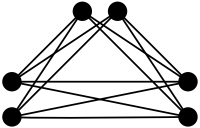
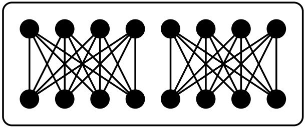
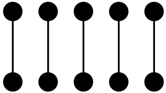


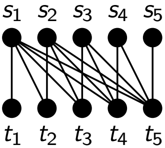
complete multipartite graphs:
in P



two disjoint bicliques:
FPT



matchings:
W[1]-hard



skew bicliques:
W[1]-hard

Maximum Disjoint $\mathcal{H}$-Paths

Questions:

- Algorithmic: FPT vs. W[1]-hard.
- Combinatorial (Erdős-Pósa): is there a function f such that there is either a set of k vertex-disjoint good paths or a set of $f(k)$ vertices covering every good path?

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Theorem [M. and Wollan]

Let $\mathcal{H}$ be a hereditary class of graphs.

1. If $\mathcal{H}$ does not contain every matching and every skew biclique, then Maximum Disjoint $\mathcal{H}$-Paths is FPT and has the Erdős-Pósa Property.
2. If $\mathcal{H}$ does not contain every matching, but contains every skew biclique, then Maximum Disjoint $\mathcal{H}$-Paths is W[1]-hard, but has the Erdős-Pósa Property.
3. If $\mathcal{H}$ contains every matching, then Maximum Disjoint $\mathcal{H}$-Paths is W[1]-hard, and does not have the Erdős-Pósa Property.

Maximum Disjoint $\mathcal{H}$-Paths

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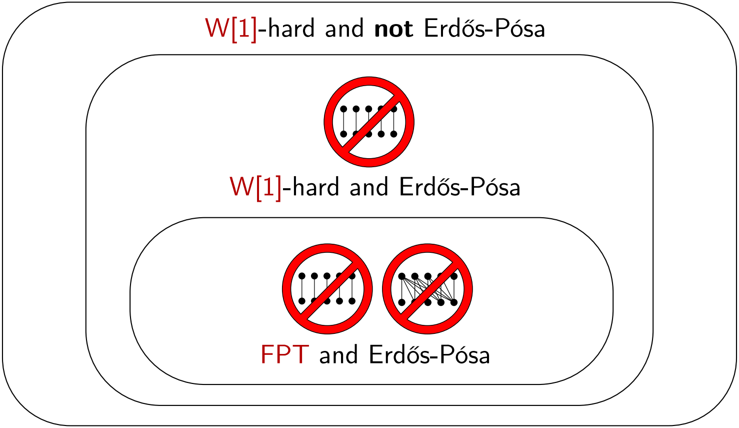

Summary

Dichotomy results:

- P vs. NP-hard or FPT vs. W[1]-hard.
- For a fixed graph H or (hereditary) class $\mathcal{H}$.

Considered problems:

- H-FACTOR
- H-COLORING
- SUB($\mathcal{H}$)
- #SUB($\mathcal{H}$)
- MAXIMUM DISJOINT $\mathcal{H}$-PATHS

Conclusions

- For numerous problems, we can prove that every fixed graph (or graph class) is either easy or hard.
- Good research programs: easy to formulate, hard to solve, but not completely impossible.
- Possible outcomes:
  - Everything is hard, except some trivial cases.
  - Everything is hard, except the famous known nontrivial positive cases.
  - Some unexpected easy cases are found.
- Requires attacking the problem both from the algorithmic and the complexity side.