A short proof of the **NP**-completeness of circular arc coloring

Dániel Marx*

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Abstract

Coloring circular arcs was shown to be **NP**-complete by Garey, Johnson, Miller and Papadimitriou [5]. Here we present a simpler proof of this result.

Keywords: computational complexity, graph coloring, circular arc graphs

1 Introduction

A graph G is a *circular arc graph* if it is the intersection graph of arcs on a circle, that is, the vertices of G can be placed in one-to-one correspondence with a set of arcs in such a way that two vertices in G are adjacent if and only if the corresponding two arcs intersect each other. Clearly, interval graphs form a subset of circular arc graphs. It is well-known that interval graphs can be colored in polynomial time (cf. [6]). On the other hand, it was shown by Garey, Johnson, Miller and Papadimitriou [5] that coloring circular arc graphs is **NP**-complete. The aim of this note is to give a somewhat shorter proof of this result.

After [5], the **NP**-completeness of circular arc coloring was used to establish **NP**-completeness for other problems. It is not very surprising that the **NP**-hardness of wavelength assignment in WDM rings can be proved by reduction from circular arc coloring [2]. It is much more surprising that the **NP**-hardness of wavelength assignment in bidirected WDM binary trees is also proved by reduction from circular arc coloring [2, 7]. Another example is [1], where the **NP**-completeness of precoloring extension on interval graphs is proved by reduction from circular arc coloring.

In Section 2 we introduce the disjoint paths problem and present a lemma of Vygen [9]. In Section 3 we reduce an **NP**-complete disjoint paths problem to circular arc coloring.

*Dept. of Computer Science and Information Theory, Budapest University of Technology and Economics (dmarx@cs.bme.hu). Research supported by OTKA grants 44733, 42559 and 42706.

2 Disjoint paths

In the disjoint paths problem a graph G and a set of source-destination pairs (s_1, t_1), (s_2, t_2), ..., (s_k, t_k) called the *terminals* are given. The goal is to find k disjoint paths $P_1, \ldots, P_k$ such that path P_i connects vertex s_i to vertex t_i. There are four basic variants of the problem: the graph can be directed or undirected, and we can require edge disjoint or vertex disjoint paths. Henceforth the edge disjoint directed version of the problem will be considered. The problem is often described in terms of a supply graph and a demand graph, as follows:

> **Disjoint Paths**
>
> *Input:* The *supply graph* G and the *demand graph* H on the same set of vertices, both directed
>
> *Goal:* Find a path P_e in G for each $e \in E(H)$ such that these paths are edge disjoint and path P_e together with edge e forms a directed circuit

The disjoint paths problem is a classical **NP**-complete problem:

Theorem 2.1 (Even, Itai and Shamir, 1976 [3]). *The disjoint path problem is* **NP***-complete even if G is acyclic.*

A directed graph is *Eulerian* if the indegree equals the outdegree at every vertex. In the Eulerian disjoint paths problem it is assumed that $G + H$ is Eulerian. For various theoretical reasons, this special case was intensively studied (see [8, 4]). However, the problem remains **NP**-complete even with the Eulerian restriction. This can be proved by a simple argument due to Vygen. For completeness, we repeat the proof here.

Lemma 2.2 (Vygen [9]). *The disjoint path problem remains* **NP***-complete even if G is acyclic and $G + H$ is Eulerian.*

Proof. By Theorem 2.1 the problem is **NP**-complete for acyclic supply graphs. In order to make $G + H$ Eulerian, we add two new vertices s, t, and new edges as follows. If the indegree of vertex x in $G + H$ is smaller than the outdegree ($\delta^-_{G+H}(x) < \delta^+_{G+H}(x)$), then add $\delta^+_{G+H}(x) - \delta^-_{G+H}(x)$ parallel copies of the edge $\overrightarrow{sx}$ to G. If $\delta^-_{G+H}(x) > \delta^+_{G+H}(x)$, then add $\delta^-_{G+H}(x) - \delta^+_{G+H}(x)$ copies of the edge $\overrightarrow{xt}$ to G. Denote by G' the modified version of G. Denote by H' the graph obtained from H after adding $\delta^+_{G'}(s) = \delta^-_{G'}(t)$ copies of the edge $\overrightarrow{ts}$. Clearly, $G' + H'$ is Eulerian.

We claim that (G, H) has a solution if and only if (G', H') has. First, it is clear that a solution to (G', H') induces a solution to (G, H), since the demands in H cannot use the new edges in G'. On the other hand, given a solution of (G, H), delete the corresponding demand and supply edges from $G' + H'$. Since $G' + H'$ is Eulerian, and we have deleted directed cycles from it, then what remains is also an Eulerian graph. An Eulerian graph can be decomposed into edge disjoint cycles. In every such cycle at most one copy of the demand edge $\overrightarrow{ts}$ can be present, hence all the remaining demands of H' can be satisfied. ∎

The following simple observation will be useful:

Lemma 2.3. *If $G + H$ is Eulerian and G is acyclic, then every solution of the disjoint path problem uses every edge of G.*

Proof. Consider a solution to the disjoint path problem. Delete from $G + H$ all the directed circuits of the solution. Since $G + H$ is Eulerian, and we delete directed circuits, hence the resulting graph is also Eulerian. However, this graph is a subgraph of G (since every edge of H is part of some deleted directed circuit), thus it is acyclic. If an Eulerian graph is acyclic, then it has no edges, which means that the disjoint paths use all the edges of G. ■

3 Circular arc coloring

In this section we show that it is **NP**-complete to decide whether a given circular arc graph can be colored with k colors. The proof is by reduction from the Eularian disjoint path problem. Note that Garey, Johnson, Miller and Papadimitriou [5] also used a disjoint paths problem as a base for their reduction.

Theorem 3.1. *Circular arc coloring is* **NP***-complete.*

Proof. We reduce the Eulerian directed disjoint path problem to circular arc coloring. Let 1, 2, $\ldots$, n be the vertices of G in a topological ordering. We construct a circular arc graph as follows. Let 0, 1, 2, $\ldots$, n be points on the circle in clockwise order. For every edge $\overrightarrow{xy} \in G$ in G, we add an arc from x to y (going in clockwise direction, i.e., not covering 0). It can be assumed that $x > y$ for every edge $\overrightarrow{xy} \in H$, otherwise there is no solution, since such an x cannot be reached from y. For every edge $\overrightarrow{xy} \in H$ $(x > y)$ we add an arc going from x to y in clockwise direction (i.e., covering 0). Let k be the number of edges in H. We claim that the constructed circular arc graph is k-colorable if and only if there is a solution for the disjoint paths problem.

First assume that the disjoint path problem has a solution. By Lemma 2.3, every edge is used by one of the paths, hence $G + H$ can be partitioned into exactly k disjoint directed circuits. For each such circuit, assign the same color to the arcs corresponding to the edges of the circuit. Every circuit contains an edge $\overrightarrow{xy}$ from H and a path going from y to x in G. The arcs corresponding to these edges do not intersect: the arc corresponding to $\overrightarrow{xy} \in H$ covers the circle from x to y (including 0), while the arcs corresponding to the path in G covers the cycle from y to x. Therefore we obtain a proper k-coloring of the circular arc graph.

Now assume that the arcs have a k-coloring. Notice first that 0 is covered by exactly k arcs. Moreover, all the other points of the circle are also covered by exactly k arcs. This follows from the fact that $G + H$ is Eulerian: the number of arcs that end at a given point $1 \le i \le n$ equals the number of arcs that start at i. The arcs corresponding to the edges in H have different colors, since they all go through point 0. For an arbitrary edge $\overrightarrow{xy} \in H$, consider those arcs that have the same color as the arc corresponding to $\overrightarrow{xy}$. Since every point of the circle is covered by every color, these arcs determine a directed path from y to x. We use this path to satisfy the demand $\overrightarrow{xy}$. Repeating this for every edge of H, we obtain a solution to the disjoint path problem. Every edge of G will be used in only one path, since every arc has only one color. ■

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