# Fixed-Parameter Tractability of Directed Multiway Cut Parameterized by the Size of the Cutset

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**Abstract**

Given a directed graph $G$, a set of $k$ terminals and an integer $p$, the Directed Vertex Multiway Cut problem asks if there is a set $S$ of at most $p$ (nonterminal) vertices whose removal disconnects each terminal from all other terminals. Directed Edge Multiway Cut is the analogous problem where $S$ is a set of at most $p$ edges. These two problems indeed are known to be equivalent. A natural generalization of the multiway cut is the *multicut* problem, in which we want to disconnect only a set of $k$ given pairs instead of all pairs. Marx (Theor. Comp. Sci. 2006) showed that in undirected graphs multiway cut is fixed-parameter tractable (FPT) parameterized by $p$. Marx and Razgon (STOC 2011) showed that undirected multicut is FPT and directed multicut is W[1]-hard parameterized by $p$. We complete the picture here by our main result which is that both Directed Vertex Multiway Cut and Directed Edge Multiway Cut can be solved in time $2^{2^{O(p)}}n^{O(1)}$, i.e., FPT parameterized by size $p$ of the cutset of the solution. This answers an open question raised by Marx (Theor. Comp. Sci. 2006) and Marx and Razgon (STOC 2011). It follows from our result that Directed Multicut is FPT for the case of $k = 2$ terminal pairs, which answers another open problem raised in Marx and Razgon (STOC 2011).

## 1 Introduction

Ford and Fulkerson [7] gave the classical result finding a minimum cut the separates two terminals $s$ and $t$ back in 1956. A natural and well-studied generalization of the minimum $s-t$ cut problem is Multiway Cut, in which given a graph $G = (V,E)$ and a set of terminals $\{s_1, s_2, \dots, s_k\}$, the task is to find a minimum subset of vertices or edges whose deletion disconnects all the terminals from one another. Dahlhaus et al. [4] showed the edge version in undirected graphs is APX-complete for $k \geq 3$. For the edge version Karger et al. [10] gave the current best known approximation ratio of 1.3438 for general $k$. The vertex version of the problem is known to be at least as hard as the edge version, and the current best approximation ratio is $2 - \frac{2}{k}$ [8].

The problem behaves very differently on directed graphs. Interestingly, for directed graphs, the edge and vertex versions turn out to be equivalent. Garg,Vazirani and Yannakakis [8] showed that computing a minimum multiway cut in directed graphs is NP-hard and MAX SNP-hard already for $k = 2$. They also give an approximation algorithm with ratio 2 log $k$, which was improved to ratio 2 later by Naor and Zosin [15].

Rather than finding approximate solutions in polynomial time, one can look for exact solutions in time that is superpolynomial, but still better than the running time obtained by brute force solutions. For example, Dahlhaus et al. [4] showed that undirected Multiway Cut can be solved in time $n^{O(k)}$ on planar graphs, which can be an efficient solution if the number of terminals is small. On the other hand, on general graphs the problem becomes NP-hard already for $k = 3$. In both the directed and the undirected version, brute force can be used to check in time $n^{O(p)}$ if a solution of size at most $p$ exists: one can go through all sets of size at most $p$. Thus the problem can be solved in polynomial time if the optimum is assumed to be small. In the undirected case, significantly better running time can be obtained: the vertex version of the problem can be solved in time $O^*(4^p)$ [2, 9], while the edge version can be solved in time $O^*(2^p)$ [18] (the $O^*$ notation hides all factors which are polynomial in size of input). That is, undirected Multiway Cut is fixed-parameter tractable parameterized by the size of the cutset we remove. Recall that a problem is *fixed-parameter tractable* (FPT) with a particular parameter $p$ if it can be solved in time $f(p)n^{O(1)}$, where $f$ is an arbitrary function depending only on $p$; see [5, 6, 16] for more background. Our main result is that the directed version of Multiway Cut is also fixed-parameter tractable:

Theorem 1.1. **(main result)** Directed Vertex Multiway Cut *and* Directed Edge Multiway Cut *can be solved in* $O^*(2^{2^{O(p)}})$ *time.*

Note that the hardness result of Garg et al. [8] shows that in

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the directed case the problem is nontrivial (in fact, NP-hard) even for $k = 2$ terminals; our result holds without any bound on the number of terminals. The question was first asked explicitly in [12] and was also stated as an open problem in [13]. Our result shows in particular that directed multiway cut is solvable in polynomial time if the size of the optimum solution is $O(\log\log n)$, where $n$ is the number of vertices in the digraph.

A more general problem is MULTICUT: Here the input contains a set $\{(s_1,t_1),\dots,(s_k,t_k)\}$ of $k$ pairs, and the task is to break every path from $s_i$ to its corresponding $t_i$ by the removal of at most $p$ vertices. Very recently, it was shown that undirected MULTICUT is FPT parameterized by $p$ [1, 13], but the directed version is unlikely to be FPT as it is W[1]-hard [13] with this parameterization. However, in the special case of $k = 2$ terminal pairs, there is a simple reduction from DIRECTED MULTICUT to DIRECTED MULTIWAY CUT, thus our result shows that the latter problem is FPT parameterized by $p$ for $k = 2$. Let us briefly sketch the reduction (Note that the reduction we sketch works only for the variant of DIRECTED MULTICUT which allows the deletion of terminals. Marx and Razgon [13] asked about the FPT status of this variant which is in fact equivalent to the one which does not allow deletion of the terminals): Let $(G,T,p)$ be a given instance of DIRECTED MULTICUT and let $T = \{(s_1,t_1),(s_2,t_2)\}$. We construct an equivalent instance of DIRECTED MULTIWAY CUT as follows: Graph $G'$ is obtained by adding two new vertices $s,t$ to the graph and adding the four edges $s \to s_1$, $t_1 \to t$, $t \to s_2$, and $t_2 \to s$. It is easy to see that the DIRECTED MULTIWAY CUT instance $(G',\{s,t\},p)$ is equivalent to the original DIRECTED MULTICUT instance.[1]

COROLLARY 1.1. DIRECTED MULTICUT *with* $k = 2$ *can be solved in time* $O^*(2^{2^{O(p)}})$.

The complexity of the case $k = 3$ remains an interesting open problem.

**Our techniques.** Our algorithm for DIRECTED MULTIWAY CUT is inspired by the algorithm of Marx and Razgon [13] for undirected MULTICUT. In particular we use the technique of "random sampling of important separators" introduced in [13] and try to ensure that there is a solution whose "isolated part" is empty. However, DIRECTED MULTIWAY CUT behaves in a significantly different way than MULTICUT: at the same time, we are dealing with a much easier and a much harder situation. The first step in [13] is to reformulate the problem in a way that the solution has to be a multiway cut of a certain set $W$ of vertices; the technique of *iterative compression* allows us to reduce the original problem to this new version. As MULTIWAY CUT is already defined in terms of finding a multiway cut, this step is not necessary in our case. Furthermore, in [13], after ensuring that there is a solution whose "isolated part" is empty, the problem is reduced to ALMOST-2SAT (Given a 2SAT formula and an integer $k$, is there an assignment satisfying all but $k$ of the clauses ?) This reduction works only if every component has at most two "legs"; a delicate branching algorithm is given to ensure this property. In the case of DIRECTED MULTIWAY CUT, the situation is much simpler: if there is a solution whose "isolated part" is empty, then the problem can be reduced to the undirected version and the known undirected algorithms can be used [2, 9].

On the other hand, the fact that we are dealing with a directed graph makes the problem significantly harder (recall that DIRECTED MULTICUT is W[1]-hard, thus it is expected that not every undirected argument generalizes to the directed case). After defining a proper notion of directed important separators, the non-trivial interaction amongst two kinds of "shadows" forces us to do the random sampling of important separators in two independent steps. In [13], the basic version of random sampling gives a running time that is double exponential in $p$; a more complicated sampling process allowed to bring down the running time from $O^*(2^{2^{O(p)}})$ to $O^*(2^{O(p^3)})$. Directed graphs have a notion of weak versus strong connectivity and this difference does not allow us to extend the more complicated version of sampling to directed graphs. Therefore, it remains an open question if single-exponential running time can be achieved for DIRECTED MULTIWAY CUT.

## 2 Preliminaries

A multiway cut is a set of edges/vertices that separate the terminal vertices from each other:

DEFINITION 2.1. **(multiway cut)** *Let $G$ be a directed graph and let $T = \{t_1,t_2,\dots,t_k\} \subseteq V(G)$ be a set of terminals.*

1. *$S \subseteq V(G)$ is a* vertex multiway cut *of $(G,T)$ if $G \setminus S$ does not have a path from $s_i$ to $s_j$ for any $i \neq j$.*
2. *$S \subseteq E(G)$ is a* edge multiway cut *of $(G,T)$ if $G \setminus S$ does not have a path from $s_i$ to $s_j$ for any $i \neq j$.*

In the edge case, it is straightforward to define the problem we are trying to solve:

| **DIRECTED EDGE MULTIWAY CUT** |
|---|
| *Input* : A directed graph $G$, an integer $p$ and a set of terminals $T$. |
| *Output* : A multiway cut $S \subseteq E(G)$ of $(G,T)$ of size at most $p$ or "NO" if such a multiway cut does not exist. |

[1] $G$ has a $s_i \to t_i$ path for some $i$ if and only if $G'$ has a $s \to t$ or $t \to s$ path. This is because $G$ has a $s_1 \to t_1$ path if and only if $G'$ has a $s \to t$ path and $G$ has a $s_2 \to t_2$ path if and only if $G'$ has a $t \to s$ path. This property of paths also holds after removing some vertices/edges and thus the two instances are equivalent.

In the vertex case, there is a slight technical issue in the definition of the problem: are the terminal vertices allowed to be deleted? We focus here on the version of the problem where the vertex multiway cut we are looking for has to be disjoint from the set of terminals. More generally, we define the problem in such a way that the graph has some *distinguished* vertices which cannot be included as part of any separator (and we assume that every terminal is a distinguished vertex). This can be modeled by considering weights on the vertices of the graph: weight of $\infty$ on each distinguished vertex and 1 on every non-distinguished vertex. We only look for solutions of finite weight. From here on, for a graph $G = (V, E)$ we will denote by $V^\infty(G)$ the set of distinguished vertices of $G$ with the meaning that these distinguished vertices cannot be part of any separator, i.e., all separators we consider are of finite weight. In fact, for any separator we can talk interchangeably about size or weight as these notions are the same since each vertex of separator has weight 1.

The main focus of the paper is the following vertex version, where we require $T \subseteq V^\infty(G)$, i.e., terminals cannot be deleted:

> **DIRECTED VERTEX MULTIWAY CUT**
> *Input* : A directed graph $G$, an integer $p$, a set of terminals $T$ and a set $V^\infty \supseteq T$ of distinguished vertices.
> *Output* : A multiway cut $S \subseteq V(G) \setminus V^\infty(G)$ of $(G, T)$ of size at most $p$ or "NO" if such a multiway cut does not exist.

We note that if we want to allow the deletion of the terminal vertices, then it is not difficult to reduce the problem to the version defined above. For each terminal $t$ we introduce a new vertex $t'$ and we add the directed edges $(t, t')$ and $(t', t)$. Let the new graph be $G'$ and let $T' = \{t' \mid t \in T\}$. Then there is a clear bijection between vertex multiway cuts which can include terminals in the instance $(G, T, p)$ and vertex multiway cuts which cannot include terminals in the instance $(G', T', p)$.

Furthermore, the vertex and edge versions of DIRECTED MULTIWAY CUT defined above are known to be equivalent. For sake of completeness, we prove the equivalence in Appendix A. Henceforth we will refer to DIRECTED VERTEX MULTIWAY CUT as DIRECTED MULTIWAY CUT.

The crucial idea in the algorithm of [13] for undirected MULTICUT is to get rid of the "isolated part" of the solution $S$. We use a similar concept here, but we use the term *shadow,* as it is more expressive for directed graphs.

DEFINITION 2.2. **(separator)** *Let $G = (V, E)$ be a directed graph and $V^\infty \supseteq T$ be the set of distinguished vertices. Given two disjoint non-empty sets $X, Y \subseteq V$ we call a set $S$ of vertices as a $X - Y$* separator *if*

1. *$S$ is disjoint from $X \cup Y$,*
2. *$S$ is disjoint from $V^\infty(G)$, and*
3. *There is no path from $X$ to $Y$ in $G \setminus S$.*

*Set $S$ is a* minimal $X - Y$ separator *if no proper subset of $S$ is an $X - Y$ separator.*

DEFINITION 2.3. **(shadow)** *Let $G$ be graph and $T$ be a set of terminals. Let $S \subseteq V(G) \setminus V^\infty(G)$ be a subset of vertices. Then for $v \in V(G)$ we say that*

1. *$v$ is in the "forward shadow" $f_{G,T}(S)$ of $S$ (with respect to $T$), if $S$ is an $T - \{v\}$ separator in $G$, and*
2. *$v$ is in the "reverse shadow" $r_{G,T}(S)$ of $S$ (with respect to $T$), if $S$ is an $\{v\} - T$ separator in $G$.*

That is, we can imagine $T$ as a light source with light spreading on the directed edges. The forward shadow is the set of vertices that remain dark if the set $S$ blocks the light. In the reverse shadow, we imagine that light is spreading on the edges backwards. We abuse the notation slightly and write $v - T$ separator instead of $\{v\} - T$ separator. We also drop $G$ and $T$ from the subscript if they are clear from the context. Note that $S$ itself is not in the shadow of $S$ (as a $T - v$ or $v - T$ separator needs to be disjoint from $T$ and $v$), that is, $S$ and $f_{G,T}(S) \cup r_{G,T}(S)$ are disjoint.

## 3 Overview of our Algorithm

We say that a solution $S$ of DIRECTED MULTIWAY CUT is *shadowless* if $f(S) \cup r(S) = \emptyset$. If $S$ is a shadowless solution, then for each vertex $v$ in $G \setminus S$, there is a $t_1 \to v$ path and a $v \to t_2$ path for some $t_1, t_2 \in T$. As $S$ is a solution, it is not possible that $t_1 \neq t_2$: this would give a $t_1 \to t_2$ path in $G \setminus S$. Therefore, if $S$ is a shadowless solution, then each vertex in the graph $G \setminus S$ belongs to the strongly connected component of exactly one terminal.

Our algorithm exploits a simple observation: if $S$ is a shadowless solution for the DIRECTED MULTIWAY CUT instance, then $S$ is also a solution for the underlying *undirected* MULTIWAY CUT instance. That is, $S$ separates the terminals from each other not only in the directed graph $G$, but also in the underlying undirected graph obtained by forgetting the orientation of the edges. Indeed, we have observed in the previous paragraph that every vertex is in the strongly connected component of some terminal and a directed edge between the strongly connected components of $t_1$ and $t_2$ would imply the existence of either a $t_1 \to t_2$ or a $t_2 \to t_1$ path. We state this in the following lemma:

LEMMA 3.1. *If $(G, T, p)$ has a shadowless solution $S$, then $S$ is also a solution for the instance $(G^*, T, p)$ where $G^*$ is the underlying undirected graph of $G$.*

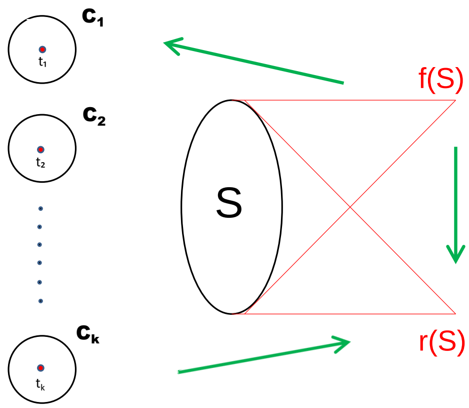


Figure 1: For a set $S$ we let $f(S), r(S)$ denote the forward, reverse shadows respectively. Let $t = \{t_1, t_2, \ldots, t_k\}$ and for every $i \in [k]$ let $C_i$ be the strong component containing $t_i$. Let $C = \cup_{i=1}^{k} C_i$. By definition, in $G \setminus S$ there can only be paths from f(S) to any $C$ and from $C$ to r(S). There cannot be paths from $C$ into $f(S)$ or from $r(S)$ into $C$. Also there can only be paths from $f(S)$ into $r(S)$. From the figure it is clear that if $r(S) \cup f(S) = \emptyset$ then $S$ is a solution of the given instance (and also of the underlying undirected instance).

Lemma 3.1 shows that if we can transform the instance in a way that ensures the existence of a shadowless solution, then we can reduce the problem to undirected MULTIWAY CUT and use the known algorithms for that problem [2, 9, 12].

Our transformation is based on two ingredients: random sampling of important separators and reduction of the instance using the torso operation. These techniques were used [13] for the undirected MULTICUT problem. In Section 4, we review these tools and adapt them for directed graphs.

**Random sampling of important separators.** In order to reduce the problem to a shadowless instance, we need a set $Z$ that has the following property:

> There is a solution $S^*$ such that $Z$ covers the shadow of $S^*$, but $Z$ is disjoint from $S^*$. (*)

Of course, when we are trying to construct this set $Z$, we do not know anything about the solutions of the instance and in particular we have no way of checking if a given set $Z$ satisfies this property. Nevertheless, we use a randomized procedure that creates a set $Z$ and we give a lower bound on the probability that $Z$ satisfies the requirements. For the construction of this set $Z$, we use a very specific probability distribution that was introduced in [13]. This probability distribution is based on randomly selecting "important separators" and taking the union of their shadows. At this point, we can consider the sampling as a black-box function RandomSet$(G,T,p)$ that returns a random subset $Z \subseteq V(G)$ according to a probability distribution that satisfies certain properties. The precise description of this function and the properties of the distribution it creates is described in Section 4.2 (see Theorem 4.1). The randomized selection can be derandomized: the randomized selection can be turned into a deterministic algorithm that returns a bounded number of sets such that at least one of them satisfies the required property. To make the description of the algorithm simpler, we focus on the randomized version of the algorithm in this section.

**Torsos.** We use the function RandomSet$(G,T,p)$ to construct a set $Z$ of vertices that we want to get rid of. The second ingredient of our algorithm is an operation that removes a set of vertices without making the problem any easier. This transformation can be conveniently described using the operation of taking the *torso* of a graph. We define this operation as follows:

DEFINITION 3.1. **(torso)** *Let $G$ be a directed graph and let $C \subseteq V(G)$. The graph* torso$(G,C)$ *has vertex set $C$ and there is (directed) edge $(a,b)$ in* torso$(G,C)$ *if there is an $a \to b$ path in $G$ whose internal vertices are not in $C$.*

In particular, if $a, b \in C$ and $(a,b)$ is a directed edge of $G$, then torso$(G,C)$ contains $(a,b)$ as well. Thus torso$(G,C)$ is a supergraph of the subgraph of $G$ induced by $C$. The following lemma shows that the *torso* operation preserves separation inside $C$.

LEMMA 3.2. **(torso preserves separation)** *Let $G$ be a directed graph and $C \subseteq V(G)$. Let $a, b \in C$, $G' =$ torso$(G,C)$ and $S \subseteq C$. Then $G \setminus S$ has an $a \to b$ path if and only if $G' \setminus S$ has an $a \to b$ path.*

*Proof.* Let $P$ be a path from $a$ to $b$ in $G$. Suppose $P$ is disjoint from $S$. Then $P$ contains vertices from $C$ and $V(G) \setminus C$. Let $u, v$ be two vertices of $C$ such that every vertex of $P$ between $u$ and $v$ is from $V(G) \setminus C$. Then by definition there is an edge $(u,v)$ in torso$(G,C)$. Using such edges we can modify $P$ to obtain a $a \to b$ path that lies completely in torso$(G,C)$ but avoids $S$.

Conversely suppose $P'$ is an $a \to b$ path in torso$(G,C)$ and it avoids $S \subseteq C$. If $P'$ uses an edge $(u,v) \notin E(G)$ then this means that there is a $u \to v$ path $P''$ whose internal vertices are not in $C$. Using such paths we modify $P$ to get an $a \to b$ path $P_0$ that only uses edges from $G$. Since $S \subseteq C$ we have that the new vertices on the path are not in $S$ and so $P_0$ avoids $S$. ■

If we want to remove a set $Z$ of vertices, then we create a new instance by taking the torso on the *complement* of $Z$:

DEFINITION 3.2. *Let $I = (G,T,p)$ be an instance of* DIRECTED MULTIWAY CUT *and $Z \subseteq V(G) \setminus T$. The reduced instance $I/Z = (G',T',p)$ is defined as*

- $G' = \text{torso}(G, V(G) \setminus Z)$
- $T' = T$

The following lemma states that the operation of taking the torso does not make the DIRECTED MULTIWAY CUT problem easier for any $Z \subseteq V(G) \setminus T$ in the sense that any solution of the reduced instance $I/Z$ is a solution of the original instance $I$. Moreover, if we perform the torso operation for a $Z$ that is large enough to cover the shadow of some solution $S^*$ but at the same time small enough to be disjoint from $S^*$, then $S^*$ remains a solution for the reduced instance $I/Z$ and in fact it is a shadowless solution for $I/Z$. Therefore, our goal is to randomly select a set $Z$ in a way that we can bound the probability that $Z$ satisfies Property (*) defined above for some hypothetical solution $S^*$.

LEMMA 3.3. **(creating a shadowless instance)** *Let $I = (G, T, p)$ be an instance of* DIRECTED MULTIWAY CUT *and $Z \subseteq V(G) \setminus T$.*

1. *If $I$ has no solution then $I/Z$ also has no solution.*
2. *If $I$ has solution $S$ with $f_{G,T}(S) \cup r_{G,T}(S) \subseteq Z$ and $S \cap Z = \emptyset$, then $S$ is a shadowless solution of $I/Z$.*

*Proof.* Let $G'$ be the graph $\text{torso}(G, V(G) \setminus Z)$ and let $C = V(G) \setminus Z$. To prove the first statement, suppose that $S' \subseteq V(G')$ is a solution for $I/Z$. We show that $S'$ is also a solution for $I$. Suppose to the contrary that $\exists x, y \in T$ such that there is an $x \to y$ path $P$ in $G \setminus S'$. As $x, y \in T$ and $Z \subseteq V(G) \setminus T$, we have that $x, y \in C$. Then by Lemma 3.2, there is an $x \to y$ path in $G' \setminus S'$, which is a contradiction as $S'$ is a solution of $I/Z$.

For the second statement, let $S$ be a solution of $I$ with $S \cap Z = \emptyset$ and $f_{G,T}(S) \cup r_{G,T}(S) \subseteq Z$. We claim $S$ is a solution of $I/Z$ as well. Suppose that $\exists\ x', y' \in T' (= T)$ such that $G' \setminus S$ has an $x' \to y'$ path. As $x', y' \in V(G) \setminus Z$, Lemma 3.2 implies $G \setminus S$ also has an $x' \to y'$ path, which is a contradiction as $S$ is a solution of $I$.

We claim that $r_{G',T}(S) = \emptyset$. Assume to the contrary that there exists $w \in r_{G',T}(S)$ (note that we have $w \in V(G')$, i.e., $w \notin Z$). So $S$ is a $w - T$ separator in $G'$, i.e., there is no $w - T$ path in $G' \setminus S$. Lemma 3.2 gives that there is no $w - T$ path in $G \setminus S$, i.e., $w \in r_{G,T}(S)$. But $r_{G,T}(S) \subseteq Z$ and so we have $w \in Z$ which is a contradiction. Thus $r_{G,T}(S) \subseteq Z$ in $G$ implies that $r_{G',T}(S)$ is empty in $I/Z$. The argument for $f_{G',T}(S) = \emptyset$ is analogous. ∎

**The algorithm.** The description of our algorithm is given in Algorithm 1. Due to the delicate way separators behave in directed graphs, we construct the set $Z$ in two phases, calling the function RandomSet twice. Our aim is to show that there is a solution $S$ such that we can give a lower bound on the probability that $Z_1$ covers $r_{G_1,T}(S)$ and $Z_2$ covers $f_{G_1,T}(S)$. Note that the graph $G_2$ obtained in Step 2 depends on the set $Z_1$ returned in Step 1 (as we made the weight of every vertex in $Z_1$ infinite), thus the distribution of the second random sampling depends on the result $Z_1$ of the first random sampling. This means that we cannot make the two calls in parallel.

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**Algorithm 1** FPT ALGORITHM FOR DIRECTED MULTIWAY CUT

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**Input**: An instance $I_1 = (G_1, T, p)$ of DIRECTED MULTIWAY CUT.

1. Let $Z_1 = \text{RandomSet}(G_1, T, p)$.
2. Let $G_2$ be obtained from $G_1$ by reversing the orientation of every edge and setting the weight of every vertex of $Z_1$ to infinity (i.e., $V^\infty(G_2) = V^\infty(G_1) \cup Z_1$).
3. Let $Z_2 = \text{RandomSet}(G_2, T, p)$.
4. Let $Z = Z_1 \cup Z_2$.
5. Let $G_3 = \text{torso}(G_1, V(G) \setminus Z)$.
6. Let $G_3^*$ be the underlying undirected graph of $G_3$.
7. Solve the *undirected* instance $(G_3^*, T, p)$.
   - If $S$ is a solution: return $S$.
   - If there is no solution: return "NO".

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We use the torso operation to remove the vertices in $Z = Z_1 \cup Z_2$ (Step 5), and then solve the undirected MULTIWAY CUT instance obtained by disregarding the orientation of the edges. For this purpose, we can use the algorithms of [2, 9] that solve the undirected problem in time $O^*(4^p)$. Note that the algorithm for undirected MULTIWAY CUT in [9] explicitly considers the variant where we have a set of distinguished vertices which cannot be deleted.

In Section 5, we analyze the algorithm and prove that it is a correct randomized algorithm by showing the following:

LEMMA 3.4. **(correctness of the algorithm)** *Let $I$ be an instance of* DIRECTED MULTIWAY CUT.

1. *If $I$ is a no-instance, then Algorithm 1 returns "NO".*
2. *If $I$ is a yes-instance, then Algorithm 1 returns a solution $S$ of $I$ with probability $2^{2^{-O(p)}}$.*

The first claim of Lemma 3.4 is easy to see: a solution $S$ of the undirected instance $(G_3^*, T, p)$ returned by Algorithm 1 is clearly a solution of the directed instance $(G_3, T, p)$ as well, and therefore it is also a solution of $(G_1, T, p)$ (by Lemma 3.2(1), the torso operation does not make the problem easier by creating new solutions). By Lemma 3.3(2), the second claim of Lemma 3.4 can be proved by showing that if $I_1$ is a yes-instance, then there exists a solution $S^*$ such that $Z$ satisfies the two requirements $Z \cap S = \emptyset$ and $f_{G_1,T}(S) \cup r_{G_1,T}(S) \subseteq Z$ with suitable probability. Section 5 is devoted to the proof of this claim. The proof requires a deeper analysis of the structure of optimum solutions and the probability distribution behind the function RandomSet$(G, T, p)$.

**Derandomization.** In Section 4.3, we present a deterministic variant of the function RandomSet$(G, T, p)$ that, instead of returning a random set $Z$, returns a deterministic set $Z_1, \ldots, Z_t$ of $O^*(2^{2^{O(p)}})$ sets. Instead of bounding the probability that the random set $Z$ has Property (*) with some probability, we prove that at least one $Z_i$ always satisfy the property. Therefore, in Steps 1 and 3 of Algorithm 1, we can replace RandomSet with this deterministic variant, and branch on the choice of one $Z_i$ from the returned sets. By the properties of the deterministic algorithm, if $I_1$ is a yes-instance, then one of the branches finds a correct solution for $I_1$. The branching increases the running time only by a factor of $(O^*(2^{2^{O(p)}}))^2$ and therefore the total running time is $O^*(2^{2^{O(p)}})$.

## 4 Important separators and random sampling

This section reviews the notion of important separators and the random sampling technique introduced in [13]. As [13] used these concepts for undirected graphs and we need them for directed graphs, we give a self-contained presentation without relying on earlier work.

**4.1 Important separators** Marx [12] introduced the concept of *important separators* to deal with the UNDIRECTED MULTIWAY CUT problem. Since then it has been used implicitly or explicitly in [2, 3, 11, 13, 17] in the design of fixed-parameter algorithms. In this section, we define and use this concept in the setting of directed graphs. Roughly speaking, an important separator is a separator of small size that is *maximal* with respect to the set of vertices on one side.

DEFINITION 4.1. **(important separator)** *Let $G$ be a directed graph and let $X, Y \subseteq V$ be two disjoint non-empty sets. A minimal $X - Y$ separator $S$ is called an* important $X - Y$ separator *if there is no $X - Y$ separator $S'$ with $|S'| \le |S|$ and $R^+_{G \setminus S}(X) \subset R^+_{G \setminus S'}(X)$, where $R^+_A(X)$ is the set of vertices reachable from $X$ in $A$.*

In undirected graphs, an upper bound of $4^p$ on the number of important $X - Y$ separators of size at most $p$ was given in [2] for any sets $X, Y$. In Appendix B, we show that the same bound holds for important separators even in directed graphs.

LEMMA 4.1. **(number of important separators)** $[\star]^2$ *Let $X, Y \subseteq V(G)$ be disjoint sets in a* directed *graph $G$. Then for every $p \ge 0$ there are at most $4^p$ important $X - Y$ separators of size at most $p$. Furthermore, we can enumerate all these separators in time $O^*(4^p)$.*

For ease of notion, we define the following set of important separators:

DEFINITION 4.2. **(impsep)** *Given a instance $(G, T, p)$ of* DIRECTED MULTIWAY CUT*, a set of vertices is called "impsep" if it is an important $v - T$ separator of size at most $p$ in $G$ for some vertex $v$ in $V(G) \setminus T$.*

It follows from Lemma 4.1 that the total number of impseps in an instance is at most $4^p \cdot |V(G)|$ and we can enumerate all of them in time $O^*(4^p)$.

We now define a special type of shadows which we use later for the random sampling:

DEFINITION 4.3. **(exact shadow)** *Let $G$ be a directed graph and $T \subseteq V(G)$ a set of terminals. Let $S \subseteq V(G) \setminus V^\infty(G)$ be a set of vertices. Then for $v \in V(G)$ we say that*

1. *$v$ is in the "exact reverse shadow" of $S$ (with respect to $T$), if $S$ is a minimal $v - T$ separator in $G$, and*
2. *$v$ is in the "exact forward shadow" of $S$ (with respect to $T$), if $S$ is a minimal $T - v$ separator in $G$.*

The exact reverse shadow of $S$ is a subset of the reverse shadow of $S$: roughly speaking, it contains a vertex $v$ only if every vertex of $S$ can be reached from $v$. This slight difference between the shadow and the exact shadow will be crucial in the analysis of the algorithm (Section 5).

The random sampling described in Section 4.2 (Theorem 4.1) randomly selects impseps and creates a subset by taking the union of the exact reverse shadows of the impseps. The following lemma will be used to give an upper bound on the probability that a vertex is covered by the union.

LEMMA 4.2. *Let $z$ be any vertex. Then there are at most $4^p$ impseps in $G$ which contain $z$ in their exact reverse shadows.*

For the proof of Lemma 4.2, we need to establish first the following:

LEMMA 4.3. *If $S$ is an impsep and $v$ is in the exact reverse shadow of $S$, then $S$ is an important $v - T$ separator.*

*Proof.* Let $w$ be the witness that $S$ is an impsep, i.e., $S$ is an important $w - T$ separator in $G$. Let $v$ be any vertex in the exact reverse shadow of $S$, which means that $S$ is a minimal $v - T$ separator in $G$. Suppose that $S$ is not an important $v - T$ separator. Then there exists a $v - T$ separator $S'$ such

[2]Proofs of results labeled with $\star$ have been moved to the appendix.

that $|S'| \le |S|$ and $R^+_{G\setminus S}(v) \subset R^+_{G\setminus S'}(v)$. We will arrive to a contradiction by showing that $R^+_{G\setminus S}(w) \subset R^+_{G\setminus S'}(w)$, i.e., $S$ is not a important $w-T$ separator.

First, we claim that $S'$ is an $(S\setminus S')-T$ separator. Suppose that there is a path $P$ from some $x \in S\setminus S'$ to $T$ that is disjoint from $S'$. As $S$ is a minimal $v-T$ separator, there is a path $Q$ from $v$ to $x$ whose internal vertices are disjoint from $S$. Furthermore, $R^+_{G\setminus S}(v) \subset R^+_{G\setminus S'}(v)$ implies that the internal vertices of $Q$ are disjoint from $S'$ as well. Therefore, concatenating $Q$ and $P$ gives a path from $v$ to $T$ that is disjoint from $S'$, contradicting the fact that $S'$ is a $v-T$ separator.

We show that $S'$ is a $w-T$ separator and its existence contradicts the assumption that $S$ is an important $w-T$ separator. First we show that $S'$ is a $w-T$ separator. Suppose that there is a $w-T$ path $P$ disjoint from $S'$. Path $P$ has to go through a vertex $y \in S\setminus S'$ (as $S$ is a $w-T$ separator). Thus by the previous claim, the subpath of $P$ from $y$ to $T$ has to contain a vertex of $S'$, a contradiction.

Finally, we show that $R^+_{G\setminus S}(w) \subseteq R^+_{G\setminus S'}(w)$. As $S \neq S'$ and $|S'| \le |S|$, this will contradict the assumption that $S$ is an important $w-T$ separator. Suppose that there is a vertex $z \in R^+_{G\setminus S}(w) \setminus R^+_{G\setminus S'}(w)$ and consider a path $w-z$ path that is fully contained in $R^+_{G\setminus S}(v)$, i.e., disjoint from $S$. As $z \notin R^+_{G\setminus S'}(v)$, path $Q$ contains a vertex $q \in S'\setminus S$. Since $S'$ is a minimal $v-T$ separator, there is a $v-T$ path that intersects $S'$ only in $q$. Let $P$ be the subpath of this path from $q$ to $T$. If $P$ contains a vertex $r \in S$, then the subpath of $P$ from $r$ to $T$ contains no vertex of $S'$ (as $z \neq r$ is the only vertex of $S'$ on $P$), contradicting our earlier claim that $S'$ is a $(S\setminus S')-T$ separator. Thus $P$ is disjoint from $S$, and hence the concatenation of the subpath of $Q$ from $w$ to $q$ and the path $P$ is a $w-T$ path disjoint from $S$, a contradiction. ■

We note that this is the point where it is crucial to distinguish between "reverse shadow" and "exact reverse shadow": Lemma 4.3 (and hence Lemma 4.2) does not remain true if we remove the word exact.

Lemma 4.2 easily follows from Lemma 4.3. Let $J$ be an impsep such that $z$ is in the exact reverse shadow of $J$. By Lemma 4.3, $J$ is an important $z-T$ separator. By Lemma 4.1, there are at most $4^p$ important $z-T$ separators and so $z$ belongs to at most $4^p$ exact reverse shadows.

**4.2 Random sampling** In this section, we adapt the random sampling of [13] to directed graphs. We try to present it in a self-contained way that might be useful for future applications.

Roughly speaking, we want to select a random set $Z$ such that for every $(S,Y)$ where $Y$ is in the reverse shadow of $S$, the probability that $Z$ is disjoint from $S$ but contains $Y$ can be bounded from below. We can guarantee such a lower bound only if $(S,Y)$ satisfies two conditions. First, it is not enough that $Y$ is in the shadow of $S$ (or in other words, $S$ is an $Y-T$ separator), but $S$ should contain important separators separating the vertices of $Y$ from $T$ (see Theorem 4.1 for the exact statement). Second, a vertex of $S$ cannot be in the reverse shadow of other vertices of $S$, this is expressed by the following technical definition:

DEFINITION 4.4. **(thin)** *Let $G$ be a directed graph and $T \subseteq V(G)$ a set of terminals. We say that a set $S \subseteq V(G)$ is* thin *in $G$ if there is no $v \in S$ such that $v$ belongs to the reverse shadow of $S \setminus v$ with respect to $T$.*

THEOREM 4.1. **(random sampling)** *There is an algorithm* RandomSet$(G,T,p)$ *that produces a random set $Z \subseteq V(G) \setminus T$ in time $O^*(4^p)$ such that the following holds. Let $S$ be a* thin *set with $|S| \le p$, and let $Y$ be a set such that for every $v \in Y$ there is a important $v-T$ separator $S' \subseteq S$. For every such pair $(S,Y)$, the probability that the following two events both occur is at least $2^{2^{-O(p)}}$:*

1. *$S \cap Z = \emptyset$, and*
2. *$Y \subseteq Z$.*

*Proof.* The algorithm RandomSet$(G,T,p)$ first enumerates every impsep of size at most $p$; let $\mathscr{X}$ be the set of all exact reverse shadows of these impseps. By Lemma 4.1, the size of $\mathscr{X}$ is $O^*(4^p)$ and can be constructed in time $O^*(4^p)$. Let $\mathscr{X}'$ be the subset of $\mathscr{X}$ where each element from $\mathscr{X}'$ occurs with probability $\frac{1}{2}$ independently at random. Let $Z$ be the union of the exact reverse shadows in $\mathscr{X}'$. We claim that the set $Z$ satisfies the requirement of the theorem.

Let us fix a pair $(S,Y)$ as in the statement of the theorem. Let $X_1, X_2, \ldots, X_d \in \mathscr{X}$ be the exact reverse shadows of every impsep that is a subset of $S$. As $|S| \le p$, we have $d \le 2^p$. By assumption that $S$ is *thin*, we have $X_j \cap S = \emptyset$ for every $j \in [d]$. Now consider the following events:

(E1) $Z \cap S = \emptyset$

(E2) $X_j \subseteq Z \; \forall j \in [d]$

Note that (E2) implies that $Y \subseteq Z$. Our goal is to show that both events (E1) and (E2) occur with probability $2^{2^{-O(p)}}$.

Let $A = \{X_1, X_2, \ldots, X_d\}$ and $B = \{X \in \mathscr{X} \mid X \cap S \neq \emptyset\}$. By Lemma 4.2, each vertex of $S$ is contained in at most $4^p$ exact reverse shadows of impseps. Thus $|B| \le |S| \cdot 4^p \le p \cdot 4^p$. If no exact reverse shadow from $B$ is selected, then event (E1) holds. If every exact reverse shadow from $A$ is selected, then event (E2) holds. Thus the probability that both (E1) and (E2) occur is bounded from below by the probability of the event that every element from $A$ is selected and no element from $B$ is selected. Note that $A$ and $B$ are disjoint: $A$ contains only sets disjoint from $S$, while $B$ contains only sets intersecting $S$. Therefore, the two events are independent and the probability that both events occur is at least

$$\left(\frac{1}{2}\right)^{2^p}\left(1-\frac{1}{2}\right)^{p\cdot 4^p}=2^{-2^{O(p)}}$$

■

**4.3 Derandomization** We now derandomize the process of choosing exact reverse shadows in Theorem 4.1 using the technique of *splitters*. A $(n,r,r^2)$-splitter is a family of functions from $[n]\to[r^2]$ such that $\forall M\subseteq[n]$ with $|M|=r$, at least one of the functions in the family is injective on $M$. Naor, Schulman and Srinivasan [14] give an explicit construction of an $(n,r,r^2)$-splitter of size $O(r^6\log(r)\log(n))$.

In the proof of Theorem 4.1, a random subset of a universe $\mathscr{X}$ of size $n_0=|\mathscr{X}|\le 4^p\cdot|V(G)|$ is selected. We argued that for a fixed $S$, there is a collection $A\subseteq\mathscr{X}$ of $a\le 2^p$ sets and a collection $B\subseteq\mathscr{X}$ of $b\le p\cdot 4^p$ sets such that if every set in $A$ is selected and no set in $B$ is selected, then events (E1) and (E2) hold. Instead of the selecting a random subset, we construct several subsets such that at least one of them satisfies both (E1) and (E2). Each subset is defined by a pair $(h,H)$, where $h$ is a function in an $(n_0,a+b,(a+b)^2)$-splitter family and $H$ is a subset of $[(a+b)^2]$ of size $a$ (there are $\binom{(a+b)^2}{a}=\binom{(2^p+p4^p)^2}{2^p}=2^{2^{O(p)}}$ such sets $H$). For a particular choice of $h$ and $H$, we select those exact shadows $S\in\mathscr{X}$ into $\mathscr{X}'$ for which $h(S)\in H$. The size of the splitter family is $O\Big((a+b)^6\log(a+b)\log(n_0)\Big)=2^{O(p)}\log|V(G)|$ and the number of possibilities for $H$ is $2^{2^{O(p)}}$. Therefore, we construct $2^{2^{O(p)}}\cdot\log|V(G)|$ subsets of $\mathscr{X}$.

By the definition of the splitter, there is a function $h$ that is injective on $A\cup B$, and there is a subset $H$ such that $h(L)\in H$ for every set $L$ in $A$ and $h(M)\notin H$ for every set $M$ in $B$. For such an $h$ and $H$, the selection will ensure that (E1) and (E2) hold. Thus at least one of the constructed subsets has the required properties, which we had to show.

## 5 Analysis of the algorithm

The goal of this section is to show the correctness of Algorithm 1 by proving Lemma 3.4. The first claim of Lemma 3.4 is easy to see:

LEMMA 5.1. *Any set $S$ returned by Algorithm 1 is a solution of $I$. Consequently, if $I$ is a no-instance, then the algorithms returns "NO".*

*Proof.* Suppose that Algorithm 1 returns as set $S$, which is a solution of the undirected instance $(G_3^*,T,p)$. Clearly, $S$ is a solution of the directed instance $I/Z=(G_3,T,p)$ as well. By Lemma 3.3(1), if $I$ has no solution, then $I/Z$ has no solution either, a contradiction. ■

To prove the second claim of Lemma 3.4, we show that if $I$ is a yes-instance, then there exists a solution $S^*$ for $I_1$ that remains a solution of the undirected $(G_3^*,T,p)$ as well with probability $2^{2^{-O(p)}}$.

Suppose that for some solution $S^*$, the following two properties hold:

1. $Z\cap S^*=\emptyset$ and
2. $r_{G_1,T}(S^*)\cup f_{G_1,T}(S^*)\subseteq Z$.

Then Lemma 3.3(2) implies that $S^*$ is a shadowless solution of $I/Z=(G_3,T,p)$. It follows by Lemma 3.1 that $S^*$ is a solution of the undirected instance $(G_3^*,T,p)$ as well. Thus our goal is to prove the existence of a solution $S^*$ for which we can give a lower bound on the probability that these two events occur.

For choosing $S^*$, we need the following definition:

DEFINITION 5.1. **(shadow-maximal solution)** *A solution $S$ for an instance $(G,T,p)$ is* minimal *if no proper subset of $S$ is a solution. A minimal solution $S$ is called* shadow-maximal *if $r_{G,T}(S)\cup f_{G,T}(S)\cup S$ is inclusion-wise maximal among all minimal solutions.*

For the rest of the proof, let us fix $S^*$ to be a shadow-maximal solution of instance $I_1=(G_1,T,p)$ such that $|r_{G_1,T}(S^*)|$ is maximum possible among all shadow-maximal solutions. We bound the probability that $Z\cap S^*=\emptyset$ and $r_{G_1,T}(S^*)\cup f_{G_1,T}(S^*)\subseteq Z$. More precisely, we bound the probability that all of the following four events occur:

1. $Z_1\cap S^*=\emptyset$,
2. $r_{G_1,T}(S^*)\subseteq Z_1$,
3. $Z_2\cap S^*=\emptyset$, and
4. $f_{G_1,T}(S^*)\subseteq Z_2$.

That is, the first random selection takes care of the reverse shadow, the second takes care of the forward shadow, and none of $Z_1$ or $Z_2$ hits $S^*$. Note that it is somewhat counter-intuitive that we choose an $S^*$ for which the shadow is large: intuitively, it seems that the larger the shadow is, the less likely that it is fully covered by $Z$. However, we need this maximality property in order to bound the probability that $Z\cap S^*=\emptyset$.

We want to invoke Theorem 4.1 to bound the probability that $Z_1$ covers $Y=r_{G_1,T}(S^*)$ and $Z_1\cap S^*=\emptyset$. First, we need to ensure that $S^*$ is a *thin* set, but this follows easily from the fact that $S^*$ is a minimal solution:

LEMMA 5.2. *If $S$ is a minimal solution for a* DIRECTED MULTWAY CUT *instance $(G,T,p)$, then no $v\in S$ is in the reverse shadow of some $S'\subseteq S\setminus\{v\}$.*

*Proof.* We claim that $S \setminus \{v\}$ is also a solution, contradicting the minimality of $S$. Suppose that there is a path $P$ from $t_1 \in T$ to $t_2 \in T$, $t_1 \neq t_2$ that intersects $S$ only in $v$. Consider the subpath of $P$ from $v$ to $t_2$. As $v$ is in $r(S')$, the set $S'$ is a $v - T$ separator. Thus $P$ goes through $S' \subseteq S \setminus \{v\}$, a contradiction. ∎

More importantly, if we want to use Theorem 4.1 with $Y = r_{G_1,T}(S^*)$, then we have to make sure that for every vertex $v$ of $r_{G_1,T}(S^*)$, there is an important $v - T$ separator that is a subset of $S^*$. The "pushing argument" of Lemma 5.3 shows that if this is not true for some $v$, then we can modify the solution in a way that increases the size of the reverse shadow. The choice of $S^*$ ensures that no such modification is possible, thus $S^*$ contains an important separator for every $v$.

LEMMA 5.3. **(pushing)** *Let $S$ be a solution of a* DIRECTED MULTIWAY CUT *instance $(G,T,p)$. For every $v \in r(S)$, either there is an $S_1 \subseteq S$ which is an important $v - T$ separator, or there is a solution $S'$ such that*

1. $|S'| \leq |S|$,
2. $r(S) \subset r(S')$,
3. $(r(S) \cup f(S) \cup S) \subseteq (r(S') \cup f(S') \cup S')$.

*Proof.* Let $S_0 \subseteq S$ be the subset of $S$ reachable from $v$ without going through any other vertices of $S$. Then $S_0$ is clearly a $v - T$ separator. Let $S_1$ be the minimal $v - T$ separator contained in $S_0$. If $S_1$ is an important $v - T$ separator, then we are done as $S$ itself contains $S_1$. Otherwise, there exists an important $v - T$ separator $S_1'$, i.e., $|S_1'| \leq |S_1|$ and $R^+_{G \setminus S_1}(v) \subset R^+_{G \setminus S_1'}(v)$. Now we show $S' = (S \setminus S_1) \cup S_1'$ is a solution for the multiway cut instance. Note that $S_1' \subseteq S'$ and $|S'| \leq |S|$.

First we claim that $r(S) \cup (S \setminus S') \subseteq r(S')$. Suppose that there is a path $P$ from $\beta$ to $T$ in $G \setminus S'$ for some $\beta \in r(S) \cup (S \setminus S')$. If $\beta \in r(S)$, then path $P$ has to go through a vertex $\beta' \in S$. As $\beta'$ is not in $S'$, it has to be in $S \setminus S'$. Therefore, by replacing $\beta$ with $\beta'$, we can assume in the following that $\beta \in S \setminus S' \subseteq S_1 \setminus S_1'$. By minimality of $S_1$, every vertex of $S_1 \subseteq S_0$ has an incoming edge from some vertex in $R^+_{G \setminus S}(v)$. This means that there is a vertex $\alpha \in R^+_{G \setminus S}(v)$ such that $(\alpha, \beta) \in E(G)$. Since $R^+_{G \setminus S}(v) \subset R^+_{G \setminus S'}(v)$, we have $\alpha \in R^+_{G \setminus S'}(v)$, implying that there is a $v \to \alpha$ path in $G \setminus S'$. The edge $\alpha \to \beta$ also survives in $G \setminus S'$ as $\alpha \in R^+_{G \setminus S'}(v)$ and $\beta \in S_1 \setminus S_1'$. By assumption, we have a path in $G \setminus S'$ from $\beta$ to some $t \in T$. Concatenating the three paths we obtain a $v \to t$ path in $G \setminus S'$ which contradicts the fact that $S'$ contains an (important) $v - T$ separator $S_1'$. Since $S \neq S'$ and $|S| = |S'|$, the set $S_1 \setminus S_1'$ is non-empty. Thus $r(S) \subset r(S')$ follows from the claim $r(S) \cup (S \setminus S') \subseteq r(S')$.

Suppose now that $S'$ is not a solution for the multiway cut instance. Then there is a $t_1 \to t_2$ path $P$ in $G \setminus S'$ for some $t_1, t_2 \in T$, $t_1 \neq t_2$. As $S$ is a solution for the multiway cut instance, $P$ must pass through a vertex $\beta \in S \setminus S' \subseteq r(S')$ (by the claim in the previous paragraph), a contradiction. Thus $S'$ is also a minimum solution.

Finally, we show that $r(S) \cup f(S) \cup S \subseteq r(S') \cup f(S') \cup S'$. We know that $r(S) \cup (S \setminus S') \subseteq r(S')$. Thus it is sufficient to consider a vertex vertex $v \in f(S) \setminus r(S)$. Suppose that $v \notin f(S')$ and $v \notin r(S')$: there are paths $P_1$ and $P_2$ in $G \setminus S'$, going from $T$ to $v$ and from $v$ to $T$, respectively. As $v \in f(S)$, path $P_1$ intersects $S$, i.e., it goes through a vertex of $\beta \in S \setminus S' \subseteq r(S')$. However, concatenating the subpath of $P_1$ from $\beta$ to $v$ and the path $P_2$ gives a path from $\beta \in r(S')$ to $T$ in $G \setminus S'$, a contradiction. ∎

Note that if $S$ is a shadow-maximal solution, then solution $S'$ in Lemma 5.3 is also shadow-maximal. Therefore, by the choice of $S^*$, applying Lemma 5.3 on $S^*$ cannot produce a shadow-maximal solution $S'$ with $r_{G_1,T}(S^*) \subset r_{G_1,T}(S')$, and hence $S^*$ contains an important $v - T$ separator for every $v \in r_{G_1,T}(S)$. Thus by Theorem 4.1 for $Y = r_{G_1,T}(S^*)$, we get:

LEMMA 5.4. *With probability at least $2^{2^{-O(p)}}$, both $r_{G_1,T}(S^*) \subseteq Z_1$ and $Z_1 \cap S^* = \emptyset$ occur.*

In the following, we assume that the events in Lemma 5.4 occur. Our next goal is to bound the probability that $Z_2$ covers $f_{G_1,T}(S^*)$. Note that $S^*$ is a solution also of the instance $(G_2,T,p)$: the vertices in $S^*$ remained finite (as $Z_1 \cap S^* = \emptyset$ by Lemma 5.4), and reversing the orientation of the edges does not change the fact that $S^*$ is a solution. Solution $S^*$ is a shadow-maximal solution also in $(G_2,T,p)$: Definition 5.1 is insensitive to reversing the orientation of the edges and making some of the weights infinite can only decrease the set of potential solutions. Furthermore, the forward shadow of $S^*$ in $G_2$ is same as the reverse shadow of $S^*$ in $G_1$, that is, $f_{G_2,T}(S^*) = r_{G_1,T}(S^*)$. Therefore, assuming that the events in Lemma 5.4 occur, every vertex of $f_{G_2,T}(S^*)$ has infinite weight in $G_2$. We show that now it holds that $S^*$ contains an important $v - T$ separator in $G_2$ for every $v \in r_{G_2,T}(S^*) = f_{G_1,T}(S^*)$:

LEMMA 5.5. *If $S$ is a shadow-maximal solution for a* DIRECTED MULTIWAY CUT *instance $(G,T,p)$ and every vertex of $f(S)$ is infinite, then $S$ contains an important $v - T$ separator for every $v \in r(S)$.*

*Proof.* Suppose to the contrary that there exists $v \in r(S)$ such that $S$ does not contain an important $v - T$ separator. Then by Lemma 5.3, there is a another shadow-maximal solution $S'$. As $S$ is shadow-maximal, it follows that $r(S) \cup f(S) \cup S = r(S') \cup f(S') \cup S'$. Therefore, the nonempty set $S' \setminus S$ is fully contained in $r(S) \cup f(S) \cup S$. However it cannot contain any vertex of $f(S)$ (as they are infinite by assumption) and cannot contain any vertex of $r(S)$ (as $r(S) \subset r(S')$), a contradiction. ■

Recall that $S^*$ is a shadow-maximal solution also in $(G_2, T, p)$. In particular, $S^*$ is a minimal solution for $G_2$ and so by Lemma 5.2 we have that $S^*$ is thin in $G_2$ also. Thus Theorem 4.1 can be used (with $Y = r_{G_2,T}(S^*)$) to bound the probability that $r_{G_2,T}(S^*) \subseteq Z_2$ and $Z_2 \cap S^* = \emptyset$. As the reverse shadow $r_{G_2,T}(S^*)$ in $G_2$ is the same as the forward shadow $f_{G_1,T}(S^*)$ in $G_1$, we have

LEMMA 5.6. *Assuming the events in Lemma 5.4 occur, with probability at least $2^{2^{-O(p)}}$ both $f_{G_1,T}(S^*) \subseteq Z_2$ and $Z_2 \cap S^* = \emptyset$ occur.*

Therefore, with probability $(2^{2^{-O(p)}})^2$, the set $Z_1 \cup Z_2$ covers $f_{G_1,T}(S^*) \cup r_{G_1,T}(S^*)$ and it is disjoint from $S^*$. By Lemma 3.2, this means that $S^*$ is a shadowless solution of $I/(Z_1 \cup Z_2)$. It follows by Lemma 3.1 that $S^*$ is a solution of the undirected instance $(G_3^*, T, p)$.

LEMMA 5.7. *With probability $2^{2^{-O(p)}}$, $S^*$ is a shadowless solution of $(G_3, T, p)$ and a solution of the undirected instance $(G_3^*, T, p)$.*

In summary, with probability $2^{2^{-O(p)}}$, Algorithm 1 returns a set $S$, which is a solution of $I$ by Lemma 5.1. This completes the proof of Lemma 3.4(2).

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## A Equivalence of DIRECTED VERTEX MULTIWAY CUT and DIRECTED EDGE MULTIWAY CUT

We first show how to solve the vertex version using the edge version. Let $(G, T, p)$ be a given instance of DIRECTED VERTEX MULTIWAY CUT and let $V^\infty(G)$ be the set of distinguished vertices. We construct an equivalent instance $(G', T', p)$ of DIRECTED EDGE MULTIWAY CUT as follows. Let $V' = \{v', v'' \mid v \in V(G)\}$ and $u' = u''$ for all $u \in V^\infty(G)$. The idea is that all incoming/outgoing edges to $v$ in $G$ will now be incoming/outgoing to $v', v''$ respectively. For every vertex $v \in V(G) \setminus V^\infty(G)$ add an edge $(v', v'')$ to $G'$. Let us call these as Type I edges. For every edge $(x, y) \in E(G)$ add $(p+1)$ parallel $(x'', y')$ edges. Let us call these as Type II edges. Define $T' = \{v' \mid v \in T\}$. Note that the number of terminals is preserved. We have the following lemma:

LEMMA A.1. DIRECTED VERTEX MULTIWAY CUT *answers YES if and only if* DIRECTED EDGE MULTIWAY CUT *answers YES.*

*Proof.* Suppose $G$ has a vertex multiway cut say $S$ of size at most $p$. Then the set $S' = \{(v', v'') \mid v \in S\}$ is clearly a edge multiway cut for $G'$ and $|S'| = |S| \leq p$.

Suppose $G'$ has an edge multiway cut say $S'$ of size at most $p$. Note that it does not help to pick in $S$ any edges of Type II as each edge has $(p+1)$ parallel copies and our budget is $p$. So let $S = \{v \mid (v', v'') \in S'\}$. Then $S$ is a vertex multiway cut for $G$ and $|S| \leq |S'| \leq p$. ■

We now show how to solve the edge version using the vertex version. Let $(G, T, p)$ be a given instance of DIRECTED EDGE MULTIWAY CUT. We construct an equivalent instance $(G', T', p)$ of DIRECTED VERTEX MULTIWAY CUT as follows. For each vertex $u \in V(G) \setminus T$, create a set $C_u$ which contains $u$ along with $p$ other copies of $u$. For $t \in T$ we let $C_t = \{t\}$. For each edge $(u,v) \in E(G)$ create a vertex $\beta_{uv}$. Add edges $(x, \beta_{uv})$ for all $x \in C_u$ and $(\beta_{uv}, y)$ for all $y \in C_v$. Define $T' = \bigcup_{t \in T} C_t = T$. Let $V^\infty(G') = T'$

LEMMA A.2. DIRECTED EDGE MULTIWAY CUT *answers YES if and only if* DIRECTED VERTEX MULTIWAY CUT *answers YES.*

*Proof.* Suppose $G$ has an edge multiway cut say $S$ of size at most $p$. Then the set $S' = \{\beta_{uv} \mid (u,v) \in S\}$ is clearly a vertex multiway cut for $G'$ and $|S'| = |S| \leq p$.

Suppose $G'$ has a vertex multiway cut say $S'$ of size at most $p$. Note that it does not help to pick in $S$ any vertices from the $C_z$ of any vertex $z \in V(G) \setminus T$ as each vertex has $(p+1)$ equivalent copies and our budget is $p$. So let $S = \{(u,v) \mid \beta_{uv} \in S'\}$. Then $S$ is a edge multiway cut for $G$ and $|S| \leq |S'| \leq p$. ■

## B Proofs omitted from Section 4

For the proof of Lemma 4.1, we need to establish first some properties of separators.

LEMMA B.1. *Let $G$ be a directed graph and $S$ be an important $X - Y$ separator. Then*

1. *For every $v \in S$, the set $S \setminus v$ is an important $X - Y$ separator in the graph $G \setminus v$.*
2. *If $S$ is an $X' - Y$ separator for some $X' \supset X$, then $S$ is also an important $X' - Y$ separator.*

*Proof.*

1. Suppose $S \setminus v$ is not a minimal $X - Y$ separator in $G \setminus v$. Let $S_0 \subset S \setminus v$ be a $X - Y$ separator in $G \setminus v$. Then $S_0 \cup v$ is a $X - Y$ separator in $G$ but $S_0 \cup v \subset S$ which contradicts the fact that $S$ is a minimal $X - Y$ separator in $G$. Now suppose $\exists\ S'$ such that $|S'| \leq |S \setminus v| = |S| - 1$ and $R^+_{(G \setminus v) \setminus (S \setminus v)}(X) \subset R^+_{(G \setminus v) \setminus S'}(X)$. But $R^+_{(G \setminus v) \setminus (S \setminus v)}(X) = R^+_{G \setminus S}(X)$ as deleting $v$ from graph is equivalent to deleting it as part of the separator. Similarly $R^+_{(G \setminus v) \setminus S'}(X) = R^+_{G \setminus (S' \cup v)}(X)$. Therefore $R^+_{G \setminus S}(X) \subset R^+_{G \setminus (S' \cup v)}(X)$ and also $|S' \cup v| = |S'| + 1 \leq |S|$ which contradicts the fact that $S$ is an important $X - Y$ separator.
2. Let $S'$ be a witness that $S$ is not important $X' - Y$ separator in $G$. Then $|S'| \leq |S|$ and $S'$ is also an $X - Y$ separator. But $S$ is important $X - Y$ separator and hence is also inclusion-wise minimal. Thus $S' \not\subseteq S$, i.e., $S' \setminus S \neq \emptyset$. Now the claim is $\exists\ s' \in S' \setminus S$ such that $s' \in R^+_{G \setminus S}(X)$. If we show this then $s' \in R^+_{G \setminus S}(X) \subseteq R^+_{G \setminus S}(X') \subset R^+_{G \setminus S'}(X')$ which is a contradiction as $s' \in S'$. So let us prove the claim. If any $X \to Y$ path $P$ contains a vertex of $S'$ before reaching a vertex of $S$ then we are done. So suppose every $X \to Y$ path reaches $S$ before $S'$. Then we have $R^+_{G \setminus S}(X) \subset R^+_{G \setminus S'}(X)$ which contradicts the fact that $S$ is an important $X - Y$ separator as $|S'| \leq |S|$.

■

We need the following claim about submodularity of the function which is size of the out-neighborhood of a set. Recall that a function $f : 2^U \to \mathbb{N} \cup \{0\}$ is submodular if for all $A, B \subseteq U$ we have $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$.

LEMMA B.2. **(submodularity)** *Let $G = (V,E)$ be a directed graph. For $A \subseteq V$, let $N^+(A)$ be the out-neighborhood of set $A$, i.e., all the vertices in $G \setminus A$ which have an incoming edge from some vertex in $A$. Then the function $\gamma(A) = |N^+(A)|$ is submodular.*

*Proof.* Let $L = \gamma(A) + \gamma(B)$ and $R = \gamma(A \cup B) + \gamma(A \cap B)$. For any vertex $x \in V$ we have following four possibilities:

1. $x \notin N^+(A)$ and $x \notin N^+(B)$
   In this case, $x$ contributes 0 to both $L$ and $R$.
2. $x \in N^+(A)$ and $x \notin N^+(B)$
   In this case, $x$ contributes 1 to $L$. Clearly $x \notin N^+(A \cap B)$. Also $x \in N^+(A \cup B)$ only if $x \notin B$ and so $x$ can contribute at most 1 to $R$.
3. $x \notin N^+(A)$ and $x \in N^+(B)$
   In this case, $x$ contributes 1 to $L$. Clearly $x \notin N^+(A \cap B)$. Also $x \in N^+(A \cup B)$ only if $x \notin A$ and so $x$ can contribute at most 1 to $R$.
4. $x \in N^+(A)$ and $x \in N^+(B)$
   In this case, $x$ contributes 2 to both $L$ and $R$.

In all four cases the contribution of $x$ to $L$ is always greater equal its contribution to $R$ and hence $L \geq R$, i.e., $\gamma$ is submodular. ■

We also require the following claim which gives a general family of $X - Y$ separators.

LEMMA B.3. *If $X \subseteq Z$ and $Y \cap Z = \emptyset$, then the set $N^+(Z)$ is an $X - Y$ separator.*

*Proof.* We have $X \subseteq Z$ and $Z \cap Y = \emptyset$. By definition of $N^+$, we have $N^+(Z)$ is $Z - Y$ separator and hence also a $X - Y$ separator. ■

## Proof of Lemma 4.1

*Proof.* To prove Lemma 4.1, we show by induction on $2p-\lambda$ that the number of important $X-Y$ separators of size at most $p$ is upper bounded by $2^{2p-\lambda}$ where $\lambda$ is the size of smallest $X-Y$ separator. Note that if $2p-\lambda<0$, then $\lambda>2p\geq p$ and so there is no (important) $X-Y$ separator of size at most $p$. If $2p-\lambda=0$, then $\lambda=2p$. Now if $p=0$ then $\lambda=p=0$ and the empty set is the unique important $X-Y$ separator of size at most $p$. If $p>0$ then $\lambda=2p>p$ and so there is no important $X-Y$ separator of size at most $p$. So we have checked the base case for induction. From now on, the induction hypothesis states that for any disjoint sets $X',Y'\subseteq V(G)$, any $k$ such that $(2k-\beta)<(2p-\lambda)$ where $\beta$ is the size of smallest $X'-Y'$ separator we have that the number of important $X'-Y'$ separators of size at most $k$ is upper bounded by $2^{2k-\beta}$.

Recall that $R^+_{G\setminus S}(X)$ is the vertices reachable from $X$ in $G\setminus S$. Now we prove a claim about uniqueness of minimum size separator whose "reach" is inclusion-wise maximal.

LEMMA B.4. *There is a unique $X-Y$ separator $S^*$ of size $\lambda$ such that $R^+_{G\setminus S^*}(X)$ is inclusion-wise maximal.*

*Proof.* Suppose to the contrary that there are two separators $S'$ and $S''$ of size $\lambda$ such that $R^+_{G\setminus S'}(X)$ and $R^+_{G\setminus S''}(X)$ are incomparable and inclusion-wise maximal. By Lemma B.2, $\gamma$ is submodular and hence $\gamma\Big(R^+_{G\setminus S'}(X)\Big)+\gamma\Big(R^+_{G\setminus S''}(X)\Big)\geq\gamma\Big(R^+_{G\setminus S'}(X)\cup R^+_{G\setminus S''}(X)\Big)+\gamma\Big(R^+_{G\setminus S'}(X)\cap R^+_{G\setminus S''}(X)\Big)$. By definition we have $\gamma(R^+_{G\setminus S'}(X))=\lambda=\gamma(R^+_{G\setminus S''}(X))$. Let $Z=R^+_{G\setminus S'}(X)\cap R^+_{G\setminus S''}(X)$. Then $X\subseteq Z$ and $Z\cap Y=\emptyset$ as $S',S''$ are both $X-Y$ separators. By Lemma B.3, $N^+\Big(R^+_{G\setminus S'}(X)\cap R^+_{G\setminus S''}(X)\Big)$ is a $X-Y$ separator and hence $\gamma\Big(R^+_{G\setminus S'}(X)\cap R^+_{G\setminus S''}(X)\Big)\geq\lambda$ which implies $\gamma\Big(R^+_{G\setminus S'}(X)\cup R^+_{G\setminus S''}(X)\Big)\leq\lambda$. Let $U=R^+_{G\setminus S'}(X)\cup R^+_{G\setminus S''}(X)$. By similar reasoning we have $X\subseteq U$ and $U\cap Y=\emptyset$. So $N^+\Big(R^+_{G\setminus S'}(X)\cup R^+_{G\setminus S''}(X)\Big)$ is also a $X-Y$ separator. But we had $\gamma\Big(R^+_{G\setminus S'}(X)\cup R^+_{G\setminus S''}(X)\Big)\leq\lambda$ which implies $N^+\Big(R^+_{G\setminus S'}(X)\cup R^+_{G\setminus S''}(X)\Big)$ is also a minimum separator which contradicts the maximality of $R^+_{G\setminus S'}(X)$ and $R^+_{G\setminus S''}(X)$. ∎

Let $S^*$ be the unique minimum separator given by Lemma B.4. The following claim shows that every important separator is "behind" this separator:

LEMMA B.5. *For every important $X-Y$ separator $S$, we have $R^+_{G\setminus S^*}(X)\subseteq R^+_{G\setminus S}(X)$.*

*Proof.* Suppose this is not true for some $S$, then by submodularity of $\gamma$ we have $\gamma\Big(R^+_{G\setminus S^*}(X)\Big)+\gamma\Big(R^+_{G\setminus S}(X)\Big)\geq\gamma\Big(R^+_{G\setminus S^*}(X)\cup R^+_{G\setminus S}(X)\Big)+\gamma\Big(R^+_{G\setminus S^*}(X)\cap R^+_{G\setminus S}(X)\Big)$. By definition, $\gamma(R^+_{G\setminus S^*}(X))=\lambda$. As before $N^+\Big(R^+_{G\setminus S^*}(X)\cap R^+_{G\setminus S}(X)\Big)$ is an $X-Y$ separator and hence $\gamma\Big(R^+_{G\setminus S^*}(X)\cap R^+_{G\setminus S}(X)\Big)\geq\lambda$ . This implies $\gamma\Big(R^+_{G\setminus S}(X)\Big)\geq\gamma\Big(R^+_{G\setminus S^*}(X)\cup R^+_{G\setminus S}(X)\Big)$ which contradicts the assumption that $S$ is important $X-Y$ separator as $N^+\Big(R^+_{G\setminus S^*}(X)\cup R^+_{G\setminus S}(X)\Big)$ is a $X-Y$ separator not larger than $S$ but $R^+_{G\setminus S^*}(X)\cup R^+_{G\setminus S}(X)$ is a proper superset of $R^+_{G\setminus S}(X)$. Therefore, for every important separator $S$ the set $R^+_{G\setminus S}(X)$ contains $R^+_{G\setminus S^*}(X)$. ∎

Let $v\in S^*$ be an arbitrary vertex. Note that $\lambda>0$ and so $S^*$ is not empty. Any important $X-Y$ separator $S$ of size at most $p$ either contains $v$ or not. If $S$ contains $v$, then by Lemma B.1 (1), the set $S\setminus\{v\}$ is an important $X-Y$ separator in $G\setminus v$ of size at most $p':=p-1$. As $v\notin X$, the size $\lambda'$ of the minimum $X-Y$ separator in $G\setminus v$ is at least $\lambda-1$. Therefore $2p'-\lambda'<2p-\lambda$ and the induction hypothesis implies that there are at most $2^{2p'-\lambda'}\leq 2^{2p-\lambda-1}$ important $X-Y$ separators of size $p'$ in $G\setminus v$. Hence there are at most $2^{2p-\lambda-1}$ important $X-Y$ separators of size at most $p$ in $G$ that contain $v$.

Now let us bound number of important $X-Y$ separators not containing $v$. By minimality of $S^*$, $v$ has an in-neighbor in $R^+_{G\setminus S^*}(X)$. For every important $X-Y$ separator $S$, we have shown that $R^+_{G\setminus S^*}(X)\subseteq R^+_{G\setminus S}(X)$. As $v\notin S$ and $v$ has an in-neighbor in $R^+_{G\setminus S^*}(X)$, even $R^+_{G\setminus S^*}(X)\cup\{v\}\subseteq R^+_{G\setminus S}(X)$ holds. Let $X'=R^+_{G\setminus S^*}(X)\cup\{v\}$. Then $S$ is an $X'-Y$ separator as $R^+_{G\setminus S^*}(X)\cup\{v\}\subseteq R^+_{G\setminus S}(X)$. Since $X\subseteq X'$ and $S$ is important $X-Y$ separator, by Lemma B.1 (2), S is in fact an important $X'-Y$ separator. Now there cannot exist an $X'-Y$ separator of size $\lambda$ as such a set $S$ would be a $X-Y$ separator of size $\lambda$ in $G$ as well with $R^+_{G\setminus S^*}(X)\cup\{v\}\subseteq R^+_{G\setminus S}(X)$ which contradicts the maximality of $R^+_{G\setminus S^*}(X)$. So the minimum size $\lambda'$ of an $X'-Y$ separator in $G$ is $>\lambda$. By the induction hypothesis, the number of important $X'-Y$ separators of size at most $p$ in $G$ is at most $2^{2p-\lambda'}\leq 2^{2p-\lambda-1}$. Hence there are at most $2^{2p-\lambda-1}$ important $X-Y$ separators of size at most $p$ in $G$ that do not contain $v$.

Adding the bounds in the two cases, we get the required bound of $2^{2p-\lambda}$.

An algorithm for enumerating all the at most $4^p$ important separators follows from the proof. First find a minimum $X-Y$ separator $S'$ in polynomial time. Then for every $w\notin R^+_{G\setminus S'}(X)$, check in polynomial time if there is an $X-Y$ separator of size $\lambda$ which does not contain any vertex from $R^+_{G\setminus S'}(X)\cup w$. This process will take us to the unique $X-Y$ separator $S^*$ of size $\lambda$ such that $R^+_{G\setminus S^*}(X)$ is inclusion-wise maximal. In the last step we branch on whether vertex $v\in S^*$

is in the important separator or not, and recursively find all possible important separators for both cases. ■